CALCULATIONAL GOAL:

Calculate phase and amplitude of wave generated by classical source

\[ h_{ij}^{TT} = G \otimes T \]

To retain analytic control consider PN sources, relevant for early stages of the inspiral

\[
h_{ij}^{TT}(t, x) = -\frac{4G}{|x|} \Lambda_{ij,kl} \left[ \left( \frac{1}{2} \frac{d^2 I^{kl}}{dt^2} + \frac{1}{6} \frac{d^3 I^{klm}}{dt^3} n_m + \frac{1}{24} \frac{d^4 I^{klmn}}{dt^4} n_n n_m + \cdots \right) - \epsilon^{ab(k} \left( \frac{2}{3} \frac{d^2 J^{l)b}}{dt^2} n_a + \frac{1}{4} \frac{d^3 J^{l)bm}}{dt^3} n_a n_m + \frac{1}{15} \frac{d^4 J^{l)bn}}{dt^4} n_a n_m n_n + \cdots \right) + \frac{1}{72} \frac{d^5 J^{l)bnmp}}{dt^5} n_a n_m n_n n_p + \cdots \right] \]

As we discuss momentarily, this expression is sufficient for computing all of the spin effects in the waveform.
IN ADDITION AT HIGHER ORDERS MUST INCLUDE NON-LINEARITIES IN RADIATION

``Tail Effect``

The primary challenge is to calculate the multipole moments in a systematic fashion in the power counting parameter (the relative velocity).
EFT Approach

Extended objects (BH/NS) + GR (the “full theory”)

Relativistic point particle + GR (ZOSO)

2-body NR problem

Composite object + radiation gravitons (NRGR)

Full Metric

\[
\frac{1}{\mu} = r_s
\]

Potential and Radiation

\[
\frac{1}{\mu} = r
\]

Radiation

\[
\frac{1}{\mu} = \frac{r}{v}
\]

Final matching stage onto a set of multiple moments whose constituents obey equations of motion determined by a set of potentials (conservative as well as dissipative) as well as radiation reaction forces (conservative as well as dissipative) coupled to radiation gravitons
MATCH FULL THEORY TO WORLDLINE EFT

\[ L = \int (m + C_R R + C_{\nu\nu} v^\mu v^\nu R_{\mu\nu} + ...) d\tau + M_{pl}^2 \int d^4 x \sqrt{g} R \]

\[ C_{(R,v)} \propto r \]

Removable via field redefinition, but play an important role in renormalizing the theory

**Leading order non-geodesic effects**

\[ L_{size} = \int (C_e E^2 + C_b B^2) d\tau \]

``Love Numbers``

\[ C_e \sim C_b \sim r^3 \]
INGREDIENTS

• Calculate potentials which determine equations of motion, including dissipative effects.

• Solve equations of motion, including radiation reaction forces.

• Plug solutions into multipole moments to generate a wave form

What's known

- Potentials: 4PN
- Multipole moments (including tail effects 3PN)
- Radiation Reaction Forces (1 PN (NLO)) use adiabatic approximation via power loss
MATCH ONTO A THEORY OF RADIATION COUPLED TO MOMENTS OF BINARY SYSTEM

\[ g_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu} + \bar{h}_{\mu\nu} \]

Calculate effective action for radiation (background field) in a systematic expansion in

\[ \Gamma[\bar{h}, x, \dot{x}] \]

Determine systematics of the power counting
NRQCD POWER COUNTING

- NRQCD Applied to onia. Relevant observable is spectrum and lifetimes, inherently quantum mechanical, but has a classical limit.

Consider generic loop diagram near threshold

\[ \frac{1}{E - \frac{p^2}{2m} + i\epsilon} \quad E \sim pv \quad \text{potential modes} \]

\[ \frac{1}{E^2 - k^2 + i\epsilon} \quad E \sim p \quad \text{Hard} \]

\[ E \sim p \sim mv \quad \text{Soft} \]

\[ E \sim p \sim mv^2 \quad \text{Ultra-soft} \]

Soft modes throw fermions off shell and are thus inherently quantum mechanical (generates RG flow)
CALCULATE NRQCD POTENTIALS

\[ \mathcal{L}_p = - \sum_{p,p'} V_{\alpha\beta\lambda\tau}(p,p') \mu^{2e} \left[ \psi_{p'\alpha}(x) \psi_{p\beta}(x) \chi_{-p'\lambda}(x) \chi_{-p\tau}(x) \right] \]

The potential is defined as the Wilson coefficient (short distance). No contributions from Usoft degrees of freedom. The Wilson loop has IR divergence at 3 loops (ADM) which is NOT part of the potential.

To do the matching one can:

a) calculate in the full theory and expand.

b) Asymptotically expand the full theory integrand around the potential, soft and Usoft regions.

c) Expand the full theory field into modes each with a definite scaling (simpler way of achieving b) )

\[ A_\mu = A^p_\mu + A^r_\mu + A^s_\mu \quad A^p_\mu \sim \nu^{3/2} \quad A^r_\mu \sim \nu^2 \]
Systematically calculate to chosen order in velocity

\[ V^{(-2)} = (T^A \otimes \bar{T}^A) \frac{V_c^{(T)}}{k^2} + (1 \otimes 1) \frac{V_c^{(1)}}{k^2} \]

\[ V^{(0)} = (T^A \otimes \bar{T}^A) \left[ \frac{V_2^{(T)}}{m^2} + \frac{V_r^{(T)}}{2m^2} \frac{p^2 + p'^2}{k^2} + \frac{V_s^{(T)}}{m^2} S^2 + \frac{V_{A}^{(T)}}{m^2} \Lambda(p', p) + \frac{V_t^{(T)}}{m^2} T(k) \right] \]

\[ \mathcal{O}_{ss}^{(0)} = -g_s^2 \mu_s \mu_t \epsilon C(q, q', p, p') \frac{1}{2} \psi^+_p [B_{\mu, q'}, B^\mu_q] \psi_p \]

\[ B^\mu(x) = -\frac{1}{g} S^+_V(x, -\infty) i D^\mu(x) S_V(x, -\infty) \]
We have also kept imaginary terms generated by the cut amplitudes in Eqs. (19) and (21) is
\[ \epsilon \]
and there is no matching correction at this order.

The real part of the amplitude agrees with Ref. [23], except for the order \[ \epsilon \] part.

Note that we disagree with Ref. [23] on the order \[ \epsilon \] part.

In Eqs. (21b), (21c), and (21d), the factors of the \[ (\gamma \Lambda) \] function from the effective theory taking two insertions of the tree level operators and the remaining terms correspond to graphs involving two factors of the \[ (\gamma \Lambda) \] function. These graphs are discussed in more detail below.

Divergence in this term comes from the Coulomb divergence in the potential regime.

Note that we disagree with Ref. [23] on the order \[ \epsilon \] part.

In the effective theory, terms from hard, soft and potential.

We have also kept imaginary terms generated by the cut amplitudes in Eqs. (19) and (21) is
\[ \epsilon \]
and there is no matching correction at this order.

The real part of the amplitude agrees with Ref. [23], except for the order \[ \epsilon \] part.

Note that we disagree with Ref. [23] on the order \[ \epsilon \] part.

In Eqs. (21b), (21c), and (21d), the factors of the \[ (\gamma \Lambda) \] function from the effective theory taking two insertions of the tree level operators and the remaining terms correspond to graphs involving two factors of the \[ (\gamma \Lambda) \] function. These graphs are discussed in more detail below.

Divergence in this term comes from the Coulomb divergence in the potential regime.

Note that we disagree with Ref. [23] on the order \[ \epsilon \] part.

In the effective theory, terms from hard, soft and potential.
IMPORTANT TECHNICAL ISSUE WITH CALCULATION OF THE SOFT LOOP

\[ I^S = \int [d^4k] \frac{1}{k_0 + i\epsilon} \frac{1}{k_0 - i\epsilon} \frac{1}{k_0^2 - \vec{k}^2 + i\epsilon} \frac{1}{k_0^2 - (\vec{k} - \vec{p})^2 + i\epsilon} \]

Ill-define pinched contour

The pinched region is actually potential, we have a mode overlap. Need to subtract out potential region

``Zero Bin Procedure” (Manohar /Stewart)

\[ I^C_3 = \int [d^4k] \frac{1}{k_0 + i\epsilon} \frac{1}{k_0 - i\epsilon} \frac{1}{-\vec{k}^2 + i\epsilon} \frac{1}{-(\vec{k} - \vec{p})^2 + i\epsilon} \]

SUBTRACTING SUB-REGION LIMIT LEADS TO WELL DEFINED INTEGRALS

A similar subtraction clarifies the 3 loop ADM IR divergence in the YM Wilson loop
TAKE CLASSICAL LIMIT OF QCD

We can choose to either take the classical limit of NQRCD or to work directly with world lines.

- Working with fluctuating lines, drop all terms down by 1/m.

**World-line**

Note: In the classical limit m becomes irrelevant.

**Potential mode scaling**

\[ E \sim \frac{v}{r}, k \sim \frac{1}{r} \]
NRGR POTENTIAL

Has been calculated to 4PN by two groups (Damour, Bini Jaranowski, Schaefer: Bernard, Bohe, Blanchet, Faye, Marsat)

Diagramatically all reduce to two point topology (1 external momentum)
AMPLITUDE METHOD  D. Neill/IZR, V. Vaidya

Upside: No Feynman diagrams.

Downside: Pick out classical pieces, must perform subtractions of TOPs, number of which grows at each order in PN expansion.

\[ V^{cl}(q, \bar{p}) = \frac{Gm^2}{q^2} \left( \sum_{n=0}^{\infty} (Gmq)^n \left( \frac{\bar{p}}{m} \right)^t \right) \]

non-linearities

Sub-leading vertices and corrections to instantaneity
FIG. 3. The time ordered product in the effective theory which must be subtracted from the full theory result. The square vertex is the order $v^2$ Coulomb potential, while the dot corresponds to the order $v^2$ kinetic term correction. The oval is the order $v^4$ Coulomb potential and the cross is the order $v^4$ correction to the kinetic term. Mirror image diagrams have been suppressed. Diagrams a and b are 1PN while c through h are 2PN and have been included only because we are interested in the metric at order $G^2$. 

VI. EXTRACTING THE METRIC BY GOING TO THE PROBE LIMIT

Given the potential we may extract the metric order by order in powers of $v$ by comparing the potential to the world-line action. Assuming a static and isotropic solution we may make the ansatz for the metric:

$$g_{00} = 1 + \frac{1}{X_i} A_i$$

$$g_{0i} = 0$$

$$g_{ij} = \epsilon_{ij} (1 + \frac{1}{X_i} B_i)$$

$$r^i r^j r^2 (1 + \frac{1}{X_i} C_i) (17)$$
PROS AND CONS OF AMPLITUDE METHOD

- No Feynman diagrams (usual simplifications)
- Must remove all the quantum pieces (no big deal)
- Must subtract Time ordered products, the number of which will grow (quickly) at each order in the PN expansion.
ISSUE OF DIVERGENCES

At 3PN we find that there both UV and IR divergences. UV divergences are acceptable as long as the requisite counter term exists in the EFT, while the only IR divergences that are acceptable are of two type:

1) Unknown phase of GW when entering the detector band

2) A coulomb phase which cancels in the power

All other IR are not acceptable and are an indication that we are not cleanly separating the radiation from the potential modes in the theory.

Subset of the UV divergences can be absorbed into unphysical operators

$$ L = \int (m + C_R R + C_v v^\mu v^\nu R_{\mu\nu} + ...) d\tau $$

True finite size effects don’t show up until 5PN (effacement the: Damour)
Also at 4PN there is a log associated with this IR divergence.

\[ \int dt V^S_{4PN}(t, \mu) = -\frac{G^2_N M}{5} \int \frac{d\omega}{2\pi} \omega^6 I^{ij}(-\omega) I^{ij}(\omega) \left( \frac{1}{\epsilon_{IR}} - 2 \log(\mu r) \right) + \text{local/finite} \]

Structure of this term belies its source
Following the methodology discussed in [42] and integration-by-parts. (Notice there are no IR divergences at terms in the effective theory.)

Hence, the zero-bin subtraction not only properly removes the IR divergences in the conservative sector, moreover, they are key to obtaining the correct renormalization group evolution for the Wilson coefficients.

Moreover, breaking which (3.13) can be set to zero in this framework. We should stress, however, that in general breaking which must be consequently also regularized using a Pauli-Villars cut-off prior to the zero-bin subtraction, whereas the log instead of (3.10). The log

The zero-bin subtraction must be implemented in all of the IR-sensitive master integrals that enter at one-loop integral in (3.8). The latter will enter in the decomposition of many of the three-loop amplitudes, and therefore IR poles may appear in principle from dimensional analysis.

After the zero-bin subtraction, the result will take the form in (3.11) simply

\[ I_{\text{zero-bin}}[n_1, n_2] = \int \frac{1}{[k^2]n_1[p^2]n_2} \frac{1}{(n_1 \rightarrow 3/2, n_2 \rightarrow 1/2)} |p|^{-1} \int \frac{1}{k^3} \propto |p|^{-1} \left( \frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \]

As expected IR divergence cancel, but leaves over a UV divergence, no counter-terms at 4PN.

It seems we need a counter-term at 4PN, of which there are none.

MUST CONSISTENTLY INCLUDE CONSERVATIVE PART OF RADISTION REACTION FORCE
SELF FORCE CALCULATIONS
(C GALLEY)

Integrate out Radiation Field in the context of doubling formalism

Generalizing Hamilton's Principle to allow for dissipation

\[ S[\vec{x}_1, \vec{x}_2, h_{\mu\nu}, h_{\mu\nu}] = S[\vec{x}_1, h_{\mu\nu}] - S[\vec{x}_2, h_{\mu\nu}] \]

\[ W[\vec{x}_a^\pm] = \int dt (L[\vec{x}_a^{(1)}] - L[\vec{x}_a^{(2)}] + R[\vec{x}_a^{(1)}, \vec{x}_a^{(2)}]) \]

Conservative piece

\[ R[\vec{x}_a^{(1)}, \vec{x}_a^{(2)}] \supset F[\vec{x}_a^{(1)}] - F[\vec{x}_a^{(2)}] \]

Dissipative piece

\[ 0 = \frac{\delta S_{\text{eff}}[\vec{x}_1^\pm, \vec{x}_2^\pm]}{\delta \vec{x}_K^-(t)} \bigg|_{\vec{x}_K^- = 0} \]

IF PART OF R CAN BE WRITTEN IN THE THIS FORM IT CAN BE ABSORBED INTO THE CONSERVATIVE POTENTIAL
BURKE THORNE POTENTIAL

\[ iW[x^\pm_a] = \begin{pmatrix} I_{ij}^A \cr I_{ij}^B \end{pmatrix} \]

\[ = \left( \frac{1}{2} \right) \left( \frac{i}{2M_{Pl}} \right)^2 \int dt \int dt' I_{ij}^A(t) \langle E_{ij}^A(t, 0) E_{kl}^B(t', 0) \rangle I_{kl}^B(t') \]

\[ (a^i_a)_{rr}(t) = -\frac{2G_N}{5} I^{(5)}_{ij}(t) x^j_a(t) \]

2.5 PN effect

Purely dissipative at this order no logs
4 PN TAIL RADIATION REACTION

(Blanchet),(Galley,Leibovich,Porto,Ross), (Foffa,Sturani)

Contains conservative piece which can be absorbed into potential

\[
\int dt V_{\text{tail}}(\mu) = \frac{G_N^2 M}{5} \int \frac{d\omega}{2\pi} \omega^6 \, I^{ij}(-\omega) I^{ij}(\omega) \left[ \frac{1}{\epsilon_{\text{UV}}} + \log \frac{\omega^2}{\mu^2} + \gamma_E - \log \pi - \frac{41}{30} \right]
\]

UV divs cancel leaving over \( \text{Log}(r\omega) \)
As a check of our result, note that upon time-averaging we are the same as in the time-ordered scheme, it is simple order in the long wavelength approximation, the \( \ell \) adapted to the classical system (see [9] for a purely classical approach). This yields an expectation value of the form:

\[
\langle \mathcal{T} \rangle = \frac{i \pi^2}{10} G^2 M \int \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} I_{ij}(\omega_1)I_{ij}(\omega_2) \times (2\pi)\delta(\omega - \omega_1 - \omega_2)P(\omega_1, \omega_2) + \cdots
\]

Mass becomes time dependent, though purely a counter-term.

\( 1/r \to \nu/r \) (Goldberger, Ross, IZR)

Renormalize theory of Multipole moments
\[ \mu \frac{d}{d\mu} M(t, \mu) = -\frac{2G^2 \bar{M}}{5} \left( 2I_{ij}^{(5)} I_{ij}^{(1)} - 2I_{ij}^{(4)} I_{ij}^{(2)} + I_{ij}^{(3)} I_{ij}^{(3)} \right) (t) \]

\[ \mu \frac{d}{d\mu} \ln \bar{M} = -2G^2 \langle I_{ij}^{(3)} I_{ij}^{(3)} \rangle_\mu \]

\[ \mu \frac{dI_{ij}}{d\mu} (\omega, \mu) = -\frac{214}{105} (G\bar{M}\omega)^2 I_{ij} (\omega, \mu) \]

Leading Log Solution

\[ \frac{\bar{M}(\mu)}{\bar{M}_0} = \exp \left[ \frac{\langle I_{ij}^{(2)} I_{ij}^{(2)} \rangle_\mu_0 - \langle I_{ij}^{(2)} I_{ij}^{(2)} \rangle_\mu}{\beta I \bar{M}_0^2} \right] \]
Interpretation of scale dependent mass

RG solution allows us to predict leading log at each order in the PN expansion

\[ \nu^4 (r_s \omega)^{2n} \ln^n \nu \]
AVOIDING ORBIT AVERAGING

- Averaging is excellent approximation save for eccentric orbits and spinning inspires.

- To avoid averaging we need radiation reaction forces to NNNLO. (for 3 PN accuracy)

- Only known right now at NLO.

- Find analytic solutions to inspiral (with spin in particular)

\[
\begin{align*}
\ddot{r} - r \omega^2 &= -\frac{M}{r^2} + \frac{64 M^3 \nu}{15 r^4} \dot{r} + \frac{16 M^2 \nu}{5 r^3} \dot{r}^3 + \frac{16 M^2 \nu}{5 r} \dot{r} \omega^2 \\
\dot{r} \omega + 2 \dot{r} \omega &= -\frac{24 M^3 \nu}{5 r^3} \omega - \frac{8 M^2 \nu}{5 r^2} \dot{r}^2 \omega - \frac{8 M^2 \nu}{5} \omega^3
\end{align*}
\]
Treating RR forces as perturbation induces secular growth in time
\[ \varepsilon \equiv \nu^5 \nu \Omega(t - t_0) \]
Utilize "Dynamical Renormalization Group" (Chen, Goldenfeld)

**Background solution:**
\[
\begin{align*}
  r(t) &= R_B + A_B \sin \left( \Omega_B(t - t_0) + \Phi_B \right) \\
  \omega(t) &= \Omega_B - \frac{2 \Omega_B A_B}{R_B} \sin \left( \Omega_B(t - t_0) + \Phi_B \right)
\end{align*}
\]

**Pert. Solution**
\[
\begin{align*}
  r(t) &= R_B - \frac{64\nu}{5} \Omega_B^6 R_B^6(t - t_0) + \frac{64\nu}{5} \Omega_B^5 R_B^6 \sin \Omega_B(t - t_0) + A \sin \left( \Omega_B(t - t_0) + \Phi \right) \\
  \omega(t) &= \Omega_B + \frac{96\nu}{5} R_B^5 \Omega_B^7(t - t_0) - \frac{128\nu}{5} R_B^5 \Omega_B^6 \sin \Omega_B(t - t_0) - \frac{2 \Omega_B A}{R_B} \sin \left( \Omega_B(t - t_0) + \Phi \right)
\end{align*}
\]

Absorb Divergence terms into bare parameters

\[
\begin{align*}
  \frac{d}{d\tau} R_R(\tau) &= -\frac{64\nu}{5} R_R^6(\tau) \Omega_R^6(\tau) \\
  \frac{d}{d\tau} \Omega_R(\tau) &= \frac{96\nu}{5} R_R^5(\tau) \Omega_R^7(\tau) \\
  \frac{d}{d\tau} \Phi_R(\tau) &= \Omega_R(\tau), \\
  \frac{d}{d\tau} A_R(\tau) &= 0.
\end{align*}
\]

**Galley/IZR**
\[
\begin{align*}
  R_R(t) &= \left( R_R^4(t_i) - \frac{256\nu}{5} M^3(t - t_i) \right)^{1/4} \\
  \Omega_R(t) &= \Omega_R(t_i) \left( \frac{R_R(t_i)}{R_R(t)} \right)^{3/2} \\
  \Phi_R(t) &= \Phi_R(t_i) + \frac{R_R^{5/2}(t_i) - R_R^{5/2}(t)}{32\nu M^{5/2}} \\
  A_R(t) &= A_R(t_i).
\end{align*}
\]
TWO LOOP SOLUTION

FIG. 2: Fractional errors for the orbital radius (top panel) and phase (bottom panel) between the numerical solution of (4.2), (4.49) and the one-loop (blue) and two-loop (orange) resummed solutions for the same system and initial conditions shown in Fig. 1.

but is much better at describing the small oscillations due to the $O(v^5)$ eccentricity that results from choosing $\dot{r}(0) = 0$ as part of the initial data.

VI. BEYOND LEADING ORDER RADIATION REACTION

In this formalism the inclusion of higher order radiation reaction forces is straightforward. The equations of motion through the 1PN correction to radiation reaction forces \cite{5, 6} are given by

\begin{align}
\ddot{r} + 2\dot{r} & = M \frac{r}{2} + \frac{64}{15} M^3 \alpha \frac{r^4}{\dot{r}^2} + \frac{16}{5} M^2 \alpha \frac{r^3}{\dot{r}^3} (\dot{r}^2 + \dot{r} \dot{r}) \\
& \quad + \frac{8}{105} M^4 \alpha \frac{r^5}{\dot{r}} (821 + 210 \alpha) \dot{r} + \frac{8}{105} M^3 \alpha \frac{r^4}{\dot{r}} \dot{r} \left(362 + 245 \alpha\right) r^2 \dot{r}^2 + 54 \dot{r}^4 \end{align}

(6.1)

and

\begin{align}
\dot{\dot{r}} + 2\dot{r} & = 245 M^3 \alpha \frac{r^3}{\dot{r}^2} \left(\dot{r}^2 + \dot{r} \dot{r}\right) + \frac{4}{105} M^4 \alpha \frac{r^4}{\dot{r}} (1325 + 546 \alpha) \dot{r}^2 + \frac{4}{35} M^2 \alpha \frac{r^2}{\dot{r}^4} \left(313 + 42 \alpha\right) r^4 \dot{r}^4 + 40 \dot{r}^6 \end{align}

(6.2)

We are interested in demonstrating how to handle higher PN order secular terms in DRG so we do not include the 1PN or higher potentials here, which do not (directly) generate secularly diverging perturbations. Of course, a fully consistent orbital solution should include all potentials that contribute to a given PN order.

As done in the previous section we expand the solution around the background including perturbations up to order $v^7_B$. Following (5.1), where now $\epsilon \sim v^7_B R_B$ and $\gamma \sim v^6_B \delta_B \sim v^7_B / R_B$, we find that the perturbed radial and angular
DRG INVARIANTS AS GENERALIZED ADIABATIC INVARIANTS

\[ J = \oint pdq \]

Constants averaged over orbits

DRG invariants are constants in time, how they related?

**TIME DEPENDENT OSCILLATOR**

\[ L = \frac{1}{2} \dot{x}^2 - \frac{1}{2} (\omega_0 + \gamma(t - t_0))^2 x^2 \]

\[ J = E/\omega \] Been shown to be EXACT in \( \gamma \)

\[ x = A_B \cos(\omega_B + \phi_B) \] DRG background solution

**DRG equations**

\[ \frac{d}{d\tau} A_R = -\frac{A_R \gamma}{2\omega_R} \]
\[ \frac{d}{d\tau} \phi_R = \omega_R \]
\[ \frac{d}{d\tau} \omega_R = \gamma \]

**DRG invariant**

\[ \kappa = A_R^2 \omega_R \]
\[ \langle A_R^2 / \omega_R \rangle = \langle E / \omega \rangle \]
• Is this an exact DRG invariant? Checked to three loops

\[
\frac{dx_R(\tau)}{d\tau} = -\frac{\gamma x_R(\tau)}{2\omega_R(\tau)} + \frac{9\gamma^3 x_R(\tau)}{16\omega_R^5(\tau)}
\]

\[
\frac{d\omega_R(\tau)}{d\tau} = \gamma - \frac{9\gamma^3}{8\omega_R^4(\tau)}
\]

\[
\frac{d\phi_R(\tau)}{d\tau} = \omega_R(\tau)
\]

• Other DRG invariants which don't follow (directly) from canonical Hamilton-Jacobi theory? Yes,

\[
I_\phi \equiv \phi_R(t) - I_\omega t - \frac{\gamma}{2} t^2
\]

\[
= \phi_R(t_i) - I_\omega t_i - \frac{\gamma}{2} t_i^2
\]

\[
I_\omega \equiv \omega_R(t) - \gamma t
\]

\[
= \text{constant}
\]

Spinning orbits have 9 DOF, invariants will play an important role in finding closed form solutions