UV/IR Polytope: carving out the space for consistent gauge and gravity theories

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Through the lens of physical observables, a new picture has arose where physical principles (unitarity, locality, symmetries) are geometrized: see Nima, Song’s and Yun-Tao’s talk.

Is there any sense in which this picture can be universal?
More precisely, can we geometrize the constraint of \textit{unitarity} + \textit{symmetries} for general QFT?

- What observables through which this geometry is realised?
- What are the content to be constrained?

\[
\text{EFT: } L + \sum_i c_i O_{i\text{Irrelevant}}
\]

- EFT: S-matrix, coefficient of higher-dimensional (irrelevant) operators
- CFT: Four point correlation function, conformal dimension and three-point coupling of primary operators.
EFT

In the IR the UV degrees of freedom are encoded in the higher dimensional operators. These information are encoded in the four-point function as

$$M(s, t) = \sum_{i,j} g_{i,j} s^i t^j$$

where \( s = (p_1 + p_2)^2 \) \( t = (p_1 + p_4)^2 \). For example:

$$\mathcal{L} = \frac{1}{2} \phi \Box \phi + a (\partial \phi \cdot \partial \phi)^2 \rightarrow M(s, t) = a(s^2 + st + t^2)$$

Naively, without an explicit UV completion, we have nothing to say. This is not true in the forward limit \( t = 0 \)

A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis and R. Rattazzi:

$$g_{i,0} = \int \frac{ds}{s^{i+1}} M(s, 0) = \int_{s_0}^{\infty} \text{Im}[M(s, 0)] \quad i \in \text{even}$$

But from the optical theorem we know that \( \text{Im}[M(s, 0)] = s \sigma > 0 \rightarrow g_{i,0} > 0 \)
For $t \neq 0$, causality bound $M(s, t)|_{s \to \infty} < s^2$. This implies that after subtraction, $M' = M - a_2 s^2 - a_1 s$, we again have a bounded function:

$$M'(s, t) = \oint_C \frac{d\nu}{v-s} M'(v, t) = -\oint_{C'} \frac{d\nu}{v-s} M'(v, t)$$

This implies an alternative representation:

$$M'(s, t) = \sum_{i=1}^{\infty} \left( \frac{1}{s-m_i^2} + \frac{1}{u-m_i^2} \right) n(m_i, t)$$
• Lorentz invariance + Unitarity dictates

\[ A_3(\phi_1, \phi_2, h^\ell) \sim i c_\ell (p_1 - p_2)^{\mu_1} (p_1 - p_2)^{\mu_2} \cdots (p_1 - p_2)^{\mu_\ell} \epsilon_{\mu_1 \mu_2 \cdots \mu_\ell} \]

The residue must take the form \((X \equiv p_1 - p_2, \ Y \equiv p_3 - p_4)\):

\[ X^{\mu_1} X^{\mu_2} \cdots X^{\mu_\ell} P_{\mu_1 \cdots \mu_\ell \nu_1 \cdots \nu_\ell} Y^{\nu_1} Y^{\nu_2} \cdots Y^{\nu_\ell} \]

where \(P_{\mu_1 \cdots \mu_\ell \nu_1 \cdots \nu_\ell}\) is symmetric traceless. This implies

\[ \Box_X f(X, Y) = \delta^{D-1}(X - Y) \rightarrow \frac{1}{|1 - \cos \theta t + t^2|^{D-3/2}} = \sum_\ell t^\ell G^D_\ell (\cos \theta) \]

\[ n(m_i, t) = c^2_i G^D_{\ell_i} \left( 1 + \frac{2t}{m_i^2} \right) \]
What constraint can we derive for the low energy EFT from the existence of a UV completion of the form:

$$M'(s, t) = - \sum_i \left( \frac{1}{s - m_i^2} + \frac{1}{u - m_i^2} \right) c_{i, \ell_i}^2 G_{\ell_i}^D \left( 1 + \frac{2t}{m_i^2} \right)$$

Note that this is applicable for massive loop corrections
• First since the is $s$, $u$-symmetric it would be convenient to switch to

$$s = -\frac{t}{2} + z, \quad u = -\frac{t}{2} - z$$

• Second, recall that low energy EFT has a four point function of the form:

$$M^{IR}(z, t) = \sum_{i,j} g_{i,j} z^{2i} t^j$$

• For a fixed mass dimension $2L$, the space of possible higher dimensional operators has dimension that correspond to the number of $(i, j)$s that satisfies $2i + j = L$. Any particular theory corresponds to a particular point in this subspace:

Exp: for $L = 4$ we have

$$\tilde{g}_{0,4} t^4 + \tilde{g}_{2,2} z^2 t^4 + \tilde{g}_{4,0} z^4$$

A given EFT is represented as a specific point $(\tilde{g}_{0,4}, \tilde{g}_{2,2}, \tilde{g}_{4,0})$ in this three-dimensional space
• Exp: for \( L = 4 \) we have

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A given EFT is represented as a specific point \((\tilde{g}_{0,4}, \tilde{g}_{2,2}, \tilde{g}_{4,0})\) in this three-dimensional space

\[
\frac{\Gamma[-\alpha' s] \Gamma[-\alpha' t] \Gamma[-\alpha' u]}{\Gamma[1 + \alpha' s] \Gamma[1 + \alpha' t] \Gamma[1 + \alpha' u]} = \cdots + \zeta_7 \left( \frac{9}{8} t^4 + 3 z^2 t^2 + 2 z^4 \right) \alpha'^4 + \cdots
\]

type-II string theory correspond to \((\frac{9}{8}, 3, 2)\)
Within this space, what is the subspace that has a UV completion of the form?

\[ A(z, t) = -\sum_i \left( \frac{1}{-\frac{t}{2} + z - m^2_i} + \frac{1}{-\frac{t}{2} - z - m^2_i} \right) c^2_{i, \ell_i} G^D_{\ell_i} \left( 1 + \frac{2t}{m^2_i} \right) \]

We can also expand in low energy. For fixed \( L \) the coefficient for \( z^{2a} t^{L-2a} \):

\[ z^{2a} t^{L-2a} : \sum_i c^2_{i, \ell_i} \left\{ \sum_{q=0}^{L-2a} \left( - \right)^q 2^{L-2i-2q+1} \frac{[2a + 1]q}{q!(L - 2a - q)!} G^D_{\ell_i,q} \right\} = \sum_i c^2_{i, \ell_i} \hat{G}^D_{\ell,a} \]

where \( G^D_{\ell,q} = \frac{\partial^q}{\partial x^q} G^D_{\ell}(1 + x) \bigg|_{x=0} \). For \( L = 4 \) we have infinite set (finite spins) of three-dimensional vectors

\[ \mathcal{V}_{\ell} = \begin{pmatrix} \hat{G}^D_{\ell,4} \\ \hat{G}^D_{\ell,2} \\ \hat{G}^D_{\ell,0} \end{pmatrix} \]

The allowed space is spanned by the convex hull of \( \mathcal{V}_\ell \)s.
We have an infinite number of vectors spanning a finite dimensional space → there probably is no constraint?
Remarkably, it does not span the full space!

Note that if we look at the last entry of the vertices

\[ \mathcal{V}_\ell = \begin{pmatrix} \hat{G}^{D}_{\ell,4} \\ \hat{G}^{D}_{\ell,2} \\ \hat{G}^{D}_{\ell,0} \end{pmatrix} \]

\[ \hat{G}^{D}_{\ell,0} \sim G^{D}_{\ell_i}(1) > 0 \]

This is nothing but the old positivity bound \( g_{i,0} > 0 \) A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis and R. Rattazzi

i.e. it’s just a tip of the iceberg.
Let’s consider the space projectively. Taking

$$v_\ell = \left( \frac{v_\ell^1 - v_\ell^2}{v_\ell^3}, \frac{v_\ell^2 - v_\ell^3}{v_\ell^3} \right)$$
$L = 6$ is spanned by a four-dimensional space: the projective three-dimensional polytope:

$$
\mathbf{v}_\ell = \left( \frac{\mathbf{v}^1_\ell - \mathbf{v}^2_\ell}{\mathbf{v}^4_\ell}, \frac{\mathbf{v}^2_\ell - \mathbf{v}^3_\ell}{\mathbf{v}^4_\ell}, \frac{\mathbf{v}^3_\ell - \mathbf{v}^4_\ell}{\mathbf{v}^4_\ell} \right)
$$

There such a “UV-IR” polytope in each dimension $\rightarrow$ infinite number of constraint
What is special about these polytopes? What is the geometric feature of Lorentz-invariance and Unitarity?

The vertices appear to lie on a moment curve:

\[ V_\ell \sim (1, t, t^2, \cdots, t^d) \]

The convex hull of points on a moment curve \( \rightarrow \) cyclic polytope!
Consider the ordered (here by spin) matrices constructed by vertex vectors. If all ordered minor is positive, then the vertices construct a cyclic polytope.

Proof:

\[
\det \begin{bmatrix} V_{\ell_1} & V_{\ell_2} & \cdots & V_{\ell_L} \end{bmatrix} = \prod_{i<j}(\ell_i - \ell_j)(D - 3 + \ell_i + \ell_j) \prod_i \Gamma[D - 3 + \ell_i] \prod_i \Gamma[D - 3 - 2 + 2i] \Gamma[1 + \ell_i]
\]

Positive if \( \ell_1 < \ell_2 < \cdots < \ell_L \)

- The gegenbauer polytope is a cyclic polytope
- All vectors are part of the vertices of the cyclic polytope
Important property of cyclic polytope:

- All boundaries are known (CP^{d-1})

  \[ d = 3 : (\mathcal{V}_{\ell_i} \mathcal{V}_{\ell_{i+1}}), \quad d = 4 : (\mathcal{V}_{\ell_1} \mathcal{V}_{\ell_i} \mathcal{V}_{\ell_{i+1}}), \quad (\mathcal{V}_{\ell_i} \mathcal{V}_{\ell_{i+1}} \mathcal{V}_{\mathcal{N}}), \quad d = 5 : (\mathcal{V}_{\ell_i} \mathcal{V}_{\ell_{i+1}} \mathcal{V}_{\ell_j} \mathcal{V}_{\ell_{j+1}}) \]

- Projecting through a vertex, the higher-dimensional cyclic polytope lands on a lower-dimensional one

  \[
  \langle \mathcal{V}_{\ell_1}, \mathcal{V}_{\ell_2}, \mathcal{V}_{\ell_3}, \mathcal{V}_{\ell_4} \rangle = \begin{pmatrix}
  * & * & * & * \\
  0 & * & * & * \\
  0 & * & * & * \\
  0 & * & * & * 
\end{pmatrix} > 0 \rightarrow \langle \bar{\mathcal{V}}_{\ell_2}, \bar{\mathcal{V}}_{\ell_3}, \bar{\mathcal{V}}_{\ell_4} \rangle > 0
  \]

- Through projection, what's inside the higher-dimensional polytope is also inside the lower dimensional one:

  \[
  \langle \mathcal{V}_{\ell_1}, \mathcal{V}_{\ell_i}, \mathcal{V}_{\ell_{i+1}}, \mathcal{X} \rangle > 0 \rightarrow \langle \mathcal{V}_{\ell_i}, \mathcal{V}_{\ell_{i+1}}, \mathcal{X} \rangle > 0
  \]
The fact that it’s a cyclic polytope resolves an apparent tension with RG

For simplicity let’s consider only s-channel contributions:

\[ A(z, t) = - \sum_i \left( \frac{1}{s - m_i^2} \right) c_{i, \ell_i}^2 G_{\ell_i}^D \left( 1 + \frac{2t}{m_i^2} \right) \rightarrow \sum_{i,j} g_{i,j} s^i t^j \]

Let’s say we have for mass-dimension four: \( a_1 s^2 + a_2 st + a_3 t^2 \). For mass-dimension six, we then have \( a_1 s^3 + a_2 s^2 t + a_3 st^2 + a_4 t^6 \)

- We will have a three-dimensional constraint for \((a_1, a_2, a_3)\), but this is not enough since
- We will also have a four-dimensional constraint for \((a_1, a_2, a_3, a_4)\) and so forth
- This would appear that the coefficients of lower mass-dimension operators are very sensitive to that of higher mass-dimension operators inconsistent with RG
However since the coefficients are bounded by a cyclic polytope

- The projection of a higher-dimensional polytope lands on a lower dimensional cyclic polytope, implies that any point in the lower dimensional polytope is guaranteed to have an image upstairs

\[(a_1, a_2, a_3) \rightarrow (a_1, a_2, a_3, a_4)\]

- The higher-dimensional polytope only constrains the new coefficient!
We can ask where does super string theory lies:
What about gauge and gravity?

In four-dimensions, 2 massless 1 massive N. Arkani-Hamed, T-z Huang, Y-t

The three-point amplitude is also unique

\[ \epsilon^{\mu_1 \mu_2 \cdots \mu_\ell} X_{\mu_1} \cdots X_{\mu_i} q_{\mu_{i+1}} \cdots q_{\mu_\ell} q^{\alpha \dot{\alpha}} = \lambda_1 \tilde{\lambda}_2 \]

The degree of \( q \) depends on the helicity \((h_1, h_2)\).

It is convenient to use manifest SL(2,C) on-shell representations

\[ M^{h_1 h_2}_{\{\alpha_1 \alpha_2 \cdots \alpha_{2S}\}} = \frac{g}{m^{2S+h_1+h_2-1}} \left( \lambda_1^{S+h_2-h_1} \lambda_2^{S+h_1-h_2} \right)_{\{\alpha_1 \alpha_2 \cdots \alpha_{2S}\}} \]
\[ M^{h_1 h_2}_{\{\alpha_1 \alpha_2 \cdots \alpha_{2S}\}} = \frac{g}{m^{2S+h_1+h_2-1}} \left( \lambda_1^{S+h_2-h_1} \lambda_2^{S+h_1-h_2} \right)_{\{\alpha_1 \alpha_2 \cdots \alpha_{2S}\}} [12]^{S+h_1+h_2}. \]

We can orient

\[
\begin{align*}
\lambda_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} \sin \frac{\theta}{2} \\ -\cos \frac{\theta}{2} \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{pmatrix}
\end{align*}
\]

The numerator is simply the Wigner $d$-matrix $d_{m',m}^j(\theta) = \langle j, m' | e^{-i\theta J_y} | j, m \rangle$, or equivalently

\[
d_{m',m}^j(\theta) = \mathcal{J}(\ell + 4h, 0, -4h, \cos \theta)
\]
We also have a universal polynomial, \( G^D_\ell(\cos \theta) \rightarrow \mathcal{J}(\ell + 4h, 0, -4h, \cos \theta) \) We again have

\[
M(s, t) = \frac{n(h_i)}{st^2} + \sum_i \left( \frac{1}{s - m_i^2} + \frac{1}{u - m_i^2} \right) \mathcal{J}(\ell + 4h, 0, -4h, \cos \theta)
\]

- The polytopes are also cyclic!
- These are real world predictions, for any theory involving weakly coupled matter.
Remarkably, the same polytope is present for CFT four-point function! W Nima, Shu-Heng Shao

Consider the a 1D four-point function:
\[
\langle \phi(1)\phi(2)\phi(3)\phi(4) \rangle \equiv F(z)
\]
\[
F(z) = \sum_{\Delta} p_{\Delta}^2 C_{\Delta}(z), \quad C_{\Delta}(z) = z^\Delta_2 F_1(\Delta, \Delta, 2\Delta, z)
\]

We can again expand the four-point function, say around \( z = \frac{1}{2} \)
\[
F \left( \frac{1}{2} + y \right) = \sum_{q=0}^{\infty} F_q y^q
\]

The 1-D blocks also yield an infinite set of vectors
\[
C_{\Delta} \left( \frac{1}{2} + y \right) = \sum_{q=0}^{\infty} c_{\Delta,q} y^q
\]

Unitarity then requires that
\[
F = \begin{pmatrix} F_0 & F_1 & \cdots & F_{L-1} \\ \end{pmatrix} \subset \sum_{\Delta} p_{\Delta}^2 \begin{pmatrix} c_{\Delta,0} \\ c_{\Delta,1} \\ \vdots \\ c_{\Delta,L-1} \end{pmatrix}
\]
Now crossing is just

\[ z^{-2\Delta \phi} F(z) = (1 - z)^{-2\Delta \phi} F(1 - z) \rightarrow F(z) = \left( \frac{z}{1 - z} \right)^{2\Delta \phi} F(1 - z) \]

Again expanded around \( z = \frac{1}{2} \) we find

\[ \sum_q F_q y^q = \left( \frac{1 + 2y}{1 - 2y} \right)^{2\Delta \phi} \sum_q (-)^q F_q y^q \]

This tells us that \( F \) must lie within the crossing plane \( X \)

We have the polytope \( P(\Delta_i) = \sum_i p_{\Delta_i}^2 c_{\Delta_i} \) and a crossing plane \( X(\Delta_{\phi}) \), and they must intersect. \( P(\Delta_i) \) is a cylic polytope! See Nima’s talk
For example:

\[
\Delta_0 = 0, \quad \Delta_1 = 2, \\
\Delta_2 = 2.5, \quad \Delta_3 = 3.1 \\
\Delta_4 = 0.2
\]

\[
\Delta_0 = 0, \quad \Delta_1 = 2, \\
\Delta_2 = 2.5, \quad \Delta_3 = 3.1 \\
\Delta_4 = 0.5
\]

\[
\Delta_0 = 0, \quad \Delta_1 = 2, \\
\Delta_2 = 2.5, \quad \Delta_3 = 3.1 \\
\Delta_4 = 0.34
\]
• The geometrization of unitarity and locality for observables in QFT
• These constraints imposed on “Factorization” (in terms of poles in the S-matrix or OPE in correlation function) ⊕ “Symmetry” (Lorentz or conformal), leads to the physical observable bounded by a cyclic polytope
• Cyclic polytope naturally leads to a RG like picture
• What other geometric structure lies beyond the real block ($z = \bar{z}$)?
• What does this imply for available models? Weak gravity conjecture?