Infrared & transcendental structure of 2-loop super-QCD amplitudes

based on work with Claude Duhr, Henrik Johansson, Gregor Kälin, Alexander Ochirov, and Bram Verbeek

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Color-kinematics duality with matter [Johansson, Ochirov '14]

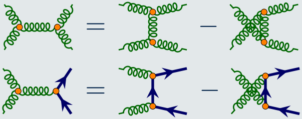
General L-loop gauge theory amplitude, $D = 4 - 2\epsilon$:

$$\mathcal{A}_m^{(L)} = \sum_{i \in \text{cubic}} \int \frac{\mathrm{d}^{LD}\ell}{(2\pi)^{LD}} \frac{N_f^{|i|}}{S_i} \frac{c_i n_i(\ell)}{D_i(\ell)}$$

We seek a duality between color and kinematics [BCJ '08]:

$$n_i = n_j - n_k \Longleftrightarrow c_i = c_j - c_k \qquad n_i = -n_j \Longleftrightarrow c_i = -c_j$$

Jacobi/commutation relations



$$\tilde{f}^{a_1 a_2 b} \tilde{f}^{b a_3 a_4} = \tilde{f}^{a_4 a_1 b} \tilde{f}^{b a_2 a_3} - \tilde{f}^{a_2 a_4} \tilde{f}^{b a_3 a_1}$$

$$\begin{aligned} \tilde{f}^{b a_3 a_4} T^b_{i_1 \bar{\imath}_2} &= T^{a_3}_{i_1 \bar{\jmath}} T^{a_4}_{j \bar{\imath}_2} - T^{a_4}_{i_1 \bar{\jmath}} T^{a_3}_{j \bar{\imath}_2} \\ &= [T^{a_3}, T^{a_4}]_{i_1 \bar{\imath}_2} \end{aligned}$$

This setup has advantages besides the double copy:

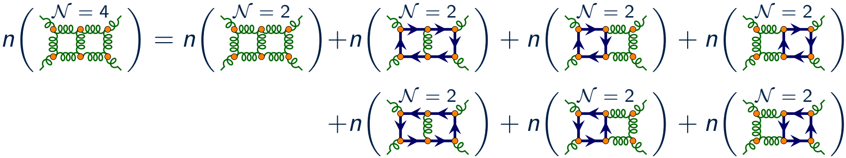
1. The gauge group G is kept entirely arbitrary.
2. All physical input comes from planar unitarity cuts.

$\mathcal{N} = 2$ super-QCD

Break up 16 states of the 4D $\mathcal{N} = 4$ SYM on-shell super-multiplet:

$$\begin{aligned}\mathcal{V}_{\mathcal{N}=4} &= A^+ + \eta^A \psi_A^+ + \frac{1}{2}\eta^A\eta^B\varphi_{AB} + \cdots + \eta^1\eta^2\eta^3\eta^4 A_- \\ &= \underbrace{V^+_{\mathcal{N}=2}}_{\text{4 states}} + \eta^3 \underbrace{\Phi_{\mathcal{N}=2}}_{\text{4 states}} + \eta^4 \underbrace{\overline{\Phi}_{\mathcal{N}=2}}_{\text{4 states}} + \eta^3\eta^4 \underbrace{V^-_{\mathcal{N}=2}}_{\text{4 states}}\end{aligned}$$

Decomposition works at the level of individual diagrams, e.g.



N_f flavors of $\Phi_{\mathcal{N}=2}$, $\overline{\Phi}_{\mathcal{N}=2}$ in the fundamental rep of gauge group G.

$$\beta(\alpha_\mathrm{s}) = -\alpha_\mathrm{s}\left(2\epsilon + \beta_0\frac{\alpha_\mathrm{s}}{2\pi}\right) \iff \alpha_\mathrm{s}^0 = \alpha_\mathrm{s}(\mu^2)\mu^{2\epsilon}\sum_{L=0}^{\infty}\left(-\frac{\beta_0}{\epsilon}\frac{\alpha_\mathrm{s}(\mu^2)}{4\pi}\right)^L$$

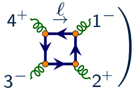
$$\beta_0 = C_A - T_F N_f \overset{\mathrm{SU}(N_c)}{=} 2N_c - N_f \implies \text{critical point at } N_f = 2N_c$$

Matter loops $\implies$ controlled IR divergences

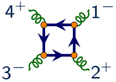
L-loop massless Feynman integrals can diverge as $1/\epsilon^{2L}$ in the IR, as there are 2 kinds of IR divergence:

1. **Soft**: when $\ell \to 0$
2. **Collinear**: when $\ell \to \tau p_i$

In diagrams with matter loops they can be prevented by ℓ-dependent numerators, e.g. $\mathcal{N}=2$ SQCD BCJ numerator:

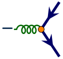


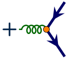
$$n\left(\qquad\right) = \frac{1}{u^2}\,\mathrm{tr}_-(1\ell(\ell+p_4)3) = \frac{1}{u}(\ell-\ell^*)^2\,, \quad \ell^* = \frac{\langle 34\rangle}{\langle 13\rangle}|1\rangle[4|$$



$$\implies \qquad = \frac{r_\Gamma}{2u^2}\left(\log^2\left(\frac{s}{t}\right)+\pi^2\right)+\mathcal{O}(\epsilon)$$

The numerator is inherently chiral, as it lives on only one 4D cut branch:

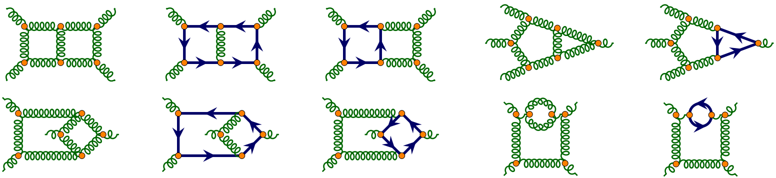
MHV :


$\overline{\text{MHV}}$:


The 2-loop 4-gluon amplitude

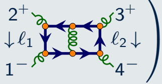
[Johansson, Kälin, GM '17]

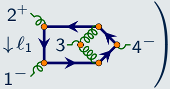
$\mathcal{A}_4^{(2)}$ consists of 19 diagrams, 10 are finite upon integration:



We found BCJ numerators by fitting an ansatz for two masters.
Expressions were originally given in terms of $\ell_i \cdot \ell_j$, $\epsilon(\ell_1, \ell_2, p_i, p_j)$, etc.

Trace-based BCJ numerators, e.g.





$$n\left(\quad\right) = n\left(\quad\right) = \frac{\mathrm{tr}_-(1\ell_1 2 3 \ell_2 4)}{t^2}$$

The double copy gives $\mathcal{N}=4$ supergravity $+$ $\mathcal{N}=4$ SYM multiplets:

$$\mathcal{V}_{\mathcal{N}=2} \otimes \mathcal{V}_{\mathcal{N}=2} = \mathcal{H}_{\mathcal{N}=4} \oplus 2\mathcal{V}_{\mathcal{N}=4}$$

$$\Phi_{\mathcal{N}=2} \otimes \overline{\Phi}_{\mathcal{N}=2} = \mathcal{V}_{\mathcal{N}=4}$$

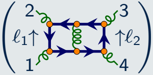
Planar $\mathcal{N}=2$ superconformal QCD (SCQCD)

Break up into $\mathcal{N}=4$ and superconformal components:

$$\mathcal{A}_n^{(L)} = \underbrace{\mathcal{A}_n^{(L)[\mathcal{N}=4]} + \mathcal{R}_n^{(L)}}_{\mathcal{N}=2\ \text{SCQCD}} + \beta_0\, \mathcal{S}_n^{(L)}$$

Planar remainder function

$$R^{(2)[2]}_{(1234)} \equiv \mathcal{R}_4^{(2)}\Big|_{N_c^2 \mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})} = \sum_{\text{cyclic}} \int \frac{\mathrm{d}^{2D}\ell}{D_{\mathrm{db}}}\, n \Bigg($$



$$\Bigg)$$

With $\tau=-t/s$, $\upsilon=-u/s$, $T=\log(\tau)$, $U=\log(\upsilon)$ and $s>0$; $t,u<0$

$$\begin{aligned} R^{(2)[2]}_{(--++)} = 12\zeta_3 + \frac{\tau}{6}\big\{ & 48\mathrm{Li}_4(\tau) - 24T\mathrm{Li}_3(\tau) - 24T\mathrm{Li}_3(\upsilon) + 24\mathrm{Li}_2(\tau)\,(\zeta_2+TU) \\ & + 24TU\mathrm{Li}_2(\upsilon) - 24U\mathrm{Li}_3(\tau) - 24S_{2,2}(\tau) + T^4 - 4T^3U + 18T^2U^2 \\ & - 12\zeta_2T^2 + 24\zeta_2TU + 24\zeta_3U - 168\zeta_4 - 4i\pi\big[6\mathrm{Li}_3(\tau) + 6\mathrm{Li}_3(\upsilon) \\ & - 6U\mathrm{Li}_2(\tau) - 6U\mathrm{Li}_2(\upsilon) - T^3 + 3T^2U - 6TU^2 - 6\zeta_2T + 6\zeta_2U\big]\big\} + \mathcal{O}(\epsilon) \end{aligned}$$

$$R^{(2)[2]}_{(-+-+)} = 12\zeta_3 + \frac{1}{6}\frac{\tau}{\upsilon^2}T^2(T+2i\pi)^2 + \mathcal{O}(\epsilon)$$

Maximal transcendentality is “slightly” violated by ζ_3 terms [Dixon, Kosower, Vergu ‘08].

Subleading color

[Duhr, Johansson, Kälin, GM, Verbeek '19]

The subleading-N_c terms contain IR divergences:

$$\underbrace{\mathcal{R}_n^{(2)}}_{\sim\epsilon^{-2}} = \left(\sum_{i<j}^{n} s_{ij} \; \mathbf{T}_i\cdot\mathbf{T}_j\right)\underbrace{\mathcal{R}_n^{(1)}}_{\sim\epsilon^{0}} + \underbrace{\mathcal{R}_n^{(2)\mathrm{fin}}}_{\sim\epsilon^{0}}$$

$\mathcal{R}_4^{(2)}$ contains weights 2,3,4; $\mathcal{R}_4^{(2)\mathrm{fin}}$ contains only weights 3,4:

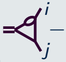
$$\begin{aligned}
R^{(2)[1]\mathrm{fin}}_{(--)(++)} = \frac{2\tau}{3}\big\{&96\mathrm{Li}_4(\tau) - 72T\mathrm{Li}_3(\tau) + 24T\mathrm{Li}_3(\upsilon) + 24T\mathrm{Li}_2(\tau)(T-U)\\
&-24U\mathrm{Li}_2(\upsilon)(T-U) + 96\mathrm{Li}_4(\upsilon) + 24U\mathrm{Li}_3(\tau) - 72U\mathrm{Li}_3(\upsilon) + T^4\\
&+4T^3U - 18T^2U^2 + 4TU^3 + U^4 + 24\zeta_2TU - 12\zeta_2T^2 - 12\zeta_2U^2\\
&-654\zeta_4 - 4i\pi\big[12\mathrm{Li}_3(\tau) + 12\mathrm{Li}_3(\upsilon) - 12T\mathrm{Li}_2(\tau) - 12U\mathrm{Li}_2(\upsilon)\\
&-T^3 - 3T^2U - 3TU^2 - U^3 - 18\zeta_2T - 18\zeta_2U\big]\big\} + \mathcal{O}(\epsilon)
\end{aligned}$$

$$\begin{aligned}
R^{(2)[1]\mathrm{fin}}_{(-+)(-+)} = \frac{2\tau}{3\upsilon^2}\big\{&48\mathrm{Li}_4(\tau) - 24T\mathrm{Li}_3(\tau) - 24S_{2,2}(\tau) + 24\zeta_2\mathrm{Li}_2(\tau) + T^4 - 84\zeta_2T^2\\
&-102\zeta_4 + 24T\zeta_3 - 8i\pi\big[3T\zeta_2 - T^3\big]\big\} - \frac{8\tau}{3\upsilon^2}\big\{6\tau\mathrm{Li}_3(\tau) - 6\tau\mathrm{Li}_3(\upsilon)\\
&-6\tau T\mathrm{Li}_2(\tau) + 6\mathrm{Li}_3(\upsilon) - 6\upsilon U\mathrm{Li}_2(\upsilon) + 3\tau TU^2 + 3T\upsilon U^2 - 3TU^2\\
&-30\tau T\zeta_2 - 30\upsilon U\zeta_2 - 6\zeta_3 + 3i\pi\big[2(\upsilon-\tau)\mathrm{Li}_2(\tau) + \tau T^2 + 2T\upsilon U + \upsilon U^2\\
&+2\tau\zeta_2\big]\big\} + \mathcal{O}(\epsilon)
\end{aligned}$$

Generalizing to $\mathcal{N}=2$ SQCD [Kälin, GM, Ochirov, Verbeek '19]

Now using $\mathcal{A}_n^{(L)}=\mathcal{A}_n^{(L)[\mathcal{N}=4]}+\mathcal{W}_n^{(L)}$, where $\beta_0\neq 0$:

$\mathcal{N}=2$ SQCD 2-loop UV/IR divergences



$$\mathcal{W}_n^{(2)}=\underbrace{\left(\sum_{i<j}^{n}s_{ij}\;\mathbf{T}_i\cdot\mathbf{T}_j\right)}_{\epsilon^{-2}}\underbrace{\mathcal{W}_n^{(1)}}_{\epsilon^{-1}}+\beta_0\underbrace{\left(\sum_{i<j}^{n}s_{ij}\left[\;-\;\right]\mathbf{T}_i\cdot\mathbf{T}_j\right)}_{\epsilon^{-3}}\mathcal{A}_n^{(0)}+\mathcal{W}_n^{(2)\mathrm{fin}}$$

All weight-0,1,2 terms cancel from $\mathcal{W}_4^{(2)\mathrm{fin}}$ for all N_c, N_f.

Expressions for the 2-loop triangles are known (e.g. [Gehrmann '99]):

$$=-\frac{r_\Gamma\Gamma^2(1-2\epsilon)\Gamma(1+2\epsilon)}{4\epsilon^3(1-2\epsilon)\Gamma(1-3\epsilon)}(-s_{ij})^{-1-2\epsilon}$$

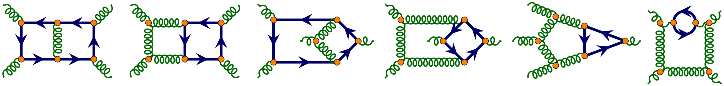
$$=-\frac{\Gamma(1+2\epsilon)\Gamma^3(1-\epsilon)}{2\epsilon^3(1-2\epsilon)\Gamma(1-3\epsilon)}(-s_{ij})^{-1-2\epsilon}$$

Our understanding of this new scheme comes from two perspectives:

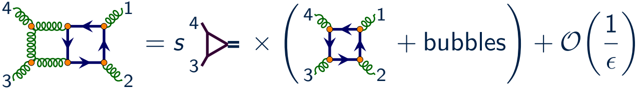
1. Examination of the 2-loop integrand prior to integration.
2. Soft-collinear factorization.

(1) Integrand perspective

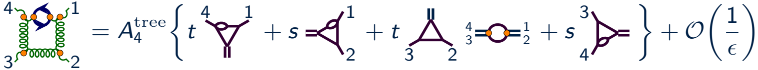
$\mathcal{W}_4^{(2)}$ consists only of diagrams with matter loops:

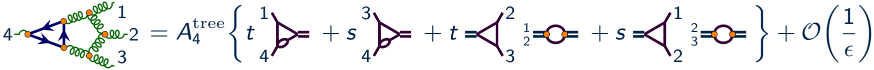


Only gluons give rise to IR divergences:



2-loop triangles come from internal matter loops, e.g.





With these **we can prove the UV/IR formula, except for $1/\epsilon$ poles**.

(2) Soft-collinear factorization

IR poles of UV-renormalized massless gauge theory amplitudes factorize away from a hard function (e.g. [Becher, Neubert; Gardi, Magnea '09]):

$$\mathcal{A}_n^{\mathrm{ren}}\left(\frac{p_i}{\mu},\alpha_{\mathrm{s}}(\mu)\right)=\mathcal{P}\exp\left\{-\int_0^{\mu}\frac{\mathrm{d}\lambda}{\lambda}\mathbf{\Gamma}\left(\frac{p_i}{\lambda},\alpha_{\mathrm{s}}(\lambda)\right)\right\}\mathcal{H}_n\left(\frac{p_i}{\mu},\alpha_{\mathrm{s}}(\mu)\right)$$

$\mathbf{\Gamma}$ includes the theory-specific anomalous dimensions $\gamma_K^{(L)}$, $\gamma_g^{(L)}$. We derive formulae for the 2-loop gluonic divergences in an arbitrary massless gauge theory, c.f. [Catani '98]:

UV/IR divergences: $\mathcal{A}_n^{(L)}=\mathcal{A}_n^{(L)[\mathcal{N}=4]}+\mathcal{W}_n^{(L)}$

$$\mathcal{W}_n^{(1)}=-\frac{\beta_0}{\epsilon}\mathcal{A}_n^{(0)}+\mathcal{O}(\epsilon^0)$$

$$\begin{aligned}\mathcal{W}_n^{(2)}=\mathbf{S}(\epsilon)\mathcal{W}_n^{(1)}&+\left(\frac{\beta_0}{\epsilon}+\frac{1}{2}\left[\gamma_K^{(2)}-\gamma_K^{(2)[\mathcal{N}=4]}\right]\right)\mathbf{S}(2\epsilon)\mathcal{A}_n^{(0)}\\&+\frac{1}{\epsilon}\left(\frac{n-2}{2}\beta_1+n\left[\gamma_g^{(2)}-\gamma_g^{(2)[\mathcal{N}=4]}\right]\right)\mathcal{A}_n^{(0)}+\mathcal{O}(\epsilon^0)\end{aligned}$$

$$\mathbf{S}(\epsilon)=\frac{1}{\epsilon^2}\sum_{i<j}^{n}\left[1-\epsilon\log(-s_{ij})\right]\mathbf{T}_i\cdot\mathbf{T}_j=\sum_{i<j}^{n}s_{ij}$$

$$\mathbf{T}_i\cdot\mathbf{T}_j+\mathcal{O}(\epsilon^0)$$

Anomalous dimensions

[Kälin, GM, Ochirov, Verbeek '19]

Anomalous dimensions $\gamma_K^{(L)}$, $\gamma_g^{(L)}$, and $\gamma_q^{(L)}$ are read off integrated loop amplitudes, including **2-loop amplitudes with matter on external legs**:

$\mathcal{N} = 2$ SQCD, $\beta_0 = C_A - T_F N_f$

$$\gamma_K^{(1)} = 4 \qquad \gamma_K^{(2)} = -2\zeta_2 C_A + 4\beta_0$$

$$\gamma_g^{(1)} = -\frac{\beta_0}{2} \qquad \gamma_g^{(2)} = \frac{1}{8} C_A\big(\zeta_3 C_A + \beta_0(\zeta_2 - 4)\big)$$

$$\gamma_q^{(1)} = \gamma_{\bar{q}}^{(1)} = 0 \qquad \gamma_q^{(2)} = \gamma_{\bar{q}}^{(2)} = \frac{1}{8} C_F\big(13\zeta_3 C_A - 12\zeta_3 C_F - \beta_0(3\zeta_2 + 4)\big)$$

Some observations:

1. When $\beta_0 = 0$, $\gamma_g^{(L)}$ and $\gamma_K^{(L)}$ equal those in $\mathcal{N} = 4$ SYM, in agreement with previous observations (e.g. [Pomoni '13]).
2. Furthermore replacing $C_F \to C_A \implies \gamma_q^{(L)} = \gamma_g^{(L)}$.
3. If $G = \mathrm{SO}(3)$, $C_A = N_c - 2 = 1$; $C_F = \frac{1}{2}(N_c - 1) = 1$, so setting $\beta_0 = 0 \implies N_f = C_A / T_F = 1$ and we recover $\mathcal{N} = 4$ SYM.

Insert into UV/IR divergence formula, and we recover 2-loop triangles.

Summary/outlook

Conclusions

- A good choice of UV/IR renormalization scheme attributes lower-weight terms to lower-loop amplitudes.
- The scheme is indicated by a good choice of integrand.

More amplitudes to calculate!

- We should check lower-weight cancellations for $n = 5$.
- Does a similar idea work at 3 loops, or for $\mathcal{N} = 1$ SQCD?

Open questions

- Does a similiar principle work on the other side of the double copy?
- Could we try bootstrapping reduced-SUSY amplitudes?

Thanks for listening!