A Penrose transform for the double copy

Three routes to the double copy: Lie-polynomials, differential forms, and the worldsheet

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Continuing work with Hadleigh Frost, 1912.04198,
building on ABHY: Arkani-Hamed, Bai, He, Yan, arxiv:1711.09102,
discussions w/ Francis Brown, Carlos Mafra, Ricardo Monteiro,
Oliver Schlotterer & an after dinner talk by Kapranov 2012.

cf also related work by Sebastian Mizera 1912.03397 and Song He.

This talk develops three interlocking mathematical structures that underpin the double copy.

1. Lie polynomials and their homomorphisms to numerators.
2. The ABHY [Arkani-Hamed, Bai, He, Yan] geometry of $(n-3)$-forms on $\mathcal{K}_n = \mathbb{R}^{n(n-3)/2}$, the space of Mandelstams s_{ij}.
3. The geometry of $\mathcal{M}_{0,n}$ the moduli space of n-marked points on $\mathbb{CP}^1$.

We further develop a twistorial correspondence between 2 & 3:

$$\begin{array}{ccccc} (s_{ij},\sigma_i) \in & & \mathcal{Y}_n & & \\ & p \swarrow & & \searrow q & \\ s_{ij} \in \mathcal{K}_n & & & & T^*_D\mathcal{M}_{0,n} \ni (\tau_i,\sigma_i) \end{array}$$

giving a transform from CHY/Ambitwistor-string half-integrands to scattering forms etc..

For n particles: let $W(n-1) = \mathbb{R}^{(n-1)!}$ with basis $n-1$-words in $n-1$ distinct letters $x_1 x_2 \ldots x_{n-1} \leftrightarrow$ i.e. permutations S_{n-1}. .

Definition

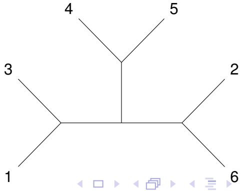
The Lie polynomials $Lie(n-1) \subset W(n-1)$ are spanned by

- *Lie monomials $\Gamma \in Lie(n-1)$, combinations of $n-1$-words made of complete commutators*

$$\Gamma = [[[x_1, x_3], [x_4, x_5]], x_2]$$

  so $[a, b] = ab - ba$ is skew and satisfies Jacobi.

- *An oriented connected trivalent tree graph Γ with $n-1$ leaves rooted at n, i.e. $n = 6$:*

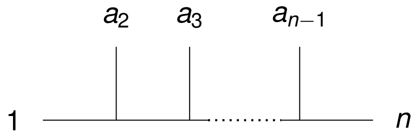


Duality for Lie polynomials

Theorem (Radford 1979)

$Lie(n-1) = \mathbb{R}^{(n-2)!}$ with 'DDM basis' half-ladders/combs:

$\Gamma_{1a} :=$



Where $a = a_2 a_3 \dots a_{n-1}$ is a permutation of $2, \dots, n-1$.

- $W(n-1) = \mathbb{R}^{(n-1)!}$ has inner product (a, b) with distinct words giving an orthonormal basis.
- $Lie(n-1) \subset W(n-1)$ by expanding all $[x_i, x_j] = x_i x_j - x_j x_i$.
- For word a, (Γ, a) = coefficient of a in expansion of Γ.
- $(\Gamma, a) = \pm 1$ iff Γ planar for ordering a with $\pm$ orientation.

Theorem (Ree 1958)

$c \in Lie(n-1) \subset W(n-1) \Leftrightarrow (c, a \sqcup b) = 0 \;\forall$ nontrivial a, b (now of different sizes $|a| + |b| = n-1$)

Geometry of Mandelstam space $\mathcal{K}_n$

- Given n null momenta k_i, $\sum k_i = 0$, set $s_{ij} = (k_i + k_j)^2$

$$\mathcal{K}_n = \{s_{ij} = s_{ji} | s_{ii} = 0, \text{ and } \sum_{j=1}^{n} s_{ij} = 0\} = \mathbb{R}^{n(n-3)/2}.$$

- **Factorization hyperplanes:** given by $s_I = 0$ where

$$s_I := \sum_{i,j \in I} s_{ij} = \left(\sum_i k_i \right)^2, \qquad I \subset \{1, 2, \ldots, n\}.$$

- $s_I = 0$ and $s_{I'} = 0$ compatible iff $I' \subset I$ or I^c, complement.
- Maximal compatible sets are in 1:1 correspondence with the $n-3$ propagators of trivalent diagrams Γ.

The pth propagator of graph Γ carries momentum $\sum_{i\in I_p} k_i$, $I_p \subset \{1,\dots,n\}$ giving propagators:

$$\frac{1}{d_\Gamma} = \frac{1}{\prod_{p=1}^{n-3} s_{I_p}}, \qquad s_I = \sum_{i<j\in I} s_{ij}$$

Definition

Abstract biadjoint scalar has amplitudes

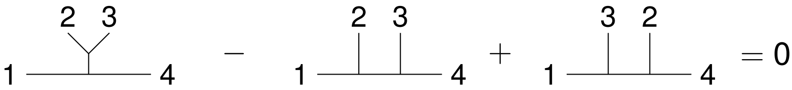
$$m = \sum_\Gamma \frac{\Gamma \otimes \Gamma}{d_\Gamma} \in \otimes^2 Lie(n-1)\,, \quad m_a = \sum_\Gamma \frac{(\Gamma, a)\Gamma}{d_\Gamma} \in Lie(n-1)$$

Standard biadjoint scalar is $m(a,b) = (m_a, b)$.

Dress abstract m with numerators to obtain favourite theories.

BCJ numerators $\{N_\Gamma\}$ give homorphism $N : Lie(n-1) \to V$:

$$\Gamma_s + \Gamma_t + \Gamma_u = 0 \quad \Rightarrow \quad N_{\Gamma_s} + N_{\Gamma_t} + N_{\Gamma_u} = 0$$



and for embeddings into larger diagrams.

Examples:

- Colour ordering: $\Gamma \to (\Gamma, a) \in W(n-1)$.
- For $g_i \in \mathfrak{g}$, a Lie algebra:

$$\Gamma \to c_\Gamma = \operatorname{tr}(g_n \Gamma(g_1, \dots, g_{n-1})) \in \{\text{invt polys on } \mathfrak{g}^n\}.$$

- Kinematic numerators: [Du, Feng Tei,...]

$$\Gamma \to N_\Gamma^{k,\epsilon} \in \{\text{invt polys in } (k_i, \epsilon_i)\}.$$

Do all such linear maps arise from choice of Lie algebra?

Theories with different combinations of numerators

Fu, Du, Huang, Feng Tei, 2017, BCJ, Chiarodoli, Roiban, 1909.01358, cf also CHY, CGMMRS 1506.08771

Amplitudes: $$\mathcal{M} = \sum_{\Gamma} \frac{N^l_\Gamma N^r_\Gamma}{d_\Gamma}$$

N^l \ N^r	$N_\Gamma^{k,\epsilon}$	$N_\Gamma^{k,k}$	$N_\Gamma^{k,\epsilon,m}$	$N_\Gamma^{k,\epsilon,g}$	c_Γ or (Γ, a)
$N_\Gamma^{k,\epsilon}$	E				
$N_\Gamma^{k,k}$	BI	Galileon			
$N_\Gamma^{k,\epsilon,m}$	*EM* $\mathrm{U}(1)^m$	DBI	*EMS* $\mathrm{U}(1)^m \times \mathrm{U}(1)^{\tilde{m}}$		
$N_\Gamma^{k,\epsilon,g}$	EYM	ext. DBI	*EYMS* $\mathrm{SU}(N) \times \mathrm{U}(1)^{\tilde{m}}$	*EYMS* $\mathrm{SU}(N) \times \mathrm{SU}(\tilde{N})$	
c_Γ or (Γ, a)	YM	Nonlinear σ	*EYMS* $\mathrm{SU}(N) \times \mathrm{U}(1)^{\tilde{m}}$	*gen. YMS* $\mathrm{SU}(N) \times \mathrm{SU}(\tilde{N})$	*Biadjoint Scalar* $\mathrm{SU}(N) \times \mathrm{SU}(\tilde{N})$

Table: Theories arising from the different choices of numerators.

$(n-3)$-forms on $\mathcal{K}_n$—the ABHY scattering forms

ABHY construct homomorphism

$$Lie(n-1) \simeq \Omega_s^{n-3}\mathcal{K}_n \subset \Omega^{n-3}\mathcal{K}_n$$

- Given Γ define $w_\Gamma = (-1)^\Gamma \bigwedge_{p=1}^{n-3} ds_{I_p}$
- They prove $w_{\Gamma_s} + w_{\Gamma_t} + w_{\Gamma_u} = 0$ so w_Γ provide numerators

$$w_\Gamma : Lie(n-1) \to \Omega_s^{n-3}\mathcal{K}_n \subset \Omega^{n-3}\mathcal{K}_n.$$

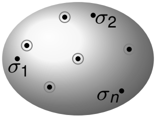
- Given other numerators N_Γ, define *scattering forms*

$$\Omega_N = \sum_\Gamma \frac{N_\Gamma w_\Gamma}{d_\Gamma} \in \Omega_s^{n-3}\mathcal{K}_n\,.$$

  E.g. $\Omega_a = \Omega_{(\Gamma,a)}$ when $N_\Gamma = (\Gamma, a)$.
- Ω_N is projective on $\mathcal{K}_n \Leftrightarrow s, t, u$ relations on N_Γ.

$$N_{\Gamma_s}+N_{\Gamma_t}+N_{\Gamma_u} = 0 \quad \Leftrightarrow \quad \Upsilon \lrcorner \Omega_N = 0\,, \qquad \Upsilon = \sum_{ij} s_{ij}\frac{\partial}{\partial s_{ij}}\,.$$

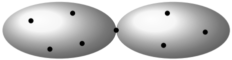
Define: Deligne-Mumford compactification



$$\mathcal{M}_{0,n} = \overline{\{\times^n \mathbb{CP}^1 - \text{diagonals}\}/\text{Möbius}}$$

with coordinates

- $(\sigma_1, \ldots, \sigma_n)$, fix Möbius by $(\sigma_1, \sigma_{n-1}, \sigma_n) = (0, 1, \infty)$.
- Or planar cross-ratios: $u_{i\,j} = \frac{\sigma_{i\,j-1}\sigma_{j\,i-1}}{\sigma_{i\,j}\sigma_{i-1\,j-1}}$, $\sigma_{i\,j} = \sigma_i - \sigma_j$.

I  $\bar{I}$

Boundary $\partial\mathcal{M}_{0,n} = D = \cup_I D_I$ labelled by $I \subset \{1, \ldots, n\}$.

- Planar Γ has propagators $\frac{1}{s_{I_p}} \leftrightarrow I_p = \{i_p, \ldots, j_p - 1\}$,
- Gives $n - 3$ cross ratio $u_{I_p} := u_{i_p\,j_p-1}$ coords for $\mathcal{M}_{0,n}$ s.t.

$$D_{I_p} = \{u_{I_p} = 0\}$$

Lie polynomials in $\mathcal{M}_{0,n}$

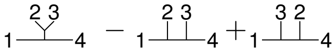
Proposition

$H_{n-3}(\mathcal{M}_{0,n} - D) = Lie(n-1)$ generated by

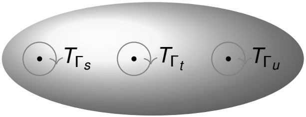
$$\Gamma \to T_\Gamma = \{|u_{I_p}| = \epsilon, p = 1, \ldots n-3\}.$$

tori around the $n-3$-fold intersection points $\leftrightarrow \Gamma$ of the D_I.

Proof: $\mathcal{M}_{0,4} = \mathbb{CP}^1$ and $D = \{\Gamma_s, \Gamma_t, \Gamma_u\}$



$= 0 \quad \leftrightarrow$



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This picture embeds in $\mathcal{M}_{0,n} \forall n, (\Gamma_s, \Gamma_t, \Gamma_u)$.

Corollary

$$\Gamma(\Omega_D^{n-3}\mathcal{M}_{0,n}) = H^{n-3}(\mathcal{M}_{0,n} - D) = Lie(n-1)^*.$$

generated by Parke-Taylors: $\quad PT_a = \frac{\prod_i d\log \sigma_{\tilde{a}_i \tilde{a}_{i-1}}}{vol(SL(2))}, \quad \tilde{a} = an.$

The correspondence between $\mathcal{K}_n$ and $T_D^*\mathcal{M}_{0,n}$

Lemma

$$\mathcal{K}_n = \{\textit{Space of sections of } T_D^*\mathcal{M}_{0,n}\}$$

Proof: Let τ_i be fiber coords on $T_D^*\mathcal{M}_{0,n}$, so $\tau = \sum_i \tau_i d\sigma_i$. Sections are

$$\tau = \sum_{ij} s_{ij} d \log \sigma_{ij} = \sum_i E_i d\sigma_i \,, \qquad E_i := \sum_j \frac{s_{ij}}{\sigma_{ij}} \,. \quad \square$$

- $E_i = 0$ are the scattering equations.
- Incidence equations are $\tau_i = E_i(s_{kl}, \sigma_j)$.
- These are incidence equations of a twistor correspondence

$$\begin{array}{ccccc} \mathcal{K}_n \times \mathcal{M}_{0,n} = & & \mathcal{Y}_n & & \ni (s_{ij}, \sigma_i) \\ & p \swarrow & & \searrow q & \\ s_{ij} \in \mathcal{K}_n & & & & T_D^*\mathcal{M}_{0,n} \ni (\tau_i = E_i(s_{kl}, \sigma_j), \sigma_i) \end{array}$$

CHY formulae as a Penrose transform

$$\begin{array}{ccccc} \mathcal{K}_n \times \mathcal{M}_{0,n} = & & \mathcal{Y}_n & & \ni (s_{ij}, \sigma_i) \\ & p \swarrow & & \searrow q & \\ s_{ij} \in \mathcal{K}_n & & & & T^*_D \mathcal{M}_{0,n} \ni (\tau_i = E_i(s_{kl}, \sigma_j), \sigma_i) \end{array}$$

- The Penrose transform by $p_* q^*$ i.e.:
- The CHY formulae are

$$\mathcal{M}(s_{ij}, \dots) = \int_{\mathcal{M}_{0,n} = p^{-1}(s_{ij})} q^* \left(\mathcal{I}_l \mathcal{I}_r \, \bar{\delta}(\tau)^{n-3} \right)$$

- Here $\mathcal{I}_l, \mathcal{I}_r \in \Omega^{n-3}(\mathcal{M}_{0,n})$ are CHY half-integrands but also often depending also on polarization data etc.,
- E.g., LHS$= m(a, b)$ for $(\mathcal{I}_l, \mathcal{I}_r) = (PT_a, PT_b)$.
- There is an empirical direct correspondence between choices of $I_{l/r}$ and numerators N_Γ.

The forms w_Γ from the symplectic volume form

- Let $\omega = d\tau_i \wedge d\sigma_i$, then ω^{n-3} has top degree and

$$q^*\omega^{n-3} \in \Omega^{n-3}\mathcal{K}_n \otimes \Omega_D^{n-3}\mathcal{M}_{0,n}\,.$$

- Define/evaluate the ABHY forms by

$$w_\Gamma := (-1)^\Gamma \bigwedge_{p=1}^{n-3} ds_{I_p} = \int_{T_\Gamma} \omega^{n-3}.$$

- Thus $w_{\Gamma_s} + w_{\Gamma_t} + w_{\Gamma_u} = 0$ follows from $T_{\Gamma_s} + T_{\Gamma_t} + T_{\Gamma_u} = 0$.
- We can expand ω^{n-3} as

$$\omega^{n-3} = \sum_{a\in S_{n-2}} w_{\Gamma_{1a}} \otimes PT_{1a}\,.$$

CHY half-integrands, scattering forms and numerators

- Just as w_Γ give numerators to give scattering forms, ω^{n-3} gives CHY half-integrand for scattering forms

$$\Omega_{\mathcal{I}_l} = \int_{\mathcal{M}_{0,n}} q^* \mathcal{I}_l \, \omega^{n-3} \, \bar{\delta}^{n-3}(\tau_i) \quad \in \Omega_s^{n-3}(\mathcal{K}_n)$$

- Projectivity follows from

$$\Upsilon = \sum_{ij} s_{ij} \frac{\partial}{\partial s_{ij}} \lrcorner \, \omega^{n-3} = \sum_i \tau_i \frac{\partial}{\partial \tau_i} \lrcorner \, \omega^{n-3} = \sum_i \tau_i d\sigma_i$$

  which vanishes against the delta functions.

- Thus

$$\Omega_{\mathcal{I}_l} = \sum_\Gamma \frac{N_\Gamma^{\mathcal{I}_l} w_\Gamma}{d_\Gamma}$$

  with $N_\Gamma^{\mathcal{I}_l}$ satisfying *stu*-relations by ABHY.

An invariant definition of associahedral $n-3$-planes.

Correspondence: planar $\Gamma \leftrightarrow P_\Gamma$ associahedral $n-3$-plane:

- Choose ordering and planar factorization channels $I_{ij} = \{i, i+1, \ldots, j-1\}$ associated $x_{I_{ij}} = \sum_{i \leq l < m < j} s_{lm}$.
- $$P_\Gamma = \bigwedge_{p=1}^{n-3} D_{I_p}\,, \qquad D_I := \frac{\partial}{\partial X_I} - \sum_{J \in I^c} \frac{\partial}{\partial X_J}.$$
  $I^c = \{\text{planar factorization channels incompatible with } I\}$,
  $I^c_{ij} = \{\text{lines that cross the line from } i \text{ to } j\}$.
- The ABHY $P_a = P_{\Gamma_{1a}}$.
  $$P_a \lrcorner\, \omega^{n-3} = PT_a \quad \text{so} \quad P_\Gamma \lrcorner\, w_{\Gamma'} = (\Gamma', a)\,.$$

Ex: biadjoint scalar follows via CHY as

$$P_b \lrcorner\, \Omega_{PT_a} = \int_{\mathcal{M}_{0,n}} q^* PT_a \, P_b \lrcorner\, \omega^{n-3} \, \bar{\delta}^{n-3}(\tau_i) = \int_{\mathcal{M}_{0,n}} q^* PT_a \, PT_b \, \bar{\delta}^{n-3}(\tau_i)$$
$$= m(a,b)\,.$$

Geometric quantization and twisted cohomology

Geometric quantization defines line-bundle $\mathcal{L} \to T^*\mathcal{M}_{0,n}$ with curvature $\alpha'\omega$, connection $\nabla = d + \alpha'\tau$.

- Wave functions are cohomology of $\mathcal{L}$ covariantly constant up fibres, e.g., PT_a.
- Pullback to $\mathcal{Y}_n$ gives twisted cohomology description of Mizera etc..
- Links into conventional string theory, twisted strings and ambitwistor strings.
- $H^1(\mathcal{M}_{0,n}, \mathbb{Z})$ gives lattice in $\mathcal{K}_n$ periodicity under which gives rise to the string KLT kernel.

Outlook

- Penrose transform: CHY $\frac{1}{2}$-integrands $\leftrightarrow$ scattering forms $\rightsquigarrow$ Lie poly structure for numerators via ABHY projectivity.
- Direct numerator formulae obscured by interdependence between Pfaffians and scattering equations (cf Mizera).
- Berends-Giele recursion $\rightsquigarrow$ field theory, Lie poly/ABHY-form based proofs of momentum kernel and numerators.
- Momentum kernel passes to $T^*\mathcal{M}_{0,n}$ using CHY treatment of KLT orthogonality.
- Quantization of $T^*\mathcal{M}_{0,n} \leftrightarrow$ ambitwistor-string path-integral. Pfaffian half-integrand for kinematic numerators arises from RNS spin field path-integral.
- Loops via nodal sphere as in [Geyer, M., Monteiro & Tourkine] series, cf. also recent 1 & 2-loop [Geyer, Monteiro] papers.

Thank You