A tale of two exponentiations in $N = 8$ supergravity

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This talk is based on two papers together with

A. Luna, S. Naculich, R. Russo, G. Veneziano and C. White, 1908.05603.
and
S. Naculich, R. Russo, G. Veneziano and C. White, 1911.11716.

Plan of the talk

Introduction

- High-energy scattering has been studied, both in field and string theories, since the end of the eighties
  ' t Hooft; Amati, Ciafaloni and Veneziano; Muzinich and Soldate
- The scattering of $2 \to 2$ scalar massless particles at high energy is dominated by the graviton exchange:

$$T(s,t) = \frac{8\pi G_N s^2}{(-t)}$$

- Since the graviton couples to energy, T diverges at high energy.
- Then, at sufficiently high energy, unitarity is violated.
- The way to restore unitarity is by summing over the contribution of loop diagrams.
- In this way the divergent contribution exponentiates in a phase, called the eikonal.
- From the eikonal one can then compute classical quantities as the deflection angle and the Shapiro time delay.

- This is by now not just the way to solve a theoretical problem.
- It may also have important applications to the study of the dynamics of binary black holes at the initial state of their merging.
- Modern quantum field theory techniques may turn out to be very efficient for extracting classical quantities needed for the study of black hole merging.
- They have allowed to compute the classical potential and the deflection angle at 3PM
  Bern, Cheung, Roiban, Shen, Solon, Zeng (2019)
- In this talk I am going to discuss the scattering of four massless particles in $N = 8$ supergravity.
- In this case the scattering amplitude has been explicitly computed up to three loops Henn and Mistlberger (HM) (2019).
- Different from CGR but should share with it the most important large-distance (infrared) features.
- In the probe analysis, by using D6-branes, it was shown that all classical corrections to the leading eikonal are vanishing
  D'Appollonio, DV, Russo and Veneziano (2010).

- Also from one loop calculations in $N = 8$ supergravity with masses it has been shown that the triangle diagrams do not contribute Caron-Huot and Zakraee (2018).
- In this talk we will see that, at two loops, one gets an additional classical contribution.

Two different kinds of exponentiation

- The UV properties of $N = 8$ supergravity have been studied to high loop order
  Bern, Carrasco, Chen, Edison, Johansson, Parra-Martinez, Roiban and Zeng (2018)
- Here we are interested in a complementary aspect: the high-energy, small angle (Regge) regime.
- In terms of the three Mandelstam variables:

$$s = -(k_1 + k_2)^2 \;\; ; \quad t = -(k_2 + k_3)^2 \;\; ; \quad u = -(k_1 + k_3)^2$$
$$s + t + u = 0$$

we work in the s-channel physical region ($s > 0; t, u < 0$) and focus on the near-forward regime $|t| << s$.

- In $N=8$ supergravity the full amplitude can be written as follows:

$$A(k_i) = \sum_{\ell=0}^{\infty} A^{(\ell)}(k_i,\dots) = A^{(0)}(k_i,\dots)\left(1+\sum_{\ell=1}^{\infty}\alpha_G^\ell \mathcal{A}^{(\ell)}(t,s)\right) .$$

- $A^{(0)}(k_i,\dots)$ is the tree level amplitude and $A^{(\ell)}$ is the ℓ-loop amplitude. Dots stand for the dependence on polarizations and flavors of external states.
- $\mathcal{A}^{(\ell)}$ is its “stripped" counterpart, and

$$\alpha_G \equiv \frac{G}{\pi\hbar}(4\pi\hbar^2)^\epsilon B(\epsilon) \; ; \qquad B(\epsilon) \equiv \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \; ,$$

  G is the Newton's constant in $D=4-2\epsilon$ dimensions.
- $A^{(\ell)}$ is infrared divergent, but all IR divergences are contained in the exponentiation of one loop amplitude.
- Therefore, it is convenient to write it in the form:

$$A(k_i) = A^{(0)}(k_i)\exp\left(\alpha_G \mathcal{A}^{(1)}(t,s,\epsilon)\right)\exp\left(\sum_{\ell=2}^{\infty}\alpha_G^\ell F^{(\ell)}(t,s,\epsilon)\right) .$$

  All remainder functions $F^{(\ell)}$ are finite in the limit of $\epsilon \to 0$.

- This is the first exponentiation and it is done in momentum space.
- The leading contribution to the ℓ-loop amplitude $A^{(\ell)}$ scales as $s^{\ell+2}$ (for large s) with subleading contributions having, modulo logarithms, lower powers of s and higher powers of t.
- At sufficiently high s unitarity is violated.
- To recover it we need another exponentiation, this time in impact parameter space $b \sim \frac{2J}{\sqrt{s}}$ rather than in momentum space.
- Let us see how that happens in the case of leading eikonal.
- The leading high energy behavior of the tree amplitude is given by

$$A_L^{(0)} = \frac{8\pi\hbar G s^2}{q^2} \quad ; \quad q^2 = -t$$

where, at high energy, q is along $D-2$ transverse directions.
- Then go to impact parameter space by

$$2i\delta_0(s,b) = \int \frac{d^{D-2}q}{(2\pi\hbar)^{D-2}} \mathrm{e}^{ibq/\hbar} \frac{iA_L^{(0)}}{2s} = -\frac{iGs}{\epsilon\hbar}\Gamma(1-\epsilon)(\pi b^2)^{\epsilon} .$$

- At one loop, we have for the leading term in s

$$A^{(1)} = A^{(0)}\alpha_G \mathcal{A}^{(1)} \longrightarrow A_L^{(0)}\alpha_G \left(\frac{-i\pi s}{\epsilon (q^2)^\epsilon} \right) \equiv A_L^{(1)} \; ,$$

- By going to impact parameter space one gets:

$$\int \frac{d^{D-2}q}{(2\pi\hbar)^{D-2}} \mathrm{e}^{\frac{ibq}{\hbar}} \frac{iA_L^{(1)}}{2s} = \int \frac{d^{D-2}q}{(2\pi\hbar)^{D-2}} \mathrm{e}^{\frac{ibq}{\hbar}} \frac{iA_L^{(0)}}{2s} \alpha_G \frac{-i\pi s}{\epsilon (q^2)^\epsilon} = -\frac{1}{2}(2\delta_0)^2 \, .$$

- Summing the two

$$\int \frac{d^{D-2}q}{(2\pi\hbar)^{D-2}} \mathrm{e}^{ibq/\hbar} \left(\frac{iA_L^{(0)}}{2s} + \frac{iA_L^{(1)}}{2s} + \dots \right)$$
$$= 2i\delta_0 - \frac{1}{2}(2\delta_0)^2 + \dots = \mathrm{e}^{2i\delta_0(s,b)} - 1 \; .$$

we see that they start to exponentiate in impact parameter space.

- Introduce a quantity related to the Schwarzschild radius:

$$R \equiv (G\sqrt{s})^{\frac{1}{1-2\epsilon}}\ , \text{i.e. } G\sqrt{s} \equiv R^{D-3}\ ,$$

- Express the scaling of different terms at a given loop order in terms of the classical quantities as b and R.
- The Fourier transform of the leading energy contribution to the ℓ-loop amplitude scales as ($A_L^{(\ell)} \sim \frac{G^{\ell+1}s^{\ell+2}}{q^2}$):

$$\int \frac{d^{D-2}q}{(2\pi\hbar)^{D-2}} \mathrm{e}^{ibq/\hbar} \frac{iA_L^{(\ell)}}{2s} \sim \left[\left(\frac{R}{b}\right)^{-2\epsilon} \frac{R\sqrt{s}}{\hbar}\right]^{\ell+1}$$

- precisely as the $(\ell+1)th$ power of the leading eikonal phase δ_0

$$\delta_0 \sim \frac{R\sqrt{s}}{\hbar} \left(\frac{R}{b}\right)^{-2\epsilon} \sim \frac{b\sqrt{s}}{\hbar} \left(\frac{R}{b}\right)^{1-2\epsilon}\ ,$$

- This confirms that the leading eikonal resums arbitrarily high powers of $\hbar^{-1}$ into an $\mathcal{O}(\hbar^{-1})$ phase provided we consider R and b as classical quantities (as in CGR).

- Let us now consider the subleading energy contributions.
- $\frac{A^{(\ell)}}{2s}$ consists of a sum of terms having powers of s all the way up to the leading power $\ell+1$.
- Each of these terms behaves in impact parameter space as follows (again neglecting possible logarithmic enhancements):

$$\int \frac{d^{D-2}q}{(2\pi\hbar)^{D-2}} \mathrm{e}^{ibq/\hbar} \frac{iA^{(\ell)}}{2s} \sim \sum_{m=0} G^{\ell+1} s^{\ell+1-m} b^{2\epsilon(\ell+1)-2m}$$

$$= \sum_{m=0} \left(\frac{R}{b}\right)^{2m-2\epsilon(\ell+1)} \left(\frac{R\sqrt{s}}{\hbar}\right)^{\ell+1-2m} .$$

- In the massless case, and in $D=4$, the amplitude $A^{(\ell)}$ cannot depend on fractional powers of s.
- Therefore the expansion above is only in terms of even powers $1/b^{2m}$.

- At each even order $A^{(2\ell)}$ we get a new contribution to the classical eikonal for $m = \frac{\ell}{2}$ and to the classical deflection angle.
- The odd-loop orders $A^{(2\ell+1)}$ do not contribute directly to the classical phase or deflection angle.
- However, they still take part in the exponentiation.

- On the basis of the previous considerations we propose the following extension of the leading eikonal to include also subleading contributions:

$$\frac{iA(k_i)}{2s} \simeq \hat{A}^{(0)}(k_i) \int d^{D-2}b \, \mathrm{e}^{-ibq/\hbar} \left[\Big(1 + 2i\Delta(s,b)\Big) \mathrm{e}^{2i\delta(s,b)} - 1 \right] ,$$

- All the terms appearing in $\mathrm{e}^{2i\delta(s,b)}$ are proportional to $\hbar^{-1}$.
- Those present in the prefactor Δ contain the contributions with non-negative powers of $\hbar$.
- Above identity is restricted to non-analytic terms as $q \to 0$ that capture long-range effects in impact parameter space.

Check of (and constraints from) the leading-eikonal

- Assuming exponentiation in impact parameter space for the leading term, we get

$$\frac{iA_L(q^2,s)}{2s} = \int d^{D-2}b\, \mathrm{e}^{-ibq/\hbar} \left(\sum_{\ell=1}^{\infty} \frac{1}{\ell!} (2i\delta_0(s,b))^{\ell} \right)$$

$$2i\delta_0 = -\frac{iGs}{\epsilon\hbar}\Gamma(1-\epsilon)(\pi b^2)^{\epsilon} .$$

- Its Fourier transform can be performed term by term:

$$\frac{iA_L(q^2,s)}{2s} = \frac{iA_L^{(0)}}{2s} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left[-\frac{iGs}{\epsilon\hbar}\Gamma(1-\epsilon)\left(\frac{4\pi\hbar^2}{q^2}\right)^{\epsilon} \right]^{\ell} \frac{\Gamma(\ell\epsilon+1)\Gamma(1-\epsilon)}{\Gamma(1-(\ell+1)\epsilon)}$$

$$= \frac{iA_L^{(0)}}{2s} \sum_{\ell=0}^{\infty} \frac{\alpha_G^{\ell}}{\ell!} \left(\frac{-i\pi s}{\epsilon (q^2)^{\epsilon}} \right)^{\ell} G^{(\ell)}(\epsilon) ,$$

- where

$$G^{(\ell)}(\epsilon) = \frac{\Gamma^{\ell}(1-2\epsilon)\Gamma(1+\ell\epsilon)}{\Gamma^{\ell-1}(1-\epsilon)\Gamma^{\ell}(1+\epsilon)\Gamma(1-(\ell+1)\epsilon)} .$$
$$= 1 - \frac{1}{3}\ell\left(2\ell^2+3\ell-5\right)\zeta_3\epsilon^3 + \mathcal{O}(\epsilon^4).$$

- We can compare this result with the other exponentiation, getting for two and three loops:

$$\frac{1}{2}(\mathcal{A}_L^{(1)})^2 + F_L^{(2)} = \frac{1}{2!}\left(\frac{-i\pi s}{\epsilon(q^2)^{\epsilon}}\right)^2 G^{(2)},$$
$$\frac{1}{3!}(\mathcal{A}_L^{(1)})^3 + F_L^{(3)} + \mathcal{A}_L^{(1)}F_L^{(2)} = \frac{1}{3!}\left(\frac{-i\pi s}{\epsilon(q^2)^{\epsilon}}\right)^3 G^{(3)},$$

- On the left-hand side we have the high energy expansion of the amplitude coming from the IR exponentiation.
- On the right-hand side we have the expression obtained from the eikonal exponentiation.

- Solving for $F_L^{(2)}$ using $\mathcal{A}_L^{(1)} = \frac{-i\pi s}{\epsilon(q^2)^\epsilon}$, we have

$$
\begin{aligned}
F_L^{(2)} &= \lim_{s\to\infty} F^{(2)} = \frac{1}{2}\left(\frac{-i\pi s}{\epsilon(q^2)^\epsilon}\right)^2 \left[G^{(2)}(\epsilon) - 1\right] \\
&= 3\pi^2 s^2 \epsilon\zeta_3 + \mathcal{O}(\epsilon^2, s)
\end{aligned}
$$

- Solving for $F_L^{(3)}$, we get

$$
\begin{aligned}
F_L^{(3)} &= \lim_{s\to\infty} F^{(3)} = \frac{1}{3!}\left(\frac{-i\pi s}{\epsilon(q^2)^\epsilon}\right)^3 \left[(G^{(3)} - 1) - 3\left(G^{(2)} - 1\right)\right] \\
&= -\frac{2i}{3}\pi^3 s^3 \zeta_3 + \mathcal{O}(\epsilon, s^2)
\end{aligned}
$$

- They agree with what is given in the paper by HM for the previous two quantities.
- The remainder functions do not have infrared divergences but are not negligible at high energy.

Exponentiation at the first subleading eikonal

- First of all we need a better approximation to $\mathcal{A}^{(1)}$ up to $\mathcal{O}(t/s)$ for general ϵ

$$
\begin{aligned}
\mathcal{A}^{(1)} \;&=\; -\frac{i\pi s}{\epsilon (q^2)^\epsilon} + \frac{q^2(1+2\epsilon)}{\epsilon (q^2)^\epsilon}\left(\log\frac{q^2}{s} + H(\epsilon)\right) \\
&\quad -\frac{2q^2(2\epsilon+1)}{\epsilon^2(\epsilon+1)s^\epsilon}\cos^2\frac{\pi\epsilon}{2} + i\frac{\pi q^2}{\epsilon}\left[\frac{1+\epsilon}{(q^2)^\epsilon} - \frac{1+2\epsilon}{s^\epsilon(1+\epsilon)}\frac{\sin\pi\epsilon}{\pi\epsilon}\right] \\
&\equiv\; -\frac{i\pi s}{\epsilon (q^2)^\epsilon} + \mathcal{A}^{(1)}_{SL} + \ldots \,,
\end{aligned}
$$

where $\mathcal{A}^{(1)}_{SL}$ is the subleading contribution and

$$
H(\epsilon) \equiv \psi(-\epsilon) - \psi(1) - 1 + \pi\cot\pi\epsilon
$$

- This expression reproduces the data of HM up to the order ϵ^4.

- The extra q^2/s factor in $\mathcal{A}_{SL}^{(1)}$ cancels the Coulomb pole in $A_L^{(0)}$ and, after Fourier transforming, we find:

$$\left(\frac{iA^{(1)}}{2s}\right)_{SL} \quad \Rightarrow \quad G^2 s b^{-2+4\epsilon} \sim \left(\frac{R}{b}\right)^{2(1-2\epsilon)}$$

- No $\hbar$ in the denominator: this confirms that it is a quantum term $\Longrightarrow$ No contribution to the eikonal but contribution to Δ.
- At two loops, we have the following hierarchy of contributions

$$\left(\frac{iA^{(2)}}{2s}\right) \quad \Rightarrow \quad (\delta_0)^3 \sim \left(\frac{b\sqrt{s}}{\hbar}\left(\frac{R}{b}\right)^{1-2\epsilon}\right)^3 ;$$

$$(\delta_0 \Delta_1) \sim \delta_2 \sim \frac{b\sqrt{s}}{\hbar}\left(\frac{R}{b}\right)^{3-6\epsilon}$$

- and similarly at three loops:

$$\left(\frac{iA^{(3)}}{2s}\right) \quad \Rightarrow \quad (\delta_0)^4 \sim \left(\frac{b\sqrt{s}}{\hbar}\left(\frac{R}{b}\right)^{1-2\epsilon}\right)^4 \quad ;$$

$$(\delta_0)^2\Delta_1 \sim (\delta_0\delta_2) \sim \left(\frac{b\sqrt{s}}{\hbar}\right)^2 \left(\frac{R}{b}\right)^{4(1-2\epsilon)} \quad ;$$

$$\Delta_3 \sim \left(\frac{R}{b}\right)^{4(1-2\epsilon)} \quad ,$$

where Δ_3 is the next term in the expansion of Δ.

- In conclusion, a new contribution to the eikonal only comes from two loops: we call it δ_2.

- From the previous one loop expression we get

$$Re(2\Delta_1) = \frac{4G^2 s}{\pi b^2} \left(\pi b^2\right)^{2\epsilon} (1+2\epsilon)\Gamma^2(1-\epsilon)$$
$$\times \left[-\log\left(\frac{sb^2}{4\hbar^2}\right) + H(\epsilon) + \psi(1-2\epsilon) + \psi(\epsilon)\right] ,$$
$$Im(2\Delta_1) = \frac{4G^2 s}{b^2} \left(\pi b^2\right)^{2\epsilon} (1+\epsilon)\Gamma^2(1-\epsilon) .$$

- From our proposed formula, at two loops, we get:

$$\frac{A_L^{(0)}}{2s} \alpha_G^2 Re\mathcal{A}_{SL}^{(2)} = \int d^{D-2}b \, \mathrm{e}^{-ibq/\hbar} \left[-Im(2\Delta_1)(2\delta_0) + Re(2\delta_2)\right] ,$$
$$\frac{A_L^{(0)}}{2s} \alpha_G^2 Im\mathcal{A}_{SL}^{(2)} = \int d^{D-2}b \, \mathrm{e}^{-ibq/\hbar} \left[Re(2\Delta_1)(2\delta_0) + Im(2\delta_2)\right] .$$

- They allow us to derive δ_2.

- We get

$$Re(2\delta_2) = \frac{4G^3 s^2}{\hbar b^2} \left(\pi b^2\right)^{3\epsilon} \frac{\Gamma^3(1-\epsilon)}{\epsilon} \left(\frac{1+2\epsilon}{G^{(2)}(\epsilon)} - (1+\epsilon) \right)$$

  We have checked this equation with the data of the paper by HM up to order ϵ^2,
- The imaginary part is given by

$$Im(2\delta_2) = -\frac{4G^3 s^2}{\pi \hbar b^2} \left(\pi b^2\right)^{3\epsilon} \frac{(1-2\epsilon)\Gamma^3(1-\epsilon)}{\epsilon}$$
$$\times \left[\left(3 - \frac{2}{G^{(2)}(\epsilon)}\right) \log\left(e^{2\gamma_E} \frac{s\, b^2}{4\hbar^2}\right) + \left(1 - 3\zeta_2 \epsilon \right.\right.$$
$$\left.\left. +(-23\zeta_3 - 32\zeta_2)\epsilon^2 + (-167\zeta_4 - 160\zeta_3 - 64\zeta_2)\epsilon^3 + \dots \right)\right] .$$

No guess for the last line.

- The term of order ϵ^0

$$\lim_{\epsilon \to 0} Re(2\delta_2) = \frac{4G^3 s^2}{\hbar b^2}$$

is identical to Eq. (5.26) of ACV(1990) where this quantity has been computed for pure gravity.
- This appears to indicate that classical quantities, such as $Re\delta_2$, are related only to large-distance physics.
- They are therefore independent of the UV behavior of the microscopic theory and thus universal.
- See also the talk by Julio Parra-Martinez.

Comparing the two exponentiations

- Show that the proposed extension of the eikonal amplitude:

$$\frac{iA(k_i,\ldots)}{2s} \simeq \hat{A}^{(0)}(k_i)\int d^{D-2}b\, \mathrm{e}^{-ibq/\hbar} \times \left[\Big(1+2i\Delta(s,b)\Big)\mathrm{e}^{2i\delta(s,b)}-1\right]$$

  agrees with the exponentiation in momentum space at first subleading order in q^2/s, to at least two orders in the Laurent expansion in ϵ.

- At the first subleading level we get

$$\frac{iA_{SL}}{2s} = \hat{A}^{(0)}(k_i,\ldots)\int d^{D-2}b\, \mathrm{e}^{-ibq/\hbar} \times \left(2i\Delta_1 \sum_{\ell=1}^{\infty}\frac{(2i\delta_0)^{\ell-1}}{(\ell-1)!} + 2i\delta_2 \sum_{\ell=2}^{\infty}\frac{(2i\delta_0)^{\ell-2}}{(\ell-2)!}\right)$$

- For the first two terms in the ϵ expansion one can neglect the remainders.
- From the eikonal, after some calculation, one gets

$$\frac{iA^{(\ell)}_{SL}}{2s} \simeq \frac{iA^{(0)}}{2s}\frac{\alpha_G^\ell}{\ell!}\left(\frac{-i\pi s}{\epsilon}\right)^\ell \frac{iq^2}{\pi s}\Big\{-\ell\log\left(q^2\right)$$
$$+\epsilon\Big[\ell(\ell-1)\log^2\left(q^2\right)-\ell(\ell-2)\log(s)\log\left(q^2\right)$$
$$-\ell^2\log\left(q^2\right)-i\pi\ell\log\left(q^2\right)\Big]\Big\}+\mathcal{O}(1/\epsilon^{\ell-2})$$

- It agrees with the part coming from the exponentiation of the one-loop diagram.
- From HM we can get the remainder functions for $\ell = 2, 3$.
- Then our procedure can be extended to additional two terms in the ϵ expansion.
- We need the constant and the term of $\mathcal{O}(\epsilon)$ of $F^{(2)}$ and the term constant of $F^{(3)}$.

- Let us start to consider the three-loop case.
- The subleading contribution $A_{SL}^{(3)}/(2s)$ scales, after Fourier transform to impact parameter space, as $(Gs/\hbar)^2(R/b)^2 \log^{n-1}(b^2)$.
- It is too singular in the classical limit (and scales too quickly with the energy) to be absorbed in a contribution to δ_3 or to Δ_3.
- Therefore it must be reproduced by the leading and subleading eikonal data.

- This implies the following relations for the real

$$\frac{A_L^{(0)}}{2s}\alpha_G^3 Re\mathcal{A}_{SL}^{(3)} = \int d^{D-2}b\, \mathrm{e}^{-ibq/\hbar} \times \left[-\frac{1}{2}(2\delta_0)^2 Re(2\Delta_1) - (2\delta_0) Im(2\delta_2)\right]$$

- and for the imaginary part

$$\frac{A_L^{(0)}}{2s}\alpha_G^3 Im\mathcal{A}_{SL}^{(3)} = \int d^{D-2}b\, \mathrm{e}^{-ibq/\hbar} \times \left[-\frac{1}{2}(2\delta_0)^2 Im(2\Delta_1) + (2\delta_0) Re(2\delta_2)\right] \; .$$

- The lhs of the previous equations can be obtained from HM paper.
- The rhs is obtained from our basic eikonal formula.
- The lhs of the imaginary part is given by five imaginary terms that are all reproduced by the rhs.
- The lhs of the real part involves 18 terms.
- All divergent terms for $\epsilon \to 0$ and all terms proportional to $\log^n q^2$ with $n \geq 2$ match.

- However, going down to the lowest order contribution (i.e of $\mathcal{O}(G^4 s^3/b^2)$ with no $\log s$ enhancement) we find a mismatch, which, in momentum space, reads:

$$(\mathrm{lhs} - \mathrm{rhs}) = \frac{16}{3} \frac{G^4 s^3}{\hbar^2} \left(3\zeta_3 - \pi^2\right) \log(q^2) \ .$$

- We can modify $F^{(2)}$ and $F^{(3)}$ and satisfy the previous relation.
- But, then, we have a mismatch at higher loops for terms that come from the exponentiation of the IR divergences.
- The only way that we have found to get rid of this mismatch is by changing the three-loop remainder by

$$\hat{F}^{(3)} = \tilde{F}^{(3)} + 2\pi^2 s^2 q^2 \frac{\zeta_3}{\epsilon} \ .$$

- Such a redefinition is not allowed if all IR divergences come from the exponentiation of the one-loop amplitude !

The deflection angle

- Having determined $2\delta_0$ and $2\delta_2$:

$$2\delta_0 = -\frac{Gs}{\epsilon\hbar}\Gamma(1-\epsilon)(\pi b^2)^{\epsilon} \quad ; \quad Re(2\delta_2) = \frac{4G^3 s^2}{\hbar b^2}$$

- we can compute the deflection angle

$$\tan\frac{\theta}{2} = -\frac{\hbar}{\sqrt{s}}\frac{\partial}{\partial b}(2\delta_0 + Re(2\delta_2)) = \frac{R}{b} + \frac{R^3}{b^3} + \dots$$

where $R \equiv 2G\sqrt{s}$.

- This is in agreement with ACV(1990).

Conclusions and outlook

- All IR divergent terms at any loop order are reproduced by the exponentiation of one-loop amplitude in momentum space.
- Consistency with unitarity at high energy requires instead an exponentiation in impact parameter space.
- At leading level in energy the two exponentiations are consistent with each other in terms of the leading eikonal $2\delta_0$.
- The leading eikonal $2\delta_0$ is universal: it is the same for all gravity theories that reduce to CGR at large distances.
- At the first subleading level we get a subleading eikonal $2\delta_2$.
- The real part of $2\delta_2$ is directly related to observable as the deflection angle and is therefore IR finite.
- The imaginary part of $2\delta_2$ is instead IR divergent.
- It would be nice to study a physical observable (as an inclusive cross section) sensitive to $Im 2\delta_2$ to see how the cancellation of IR divergences work at this high order.

- At higher loops we find a lot of agreement and a single mismatch.
- It can be cured by including an IR divergent term in the remainder $F^{(3)}$.
- This is of course inconsistent with the IR exponentiation !
- May be one has to restrict the comparison of the two results only to physical/IR finite observables.
- One introduces a finite ϵ to regulate IR divergences.
- The infrared regulator is the smallest scale in the problem.
- In the eikonal we keep ϵ fixed and then consider all values of exchanged momentum $|q|$.
- Actually, the most important contributions to the large distance Regge regime are those that are divergent as $|q| \to 0$.
- It would be interesting to understand whether the discrepancy mentioned above is related to the different kinematics where the two exponentiations are valid.