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UCLA Mani L. Bhaumik Institute for Theoretical Physics

The Born Regime: From Love Numbers to the Analyticity of Gravitational Scattering

2511.xxxxx [M. Correia, T. Gopalka GI, A. Wolz]

2503.13593 [S. Caron-Huot, M. Correia, GI, M. Solon]

2406.13737 [M. Correia, GI]

Giulia Isabella

8th November 2025

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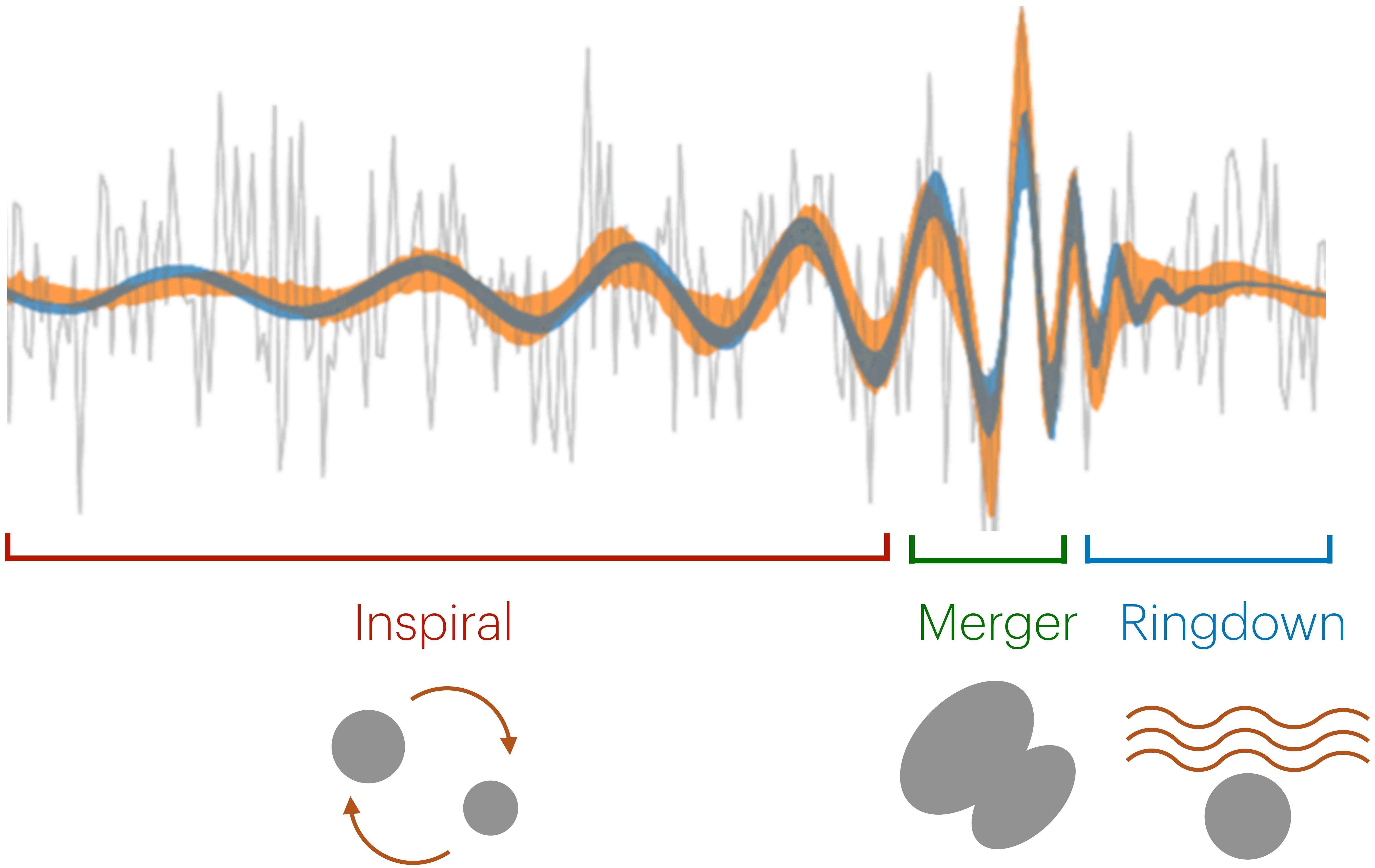
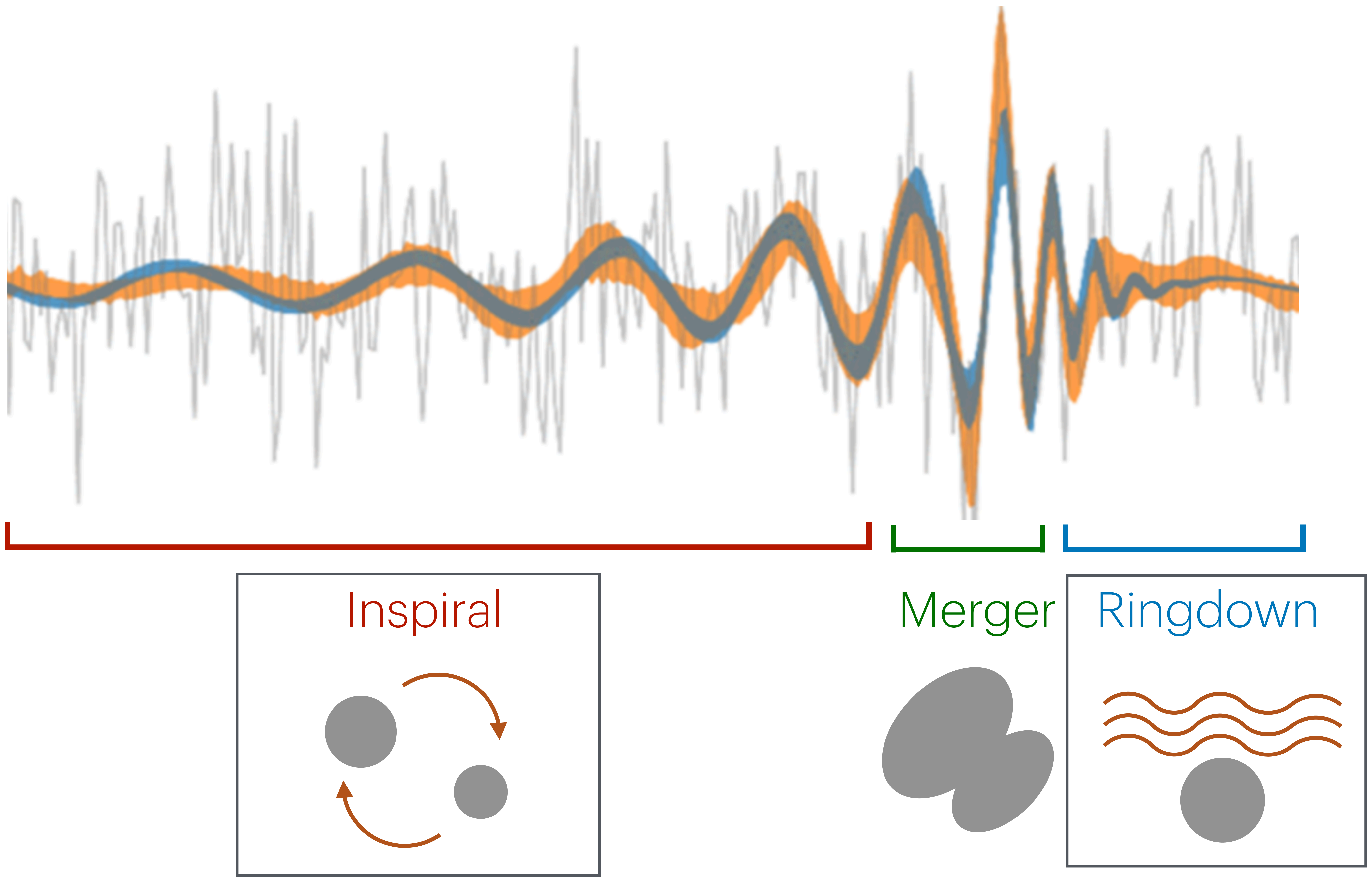
Introduction



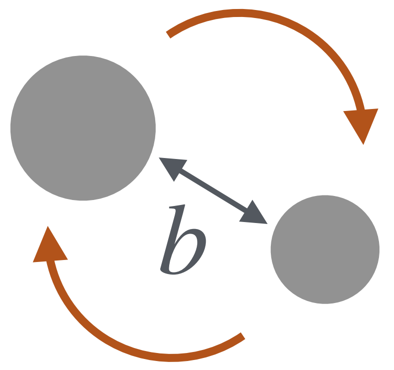
Introduction



Introduction

Eikonal regime

$b \gg \lambda_i$



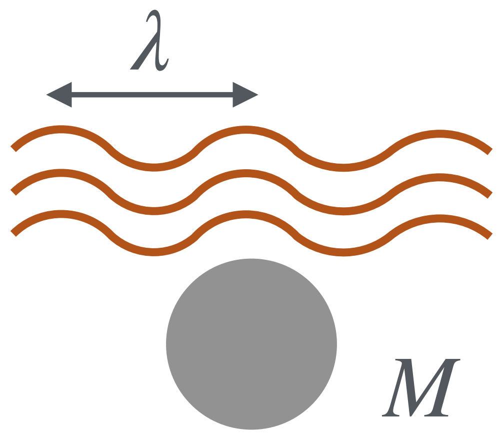
$$S(s,b) \sim e^{\frac{i}{\hbar} I(s,b)}$$

Extracting observables

Computational simplification

Born regime

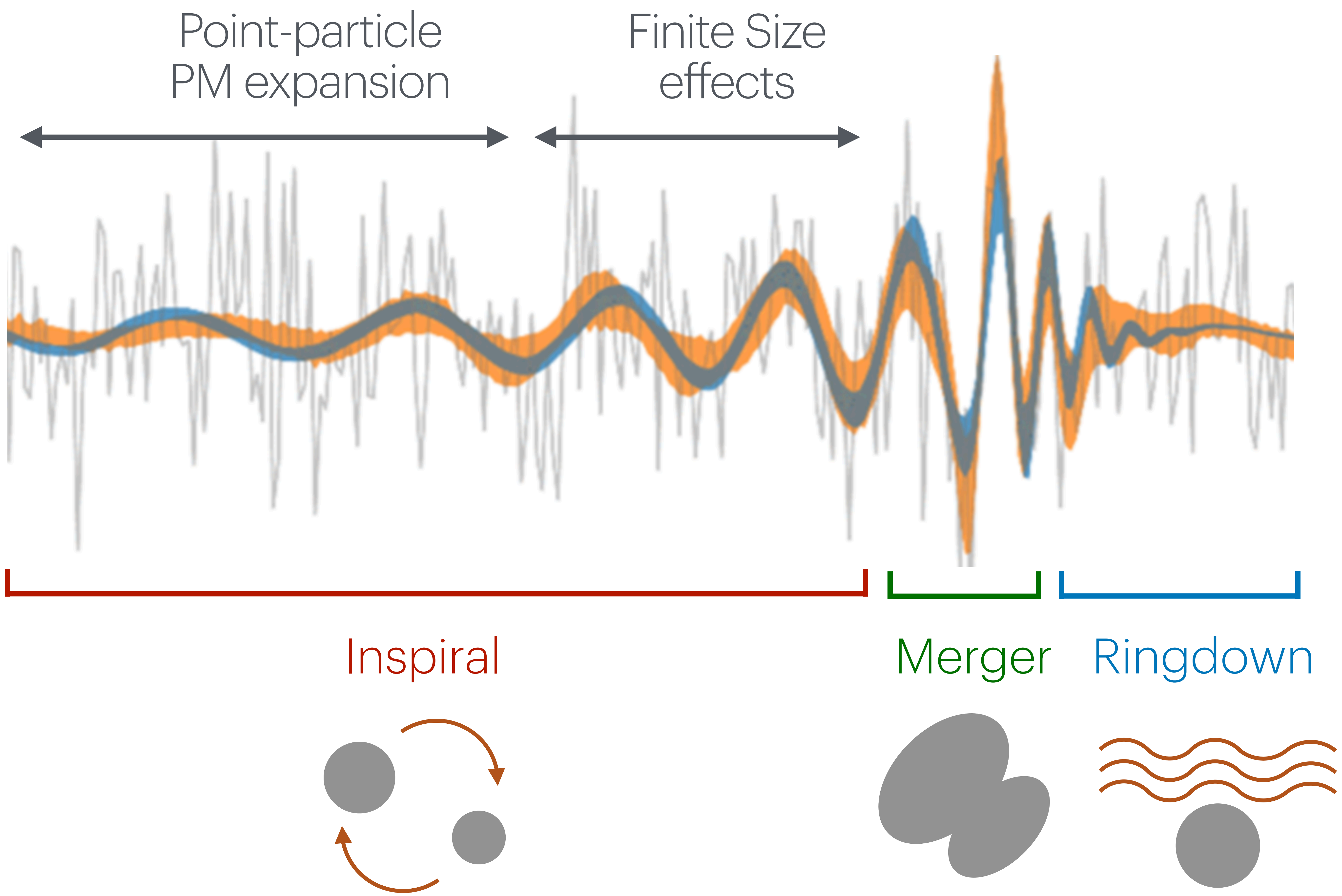
$\lambda \gg \lambda_i$



$$\boxed{\omega \ll M}$$

?

Introduction



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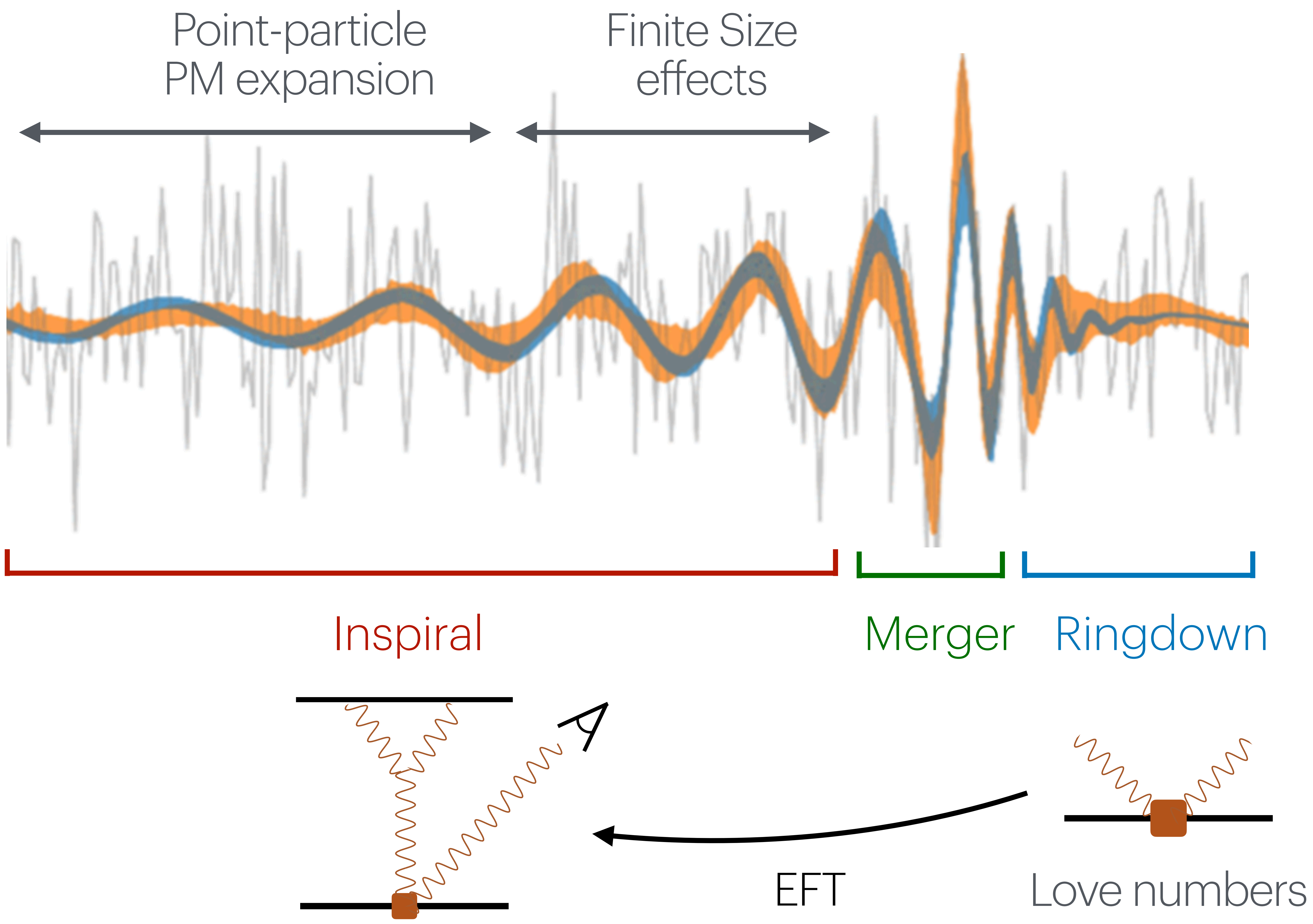

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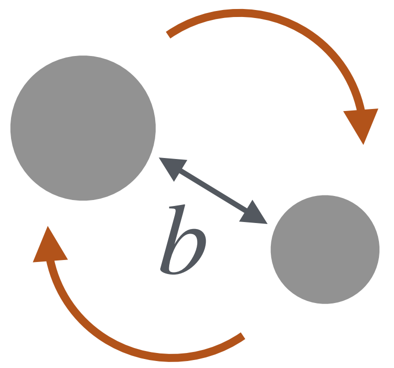

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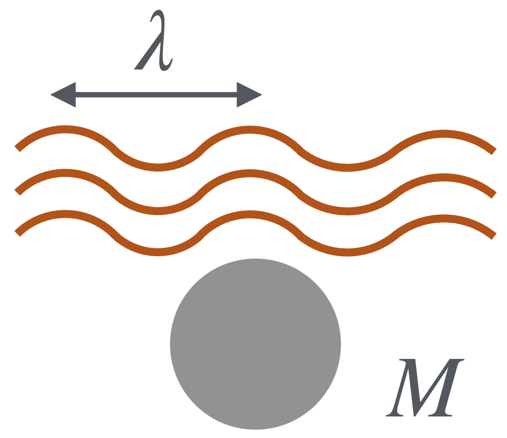


$$\boxed{\omega \ll M}$$

?

The Born regime

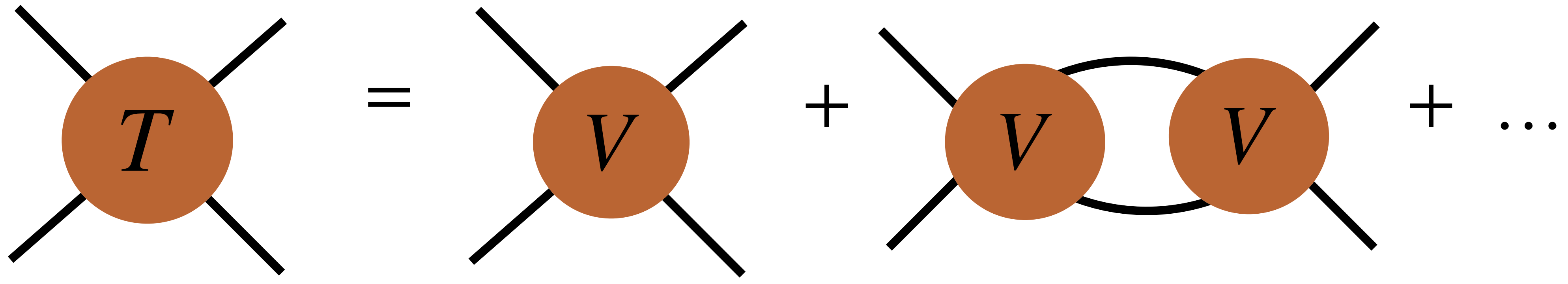
Inspiration from the eikonal

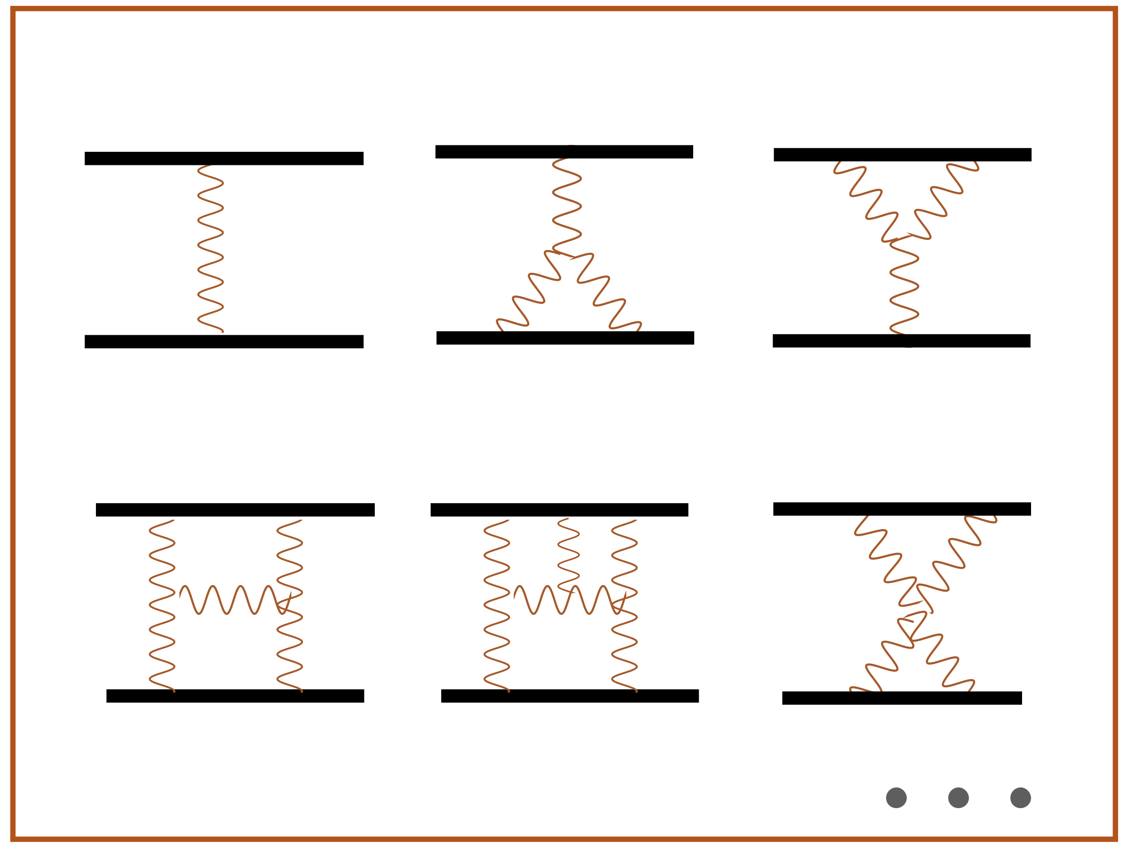
Todorov, 1970

1808.02489 [C. Cheung, I. Rothstein, M. Solon]

2406.13737 [M. Correia, GI]

Amplitude in the eikonal regime





Effective-one-body equation

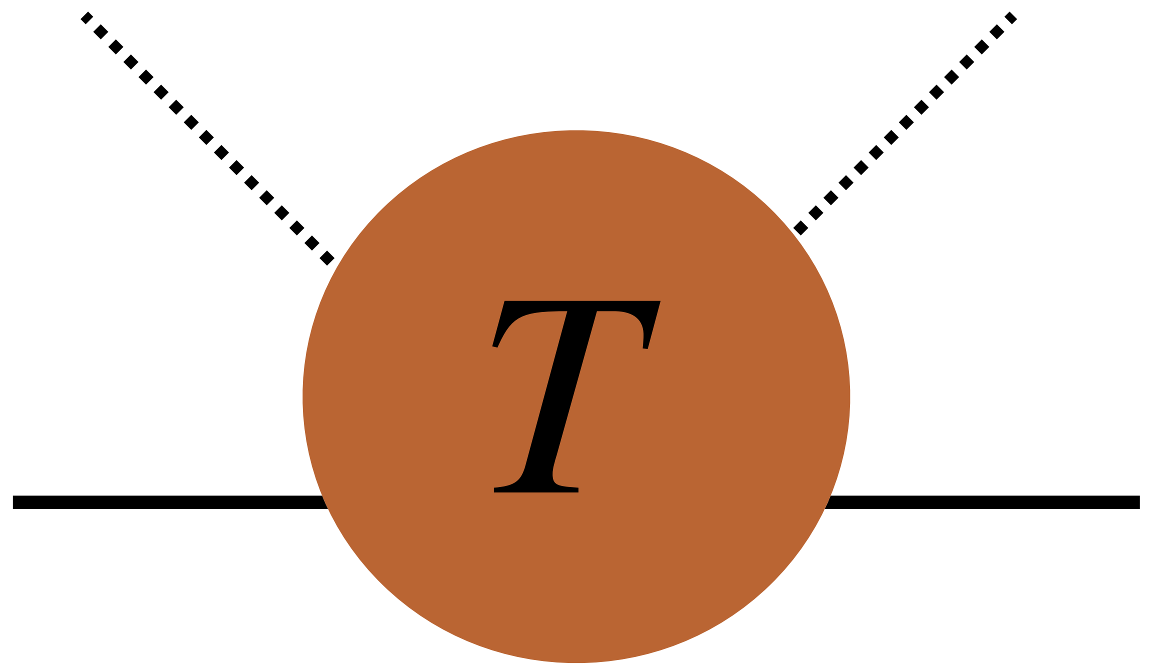
$$\left(\nabla^2 + |\mathbf{p}|^2 \right) \psi(\mathbf{x}) = V(\mathbf{x})\, \psi(\mathbf{x})$$

$$V = ?$$

Relation to radial action:

$$\psi \sim e^{\frac{i}{\hbar} I(s,b)}$$

Organising diagrams

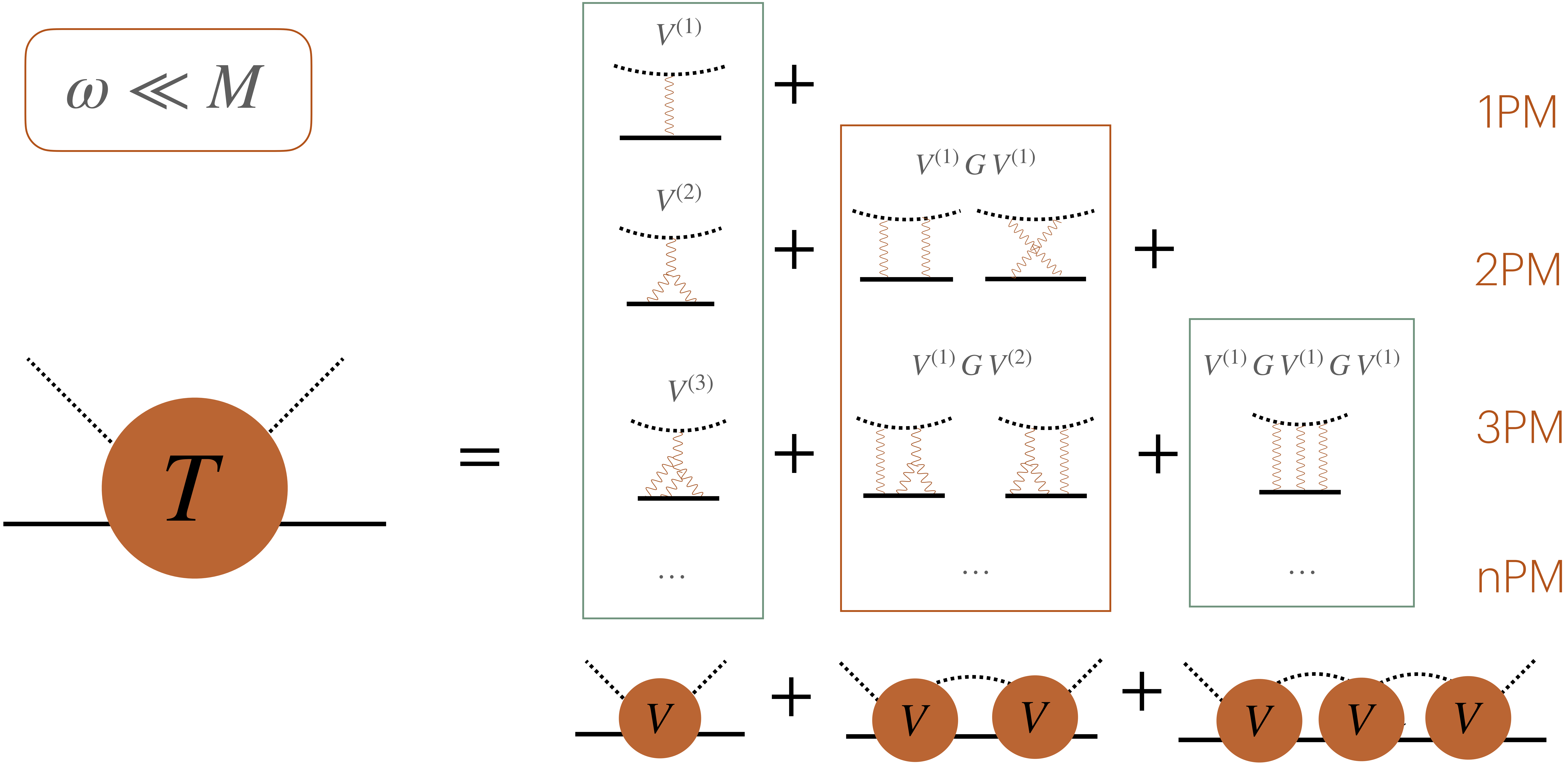


$$\boxed{\omega \ll M}$$

Organising diagrams

2406.13737 [M. Correia, GI]

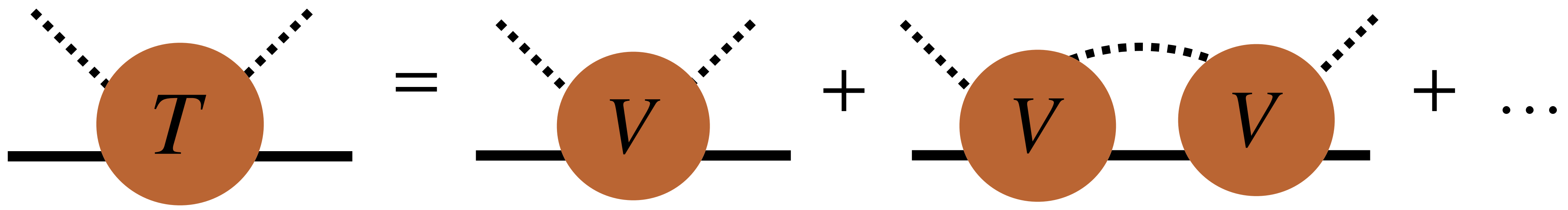
$$\omega \ll M$$

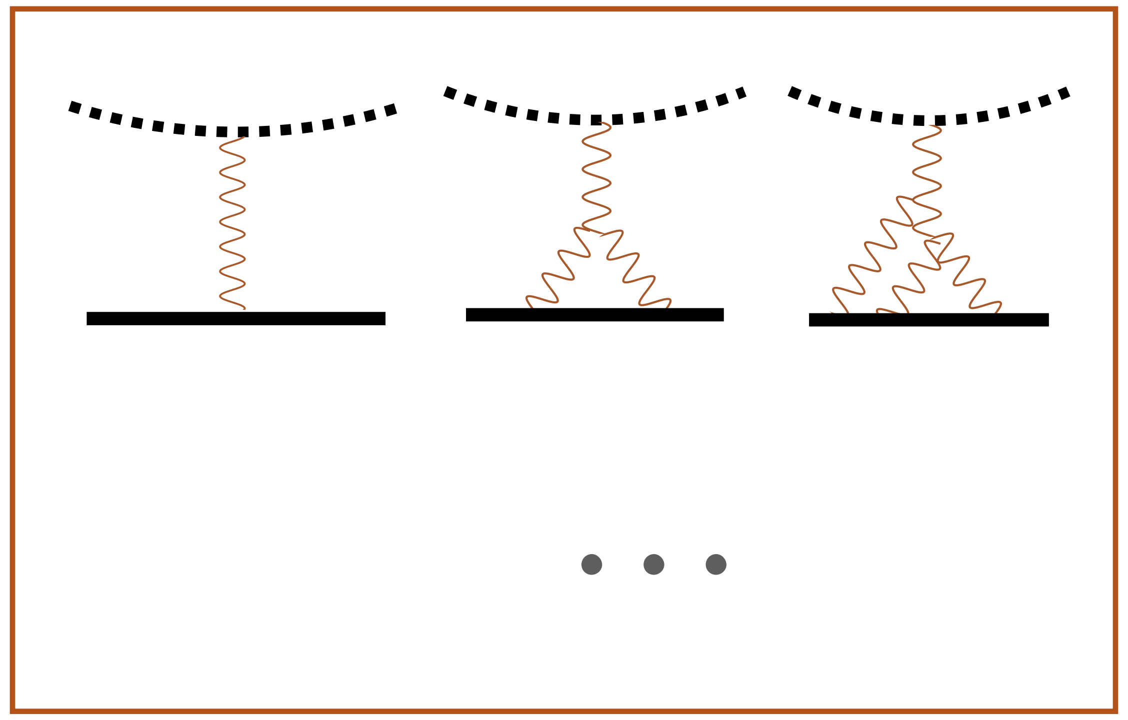


Amplitude vs Potential

2406.13737 [M. Correia, GI]

Amplitude in the Born regime





Wave equation

$$\left(\nabla^2 + \omega^2\right)\phi(\mathbf{x}) = V(\mathbf{x})\,\phi(\mathbf{x})$$

Schwarzschild potential in isotropic coordinates:

$$V = -\left(\frac{4MG}{r} + \frac{15G^2M^2}{2r^2}\right)\omega^2 - \frac{G^2M^2}{2r^3}\partial_r + \mathcal{O}(G^3)$$

Amplitude vs Potential

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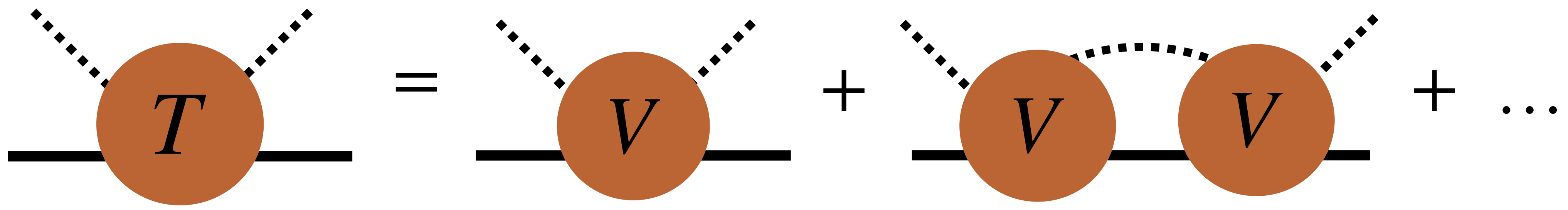

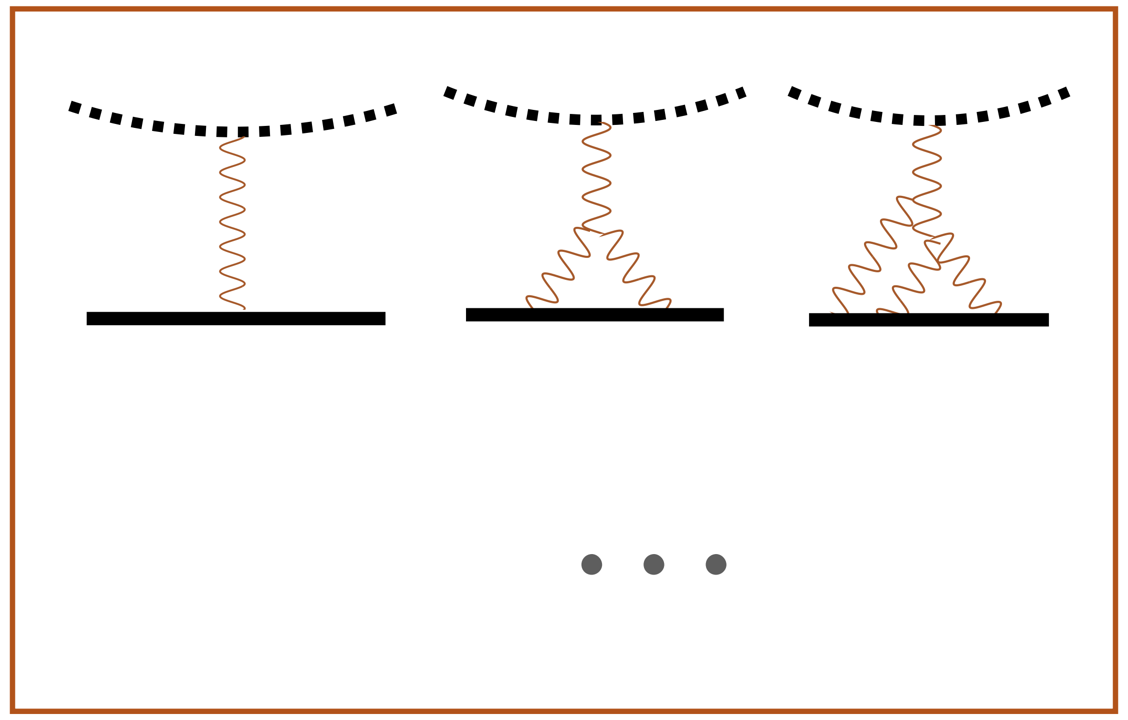

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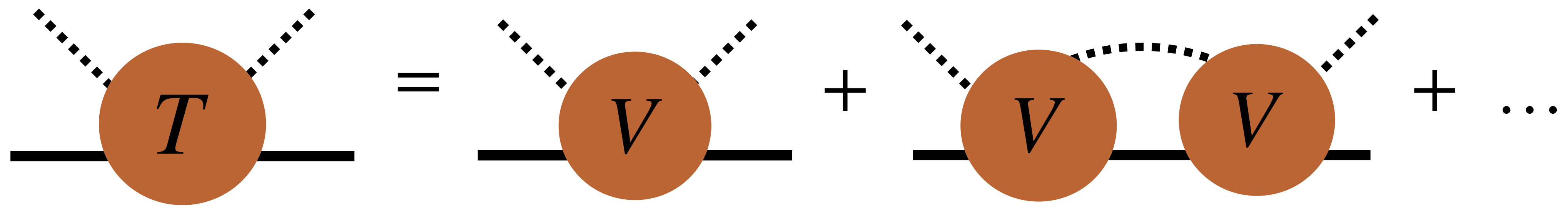

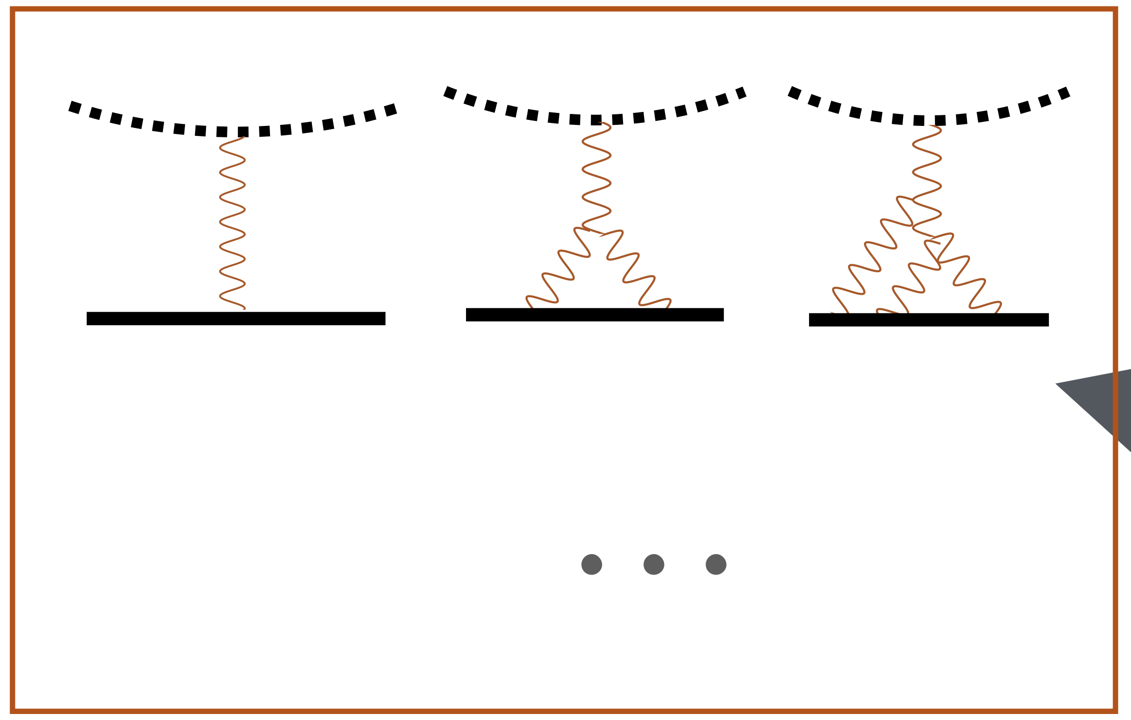

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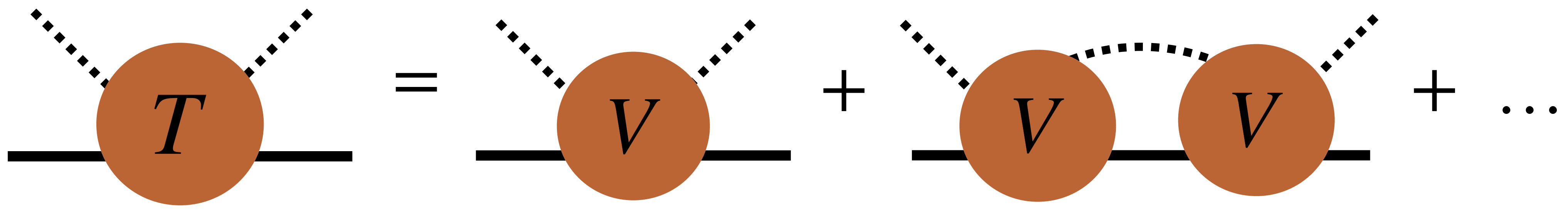

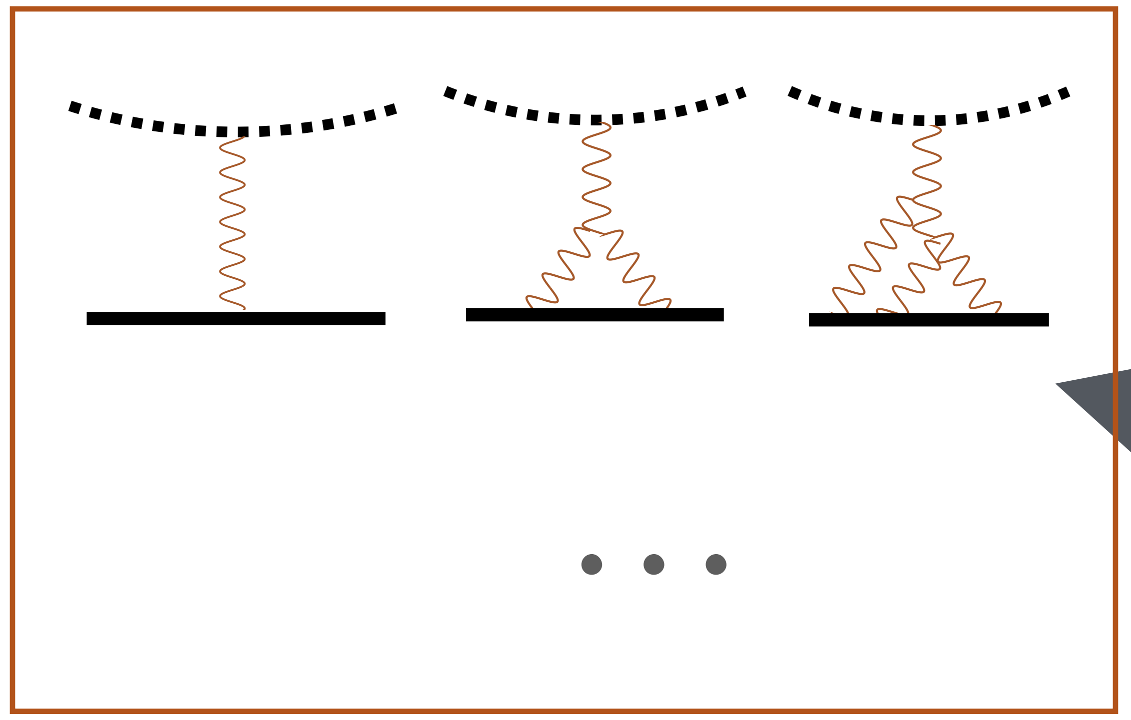

Almost…

Wave equation

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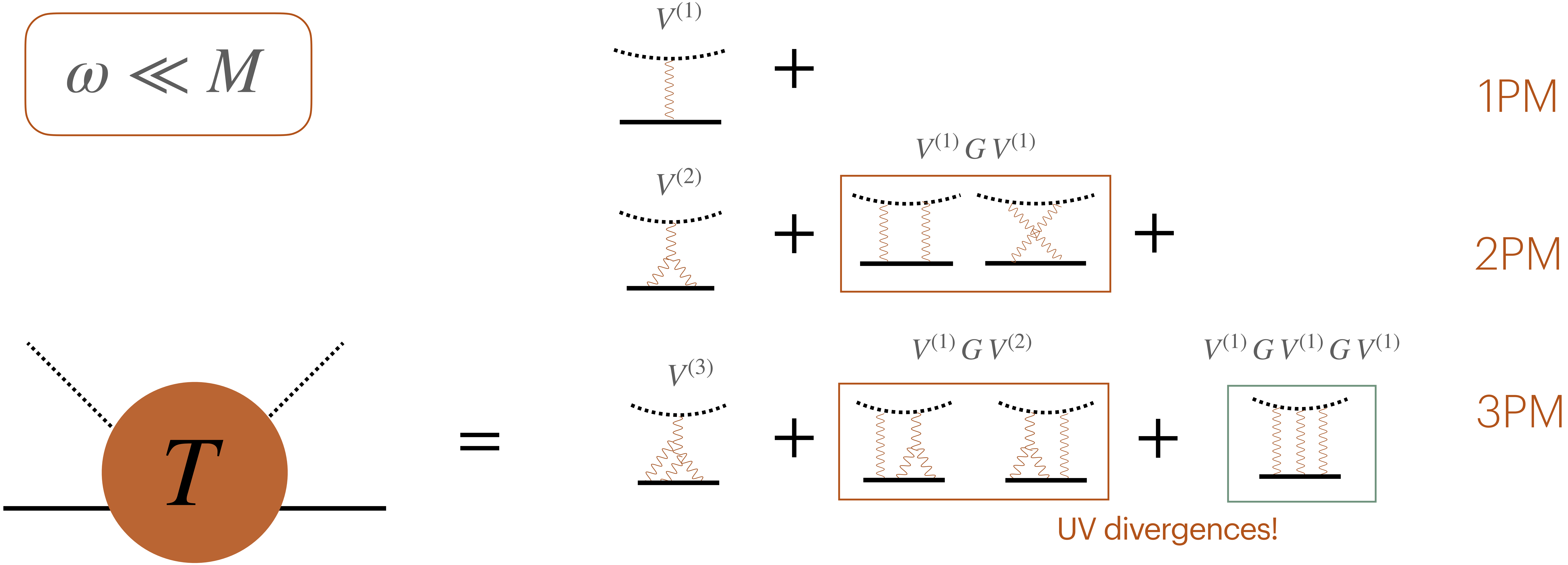
Organising diagrams

$\omega \ll M$



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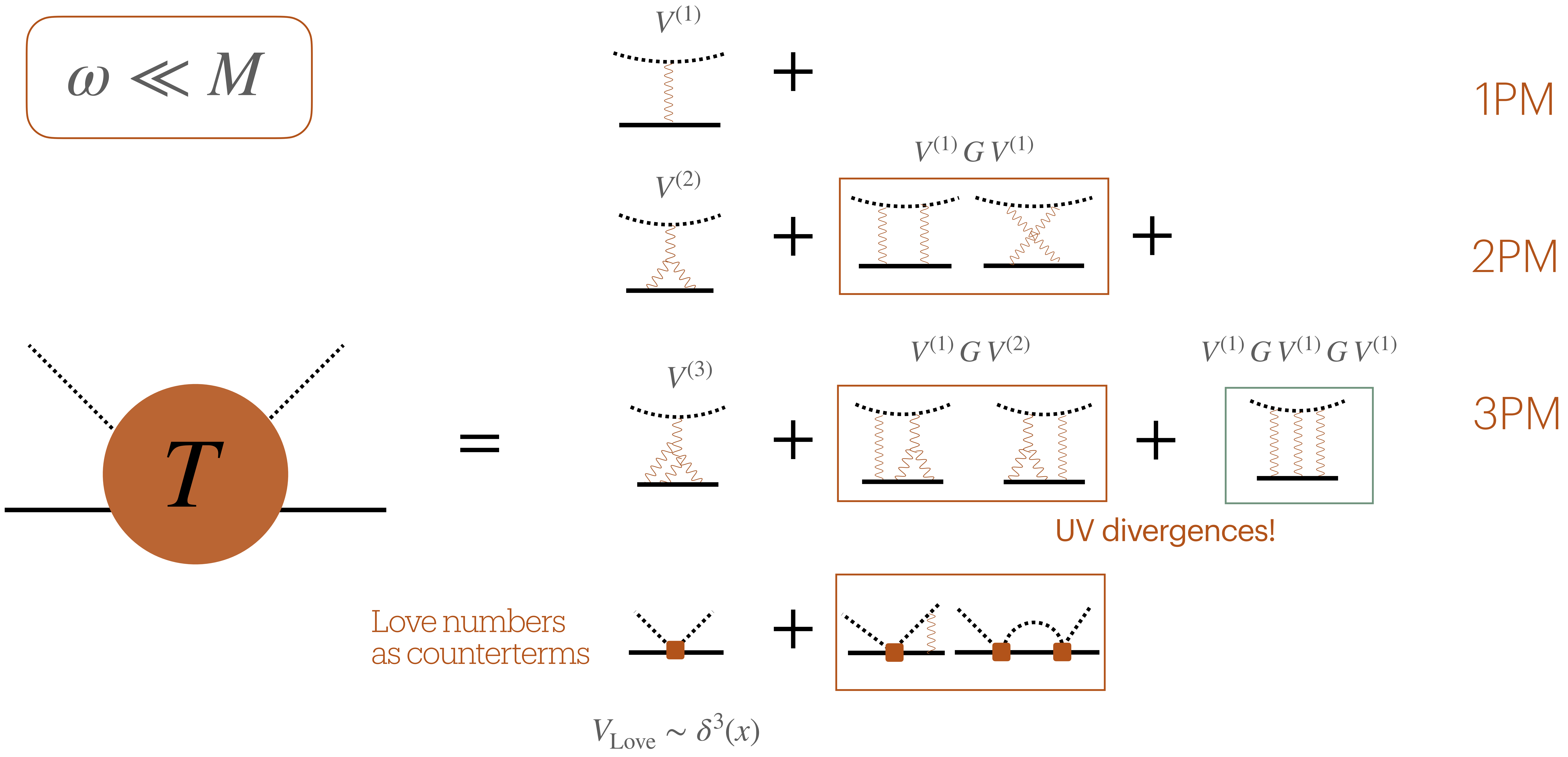
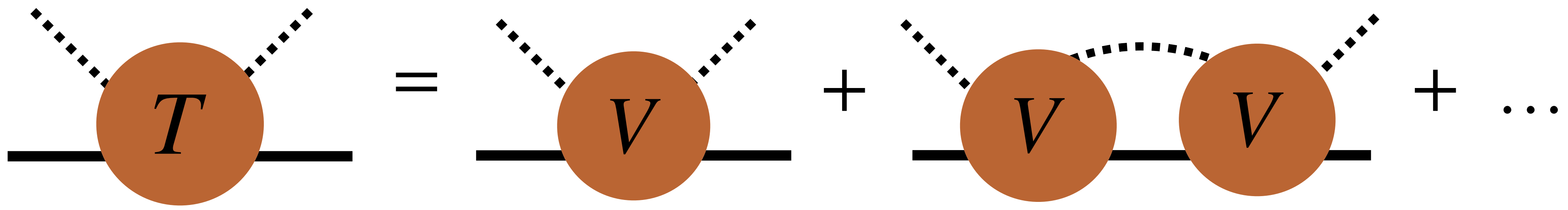
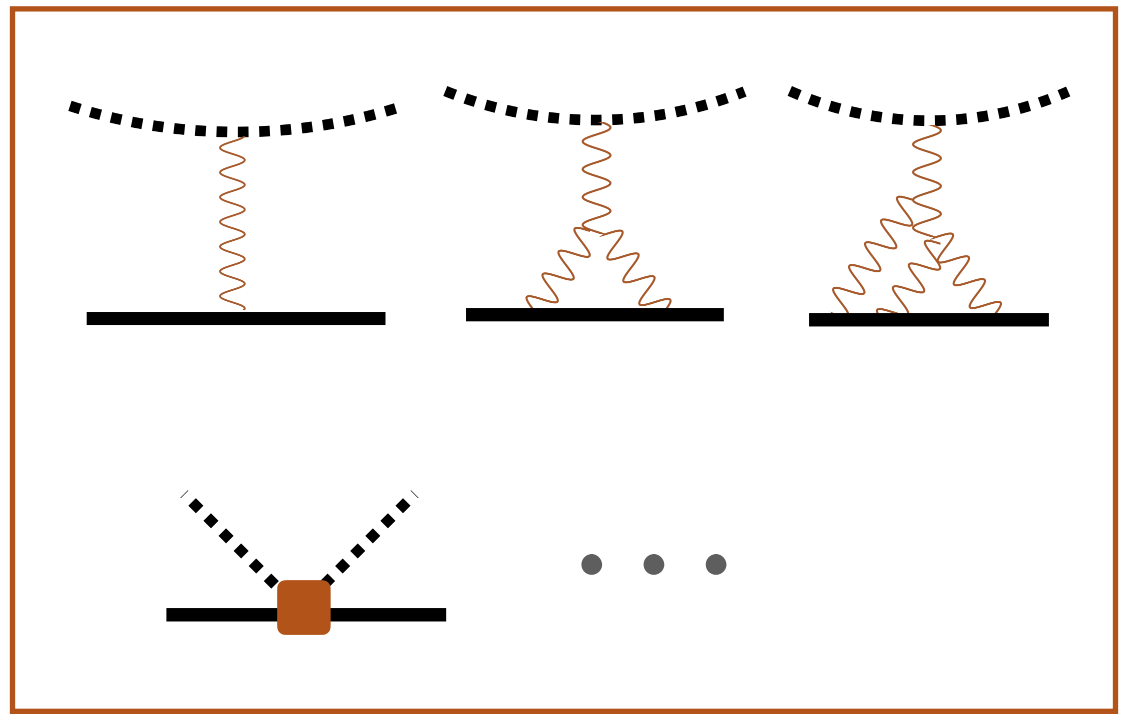


Amplitude vs Potential

Todorov, 1970
2406.13737 [M. Correia, GI]

Amplitude in the Born regime





Effective wave equation

$$\left(\nabla^2 + \omega^2\right)\phi(\mathbf{x}) = V(\mathbf{x})\,\phi(\mathbf{x})$$

Amplitude vs Potential

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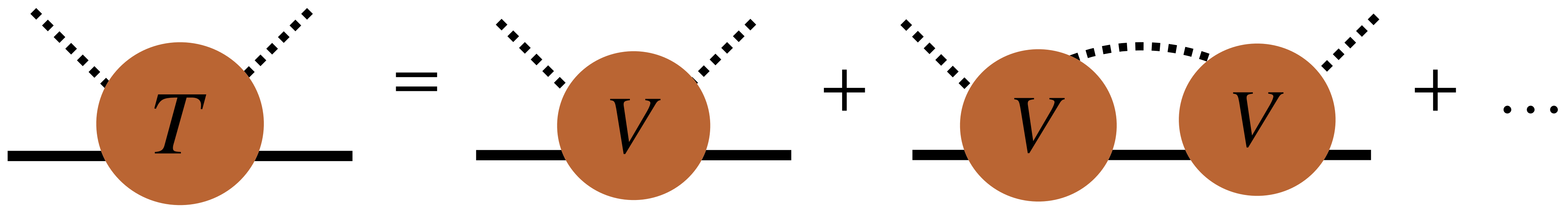
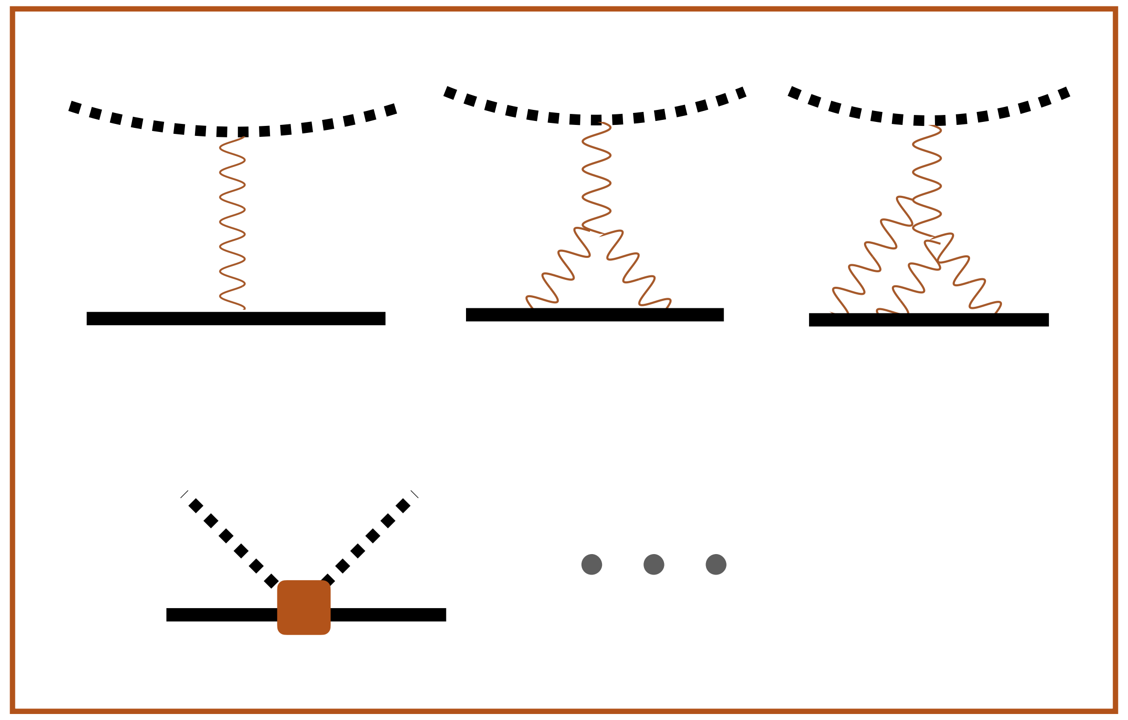
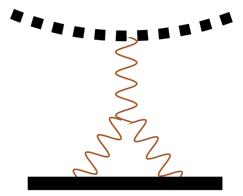
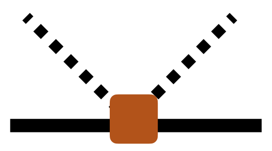




Effective wave equation

$$\left(\nabla^2 + \omega^2\right)\phi(\mathbf{x}) = V(\mathbf{x})\,\phi(\mathbf{x})$$

$$V = V_{\rm grav} + V_{\rm Love}$$

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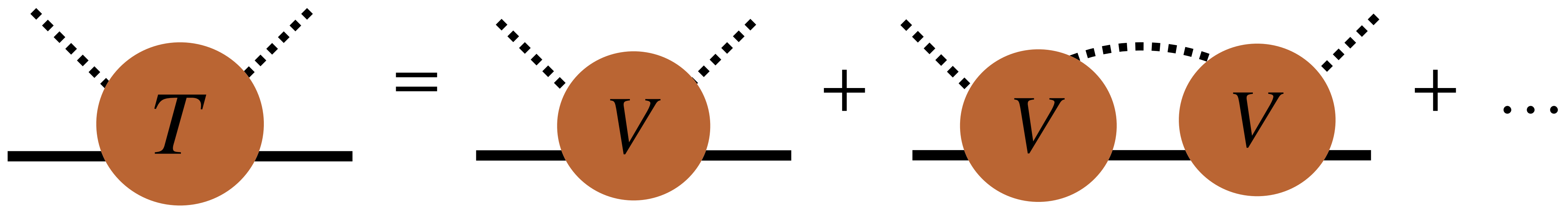
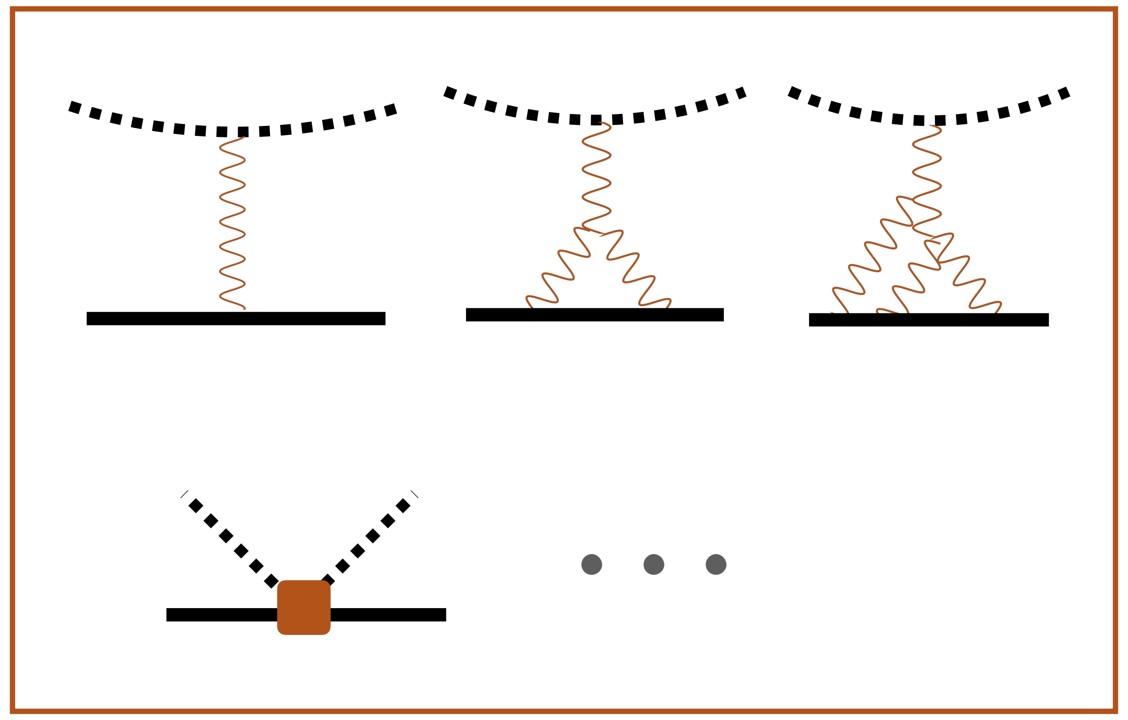


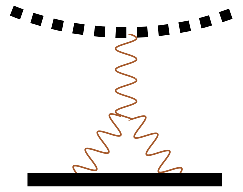


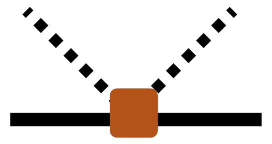
Yes, we can

Effective wave equation

$$\left(\nabla^2 + \omega^2\right)\phi(\mathbf{x}) = V(\mathbf{x})\,\phi(\mathbf{x})$$

$$V = V_{\text{grav}} + V_{\text{Love}}$$





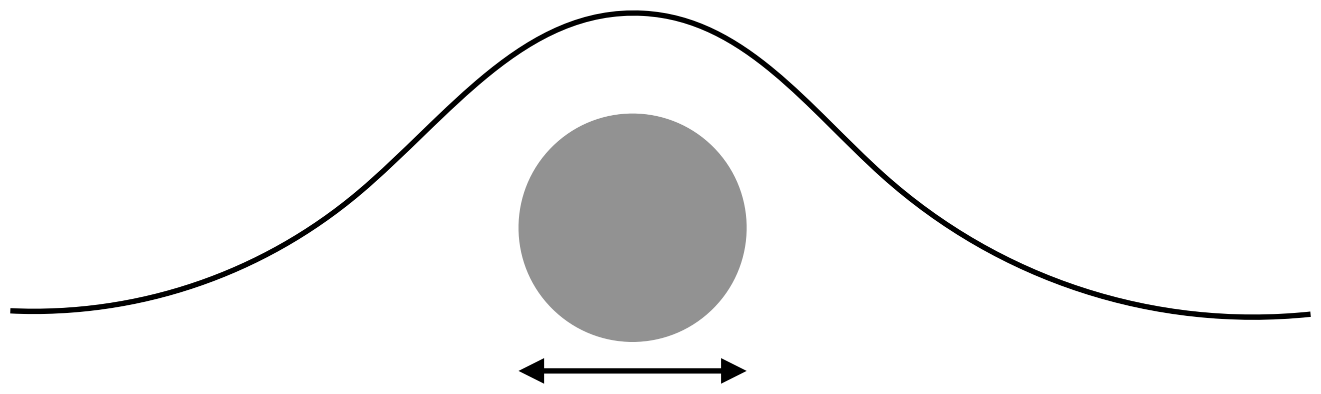
We can solve a differential equation instead of computing the amplitude directly!

Love numbers matching

EFT Matching UV

EFT — Matching — UV

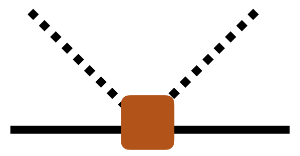
Point particle



$$\omega \ll M \qquad \lambda = \frac{1}{\omega} \gg L$$

$$S_{world} = -M\int d\tau + \sum_{n,\ell} \boxed{C_{\ell n}} (\nabla_\ell \phi_+)\, \partial_\tau^n \,(\nabla_\ell \phi_-)$$

Love numbers

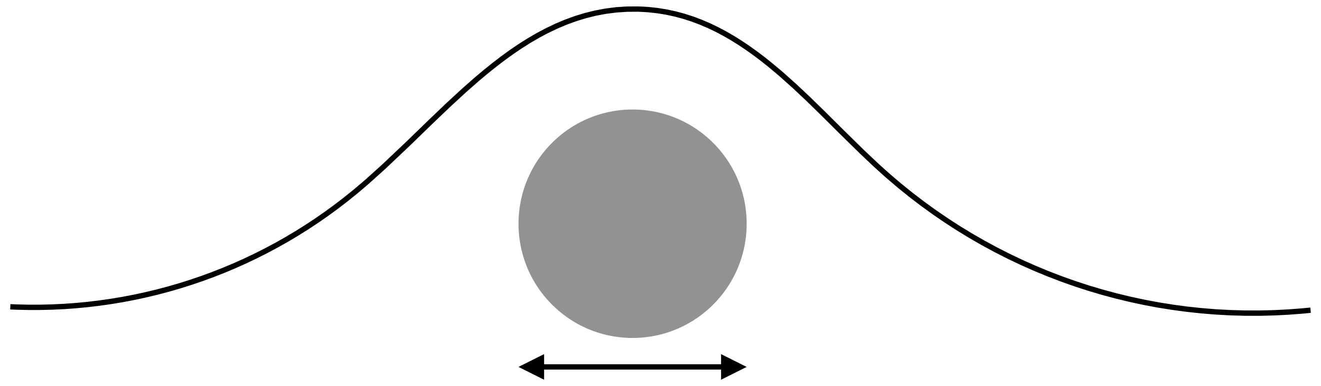


$C_{\ell n}$

EFT ⟷ Matching ⟷ UV

EFT

Point particle

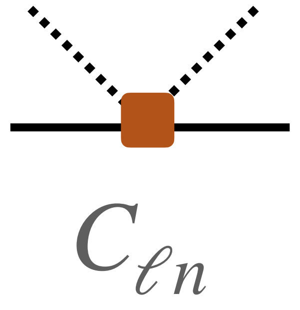


$\omega \ll M$ $\qquad \lambda = \dfrac{1}{\omega} \gg L$

⇨ $GM\omega \ll 1$

$$S_{world} = -M\int d\tau + \sum_{n,\ell} \boxed{C_{\ell n}} (\nabla_\ell \phi_+)\, \partial_\tau^n (\nabla_\ell \phi_-)$$

Love numbers

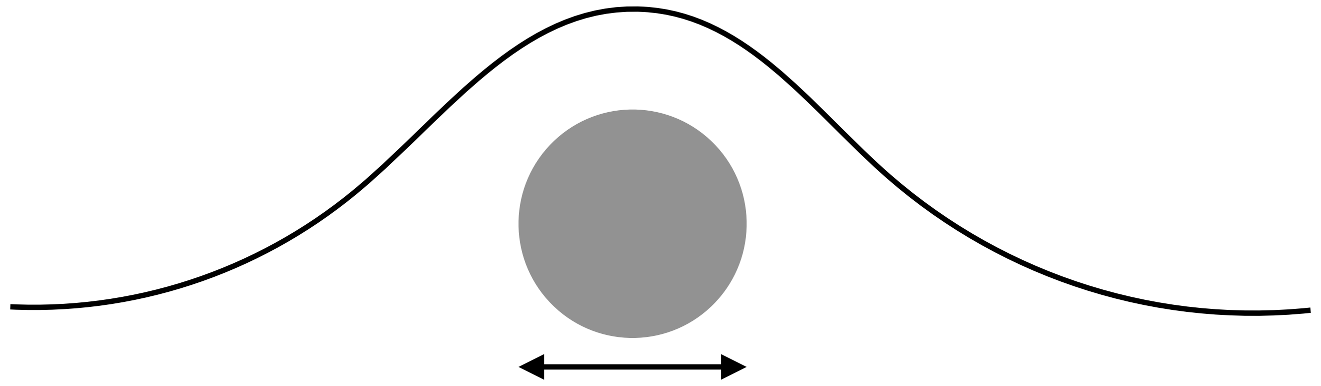


UV

Schwarzschild Black hole

EFT　　Matching　　UV

Point particle

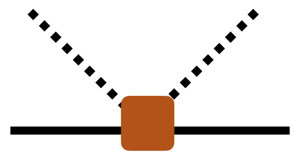


$\omega \ll M$　　　$\lambda = \dfrac{1}{\omega} \gg L$

$\Rightarrow \quad GM\omega \ll 1$

$$S_{world} = -M\int d\tau + \sum_{n,\ell} \boxed{C_{\ell n}} (\nabla_\ell \phi_+)\, \partial_\tau^n\, (\nabla_\ell \phi_-)$$

Love numbers



$C_{\ell n}$

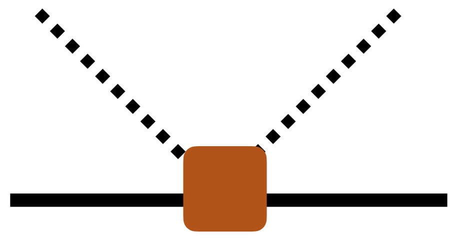
Schwarzschild Black hole

Regge-Wheeler　$\partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \phi \right) = 0$

- 2nd order ODE
- BCs at horizon
- Solution known in partial waves **(BHPT)**

State of the art for Black Hole Love number matching

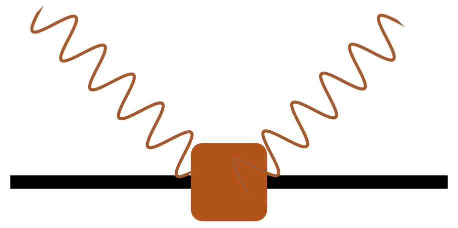
Scalar wave



2401.08752 [M. Ivanov, Y. Li, J. Parra-Martinez, Z. Zhou] $\mathcal{O}(G^3)$

2503.13593 [S. Caron-Huot, M. Correia, GI, M. Solon] $\mathcal{O}(G^7)$

Gravitational wave

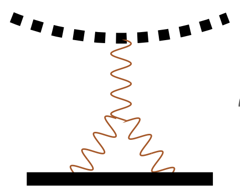


2511.02372 [O. C-S, D. Glazer, A. Joyce, M. J. Rodriguez, L. Santoni]

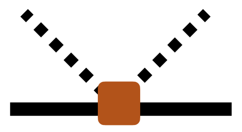
$\mathcal{O}(G^7)$

The effective wave equation

$$\left(\frac{d^2}{dr^2} - \frac{(\ell-\epsilon)(\ell-\epsilon+1)}{r^2} + \omega^2\right)\psi(r) = V(r)\,\psi(r)$$



$\sim 1/r^n$



$\sim \delta^3(x)$

Solve the effective wave equation with
a Born series

Results for the EFT

$S_\ell = e^{2i\delta_\ell}$

Without Love numbers for simplicity

- $\ell = 0$

$$
\begin{aligned}
\delta^{\ell=0}_{\text{Grav}}(\omega) &= GM\omega\left(\frac{1}{\epsilon}+\log\frac{4\omega^2}{\bar{\mu}_{\text{IR}^2}}-1+2\gamma_E\right)+\frac{11\pi}{3}(GM\omega)^2\\
&+(GM\omega)^3\left(\frac{2}{\epsilon}-6\log\left(\frac{4\omega^2}{\bar{\mu}^2}\right)+\frac{62}{3}+\frac{22\pi^2}{9}-\frac{8}{3}\zeta_3\right)\\
&+(GM\omega)^4\left(\frac{4\pi}{\epsilon}-16\pi\log\left(\frac{4\omega^2}{\bar{\mu}^2}\right)+\frac{6799\pi}{135}\right)+(GM\omega)^5\left[\frac{11}{3\epsilon^2}+\frac{1}{3\epsilon}\left(194+8\pi^2-55\log\left(\frac{4\omega^2}{\bar{\mu}^2}\right)\right)\right.\\
&\left.\quad+\frac{275}{6}\log^2\left(\frac{4\omega^2}{\bar{\mu}^2}\right)-\frac{10}{3}\left(97+4\pi^2\right)\log\left(\frac{4\omega^2}{\bar{\mu}^2}\right)+\frac{105458}{135}+\frac{3407\pi^2}{1620}-\frac{616}{9}\zeta_3-\frac{88\pi^4}{135}+\frac{32}{5}\zeta_5\right]\\
&+(GM\omega)^6\left[\frac{22\pi}{3\epsilon^2}-\frac{4\pi}{3\epsilon}\left(33\log\left(\frac{4\omega^2}{\bar{\mu}^2}\right)-82\right)+132\pi\log^2\left(\frac{4\omega^2}{\bar{\mu}^2}\right)-656\pi\log\left(\frac{4\omega^2}{\bar{\mu}^2}\right)\right.\\
&\left.\quad-\frac{176}{3}\pi^3-192\pi\zeta_3-33\pi^3+\frac{52073626\pi}{42525}\right]\\
&+(GM\omega)^7\left[\frac{157}{27\epsilon^3}+\frac{1}{405\epsilon^2}\left(-16485\log\left(\frac{4\omega^2}{\bar{\mu}^2}\right)+1980\pi^2+56663\right)\right.\\
&\quad+\frac{1}{8100\epsilon}\left(1153950\log^2\left(\frac{4\omega^2}{\bar{\mu}^2}\right)-140\left(56663+1980\pi^2\right)\log\left(\frac{4\omega^2}{\bar{\mu}^2}\right)-2138400\zeta_3\right.\\
&\left.\quad-5760\pi^4-314325\pi^2+17790348\right)\\
&\quad-\frac{53851}{162}\log^3\left(\frac{4\omega^2}{\bar{\mu}^2}\right)+\frac{49}{810}\left(56663+1980\pi^2\right)\log^2\left(\frac{4\omega^2}{\bar{\mu}^2}\right)\\
&\quad+\frac{7\log\left(\frac{4\omega^2}{\bar{\mu}^2}\right)}{2700}\left(712800\zeta_3+1920\pi^4+104775\pi^2-5930116\right)-\frac{320}{3}\pi^2\zeta_3+\frac{2048\zeta_5}{9}-\frac{128\zeta_7}{7}\\
&\left.\quad-\frac{266929\zeta_3}{45}+\frac{18572969659}{637875}+\frac{704\pi^6}{2835}-\frac{78613\pi^4}{675}-\frac{937910047\pi^2}{510300}\right],
\end{aligned}
$$

- $\ell = 1$

$$
\begin{aligned}
\delta^{\ell=1}_{\text{Grav}}(\omega) &= GM\omega\left(\frac{1}{\epsilon}+\log\frac{4\omega^2}{\bar{\mu}_{\text{IR}^2}}-3+2\gamma_E\right)+(GM\omega)^2\frac{19\pi}{15}\\
&+(GM\omega)^3\left(\frac{38\pi^2}{45}-\frac{8\zeta_3}{3}\right)+(GM\omega)^4\frac{78037\pi}{23625}\\
&+(GM\omega)^5\left[\frac{1}{9\epsilon}-\frac{5}{9}\log\left(\frac{4\omega^2}{\bar{\mu}^2}\right)+\frac{3022}{675}+\frac{156074\pi^2}{70875}-\frac{152\pi^4}{675}+\frac{2888\zeta_3}{225}+\frac{32\zeta_5}{5}\right]\\
&+(GM\omega)^6\left[\frac{2\pi}{9\epsilon}-\frac{4}{3}\pi\log\left(\frac{4\omega^2}{\bar{\mu}^2}\right)+\frac{909340802\pi}{37209375}\right]\\
&+(GM\omega)^7\left[\frac{19}{405\epsilon^2}+\frac{1}{6075\epsilon}\left(-1995\log\left(\frac{4\omega^2}{\bar{\mu}^2}\right)+900\pi^2+16739\right)+\frac{931}{810}\log^2\left(\frac{4\omega^2}{\bar{\mu}^2}\right)\right.\\
&\quad-\frac{7\left(16739+900\pi^2\right)\log\left(\frac{4\omega^2}{\bar{\mu}^2}\right)}{6075}+\frac{93540397}{1063125}+\frac{7014260791\pi^2}{446512500}-\frac{8896\pi^4}{39375}\\
&\left.\quad+\frac{1216\pi^6}{14175}+\frac{18368248\zeta_3}{354375}-\frac{23104\zeta_5}{225}-\frac{128\zeta_7}{7}\right],
\end{aligned}
$$

- $\ell = 2$

$$
\begin{aligned}
\delta^{\ell=2}_{\text{Grav}}(\omega) &= GM\omega\left(\frac{1}{\epsilon}+\log\frac{4\omega^2}{\bar{\mu}_{\text{IR}^2}}-4+2\gamma_E\right)+(GM\omega)^2\frac{79\pi}{105}+(GM\omega)^3\left(\frac{158\pi^2}{315}-\frac{8\zeta_3}{3}\right)\\
&+(GM\omega)^4\frac{708247\pi}{1157625}+(GM\omega)^5\left[-\frac{16903}{33075}+\frac{1416494\pi^2}{3472875}-\frac{632\pi^4}{4725}+\frac{49928\zeta_3}{11025}+\frac{32\zeta_5}{5}\right]\\
&+(GM\omega)^6\frac{173197070512\pi}{140390971875}+(GM\omega)^7\left[\frac{8}{6075\epsilon}-\frac{56\log\left(\frac{4\omega^2}{\bar{\mu}^2}\right)}{6075}-\frac{649093993}{2187911250}\right.\\
&\left.\quad+\frac{346394141024\pi^2}{421172915625}-\frac{3904\pi^4}{118125}+\frac{5056\pi^6}{99225}+\frac{895224208\zeta_3}{121550625}-\frac{399424\zeta_5}{11025}-\frac{128\zeta_7}{7}\right].
\end{aligned}
$$

Results for the EFT

$S_\ell = e^{2i\delta_\ell}$

Without Love numbers for simplicity

Match with 2401.08752
[Parra-Martinez, Ivanov, Li, Zhou]

- $\ell = 0$

$$\begin{aligned}
\delta^{\ell=0}_{\rm Grav}(\omega) &= GM\omega\left(\frac{1}{\epsilon}+\log\frac{4\omega^2}{\bar\mu_{\rm IR^2}}-1+2\gamma_E\right)+\frac{11\pi}{3}(GM\omega)^2 \\
&+(GM\omega)^3\left(\frac{2}{\epsilon}-6\log\left(\frac{4\omega^2}{\bar\mu^2}\right)+\frac{62}{3}+\frac{22\pi^2}{9}-\frac{8}{3}\zeta_3\right) \\
&+(GM\omega)^4\left(\frac{4\pi}{\epsilon}-16\pi\log\left(\frac{4\omega^2}{\bar\mu^2}\right)+\frac{6799\pi}{135}\right)+(GM\omega)^5\Bigg[\frac{11}{3\epsilon^2}+\frac{1}{3\epsilon}\left(194+8\pi^2-55\log\left(\frac{4\omega^2}{\bar\mu^2}\right)\right) \\
&\quad+\frac{275}{6}\log^2\left(\frac{4\omega^2}{\bar\mu^2}\right)-\frac{10}{3}\left(97+4\pi^2\right)\log\left(\frac{4\omega^2}{\bar\mu^2}\right)+\frac{105458}{135}+\frac{3407\pi^2}{1620}-\frac{616}{9}\zeta_3-\frac{88\pi^4}{135}+\frac{32}{5}\zeta_5\Bigg] \\
&+(GM\omega)^6\Bigg[\frac{22\pi}{3\epsilon^2}-\frac{4\pi}{3\epsilon}\left(33\log\left(\frac{4\omega^2}{\bar\mu^2}\right)-82\right)+132\pi\log^2\left(\frac{4\omega^2}{\bar\mu^2}\right)-656\pi\log\left(\frac{4\omega^2}{\bar\mu^2}\right) \\
&\quad-\frac{176}{3}\pi^3-192\pi\zeta_3-33\pi^3+\frac{52073626\pi}{42525}\Bigg] \\
&+(GM\omega)^7\Bigg[\frac{157}{27\epsilon^3}+\frac{1}{405\epsilon^2}\left(-16485\log\left(\frac{4\omega^2}{\bar\mu^2}\right)+1980\pi^2+56663\right) \\
&\quad+\frac{1}{8100\epsilon}\Bigg(1153950\log^2\left(\frac{4\omega^2}{\bar\mu^2}\right)-140\left(56663+1980\pi^2\right)\log\left(\frac{4\omega^2}{\bar\mu^2}\right)-2138400\zeta_3 \\
&\quad-5760\pi^4-314325\pi^2+17790348\Bigg) \\
&\quad-\frac{53851}{162}\log^3\left(\frac{4\omega^2}{\bar\mu^2}\right)+\frac{49}{810}\left(56663+1980\pi^2\right)\log^2\left(\frac{4\omega^2}{\bar\mu^2}\right) \\
&\quad+\frac{7\log\left(\frac{4\omega^2}{\bar\mu^2}\right)}{2700}\left(712800\zeta_3+1920\pi^4+104775\pi^2-5930116\right)-\frac{320}{3}\pi^2\zeta_3+\frac{2048\zeta_5}{9}-\frac{128\zeta_7}{7} \\
&\quad-\frac{266929\zeta_3}{45}+\frac{18572969659}{637875}+\frac{704\pi^6}{2835}-\frac{78613\pi^4}{675}-\frac{937910047\pi^2}{510300}\Bigg],
\end{aligned}$$

- $\ell = 1$

$$\begin{aligned}
\delta^{\ell=1}_{\rm Grav}(\omega) &= GM\omega\left(\frac{1}{\epsilon}+\log\frac{4\omega^2}{\bar\mu_{\rm IR^2}}-3+2\gamma_E\right)+(GM\omega)^2\frac{19\pi}{15} \\
&+(GM\omega)^3\left(\frac{38\pi^2}{45}-\frac{8\zeta_3}{3}\right)+(GM\omega)^4\frac{78037\pi}{23625} \\
&+(GM\omega)^5\left[\frac{1}{9\epsilon}-\frac{5}{9}\log\left(\frac{4\omega^2}{\bar\mu^2}\right)+\frac{3022}{675}+\frac{156074\pi^2}{70875}-\frac{152\pi^4}{675}+\frac{2888\zeta_3}{225}+\frac{32\zeta_5}{5}\right] \\
&+(GM\omega)^6\left[\frac{2\pi}{9\epsilon}-\frac{4}{3}\pi\log\left(\frac{4\omega^2}{\bar\mu^2}\right)+\frac{909340802\pi}{37209375}\right] \\
&+(GM\omega)^7\Bigg[\frac{19}{405\epsilon^2}+\frac{1}{6075\epsilon}\left(-1995\log\left(\frac{4\omega^2}{\bar\mu^2}\right)+900\pi^2+16739\right)+\frac{931}{810}\log^2\left(\frac{4\omega^2}{\bar\mu^2}\right) \\
&\quad-\frac{7\left(16739+900\pi^2\right)\log\left(\frac{4\omega^2}{\bar\mu^2}\right)}{6075}+\frac{93540397}{1063125}+\frac{7014260791\pi^2}{446512500}-\frac{8896\pi^4}{39375} \\
&\quad+\frac{1216\pi^6}{14175}+\frac{18368248\zeta_3}{354375}-\frac{23104\zeta_5}{225}-\frac{128\zeta_7}{7}\Bigg],
\end{aligned}$$

- $\ell = 2$

$$\begin{aligned}
\delta^{\ell=2}_{\rm Grav}(\omega) &= GM\omega\left(\frac{1}{\epsilon}+\log\frac{4\omega^2}{\bar\mu_{\rm IR^2}}-4+2\gamma_E\right)+(GM\omega)^2\frac{79\pi}{105}+(GM\omega)^3\left(\frac{158\pi^2}{315}-\frac{8\zeta_3}{3}\right) \\
&+(GM\omega)^4\frac{708247\pi}{1157625}+(GM\omega)^5\left[-\frac{16903}{33075}+\frac{1416494\pi^2}{3472875}-\frac{632\pi^4}{4725}+\frac{49928\zeta_3}{11025}+\frac{32\zeta_5}{5}\right] \\
&+(GM\omega)^6\frac{173197070512\pi}{140390971875}+(GM\omega)^7\Bigg[\frac{8}{6075\epsilon}-\frac{56\log\left(\frac{4\omega^2}{\bar\mu^2}\right)}{6075}-\frac{649093993}{2187911250} \\
&\quad+\frac{346394141024\pi^2}{421172915625}-\frac{3904\pi^4}{118125}+\frac{5056\pi^6}{99225}+\frac{895224208\zeta_3}{121550625}-\frac{399424\zeta_5}{11025}-\frac{128\zeta_7}{7}\Bigg].
\end{aligned}$$

Scalar Black Hole Love numbers

	$\ell = 0$	$\ell = 1$	$\ell = 2$
G^2	$4\pi R_s^2$	Match with 2401.08752 [Parra-Martinez, Ivanov, Li, Zhou]	
G^3	$-4\pi R_s^3\left(\frac{1}{4\epsilon}+\log R_s\bar{\mu}+\frac{19}{12}+\gamma_E\right)$	0	
G^4	$-\frac{44}{3}\pi R_s^4\left(\frac{1}{4\epsilon}+\log R_s\bar{\mu}-\frac{13}{66}-\frac{\pi^2}{11}+\gamma_E\right)$	$\pi\frac{R_s^4}{3}$	New results
G^5	$\frac{11}{6}\pi R_s^5\left[\frac{1}{4\epsilon^2}+\frac{2}{\epsilon}\left(\log R_s\bar{\mu}+\gamma_E+\frac{56}{33}\right)\left(2\log R_s\bar{\mu}+2\gamma_E+\frac{167}{66}\right)^2-\frac{338611}{21780}-\frac{8\pi^2}{3}-\frac{48\,\zeta_3}{11}\right]$	$-\frac{1}{6}\pi R_s^5\left(\frac{1}{4\epsilon}+2\log R_s\bar{\mu}-\frac{19}{30}+2\gamma_E\right)$	0

RG equations for Love numbers

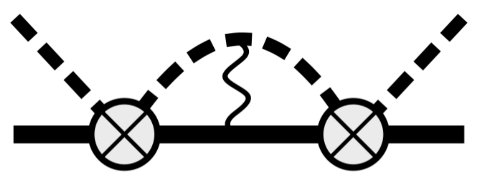
$$F_\ell(\omega) = \sum_{n=0}^{\infty} (i\omega)^n C_{\ell n}$$

- $\ell = 0$

$$\mu \frac{d}{d\mu}\bar{F}_0(\omega) = -4\pi GM\left[8(GM\omega)^2 + 72(GM\omega)^4 + \frac{520408(GM\omega)^6}{675}\right] - 2\bar{F}_0(\omega)\left[\frac{22(GM\omega)^2}{3} + \frac{7118(GM\omega)^4}{135} + \frac{22329152(GM\omega)^6}{42525}\right]$$

$$-\frac{\bar{F}_0(\omega)^2}{4\pi}\omega\left[4GM\omega + \frac{112(GM\omega)^3}{3} + \frac{125402(GM\omega)^5}{405} + \frac{6707817458(GM\omega)^7}{1913625}\right]$$

Quadratic RG



- $\ell = 1$

$$\mu \frac{d}{d\mu}\bar{F}_1(\omega) = -\frac{4\pi}{3}G^3M^3\left[8(GM\omega)^2 + \frac{5804}{75}(GM\omega)^4\right] - 2\bar{F}_1(\omega)\left[\frac{38}{15}(GM\omega)^2 + \frac{156074}{23625}(GM\omega)^4 + \frac{1504469104}{37209375}(GM\omega)^6\right]$$

$$-\frac{3\bar{F}_1(\omega)^2}{4\pi}\omega^3\left[\frac{4}{9}GM\omega + \frac{7648}{2025}(GM\omega)^3 + \frac{463800938}{22325625}(GM\omega)^5 + \frac{4484979447502}{35162859375}(GM\omega)^7\right]$$

- $\ell = 2$

$$\mu \frac{d}{d\mu}\bar{F}_2(\omega) = -\frac{4\pi}{15}G^5M^5\left[\frac{32}{9}(GM\omega)^2\right] - 2\bar{F}_2(\omega)\left[\frac{158}{105}(GM\omega)^2 + \frac{1416494}{1157625}(GM\omega)^4 + \frac{346394141024}{140390971875}(GM\omega)^6\right]$$

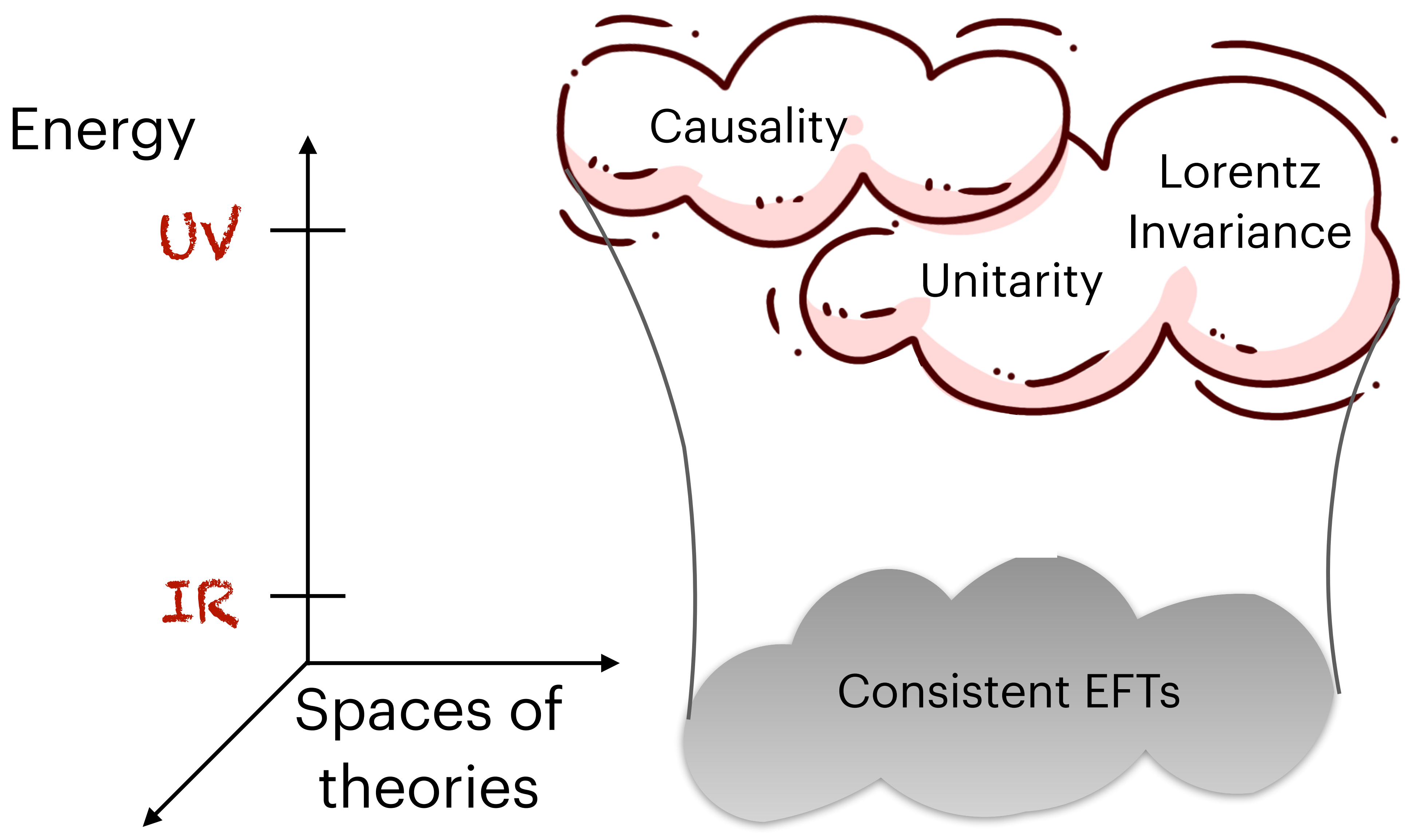
$$-\frac{15\bar{F}_2(\omega)^2}{4\pi}\omega^5\left[\frac{4}{225}GM\omega + \frac{343552}{2480625}(GM\omega)^3 + \frac{14735507852}{27348890625}(GM\omega)^5 + \frac{128766007938881974}{7296820763203125}(GM\omega)^7\right]$$

We have an EFT, now what?

Motivation

Applications

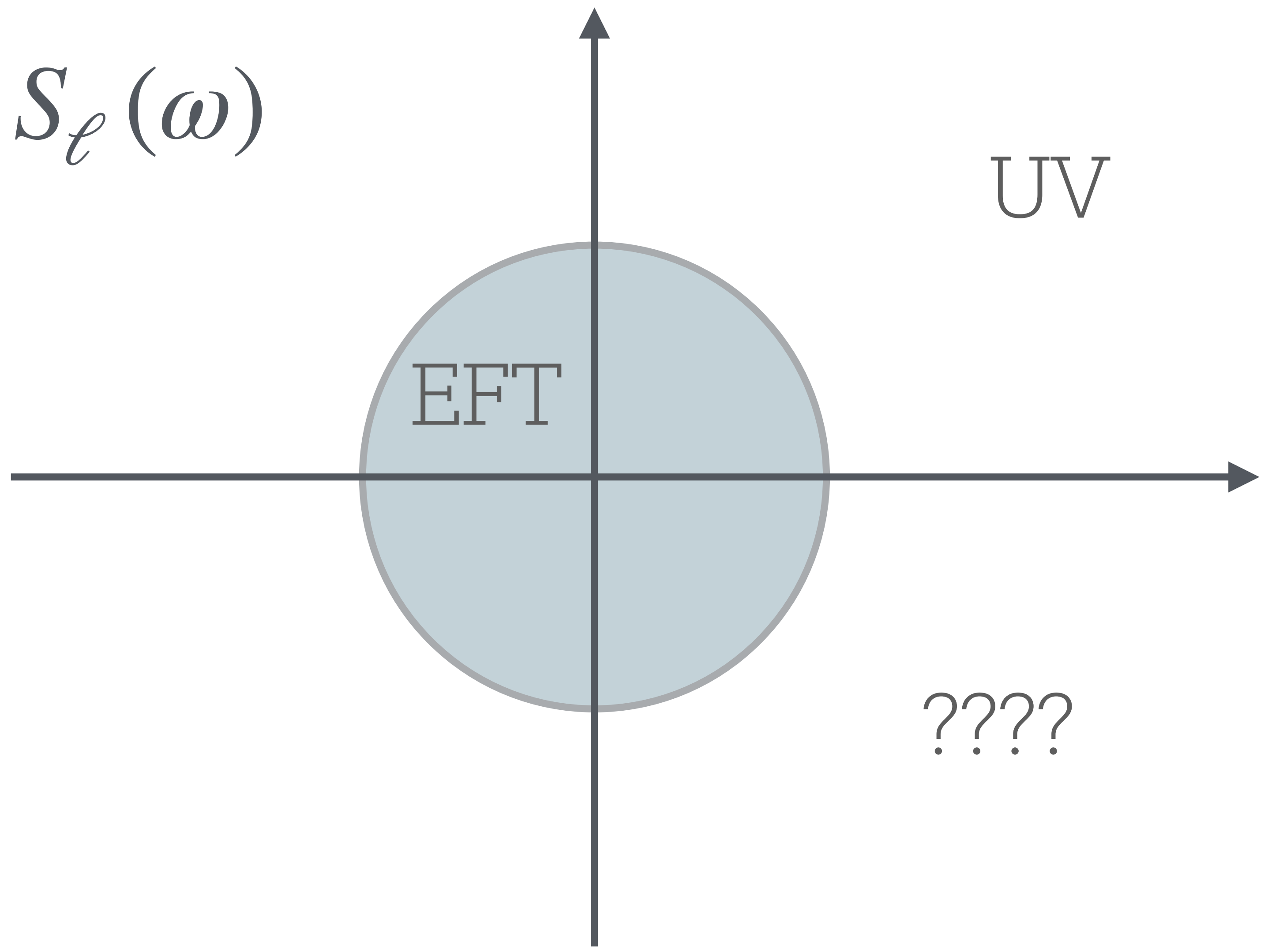
- Positivity bounds on Love numbers



Motivation

Applications

- Positivity bounds on Love numbers

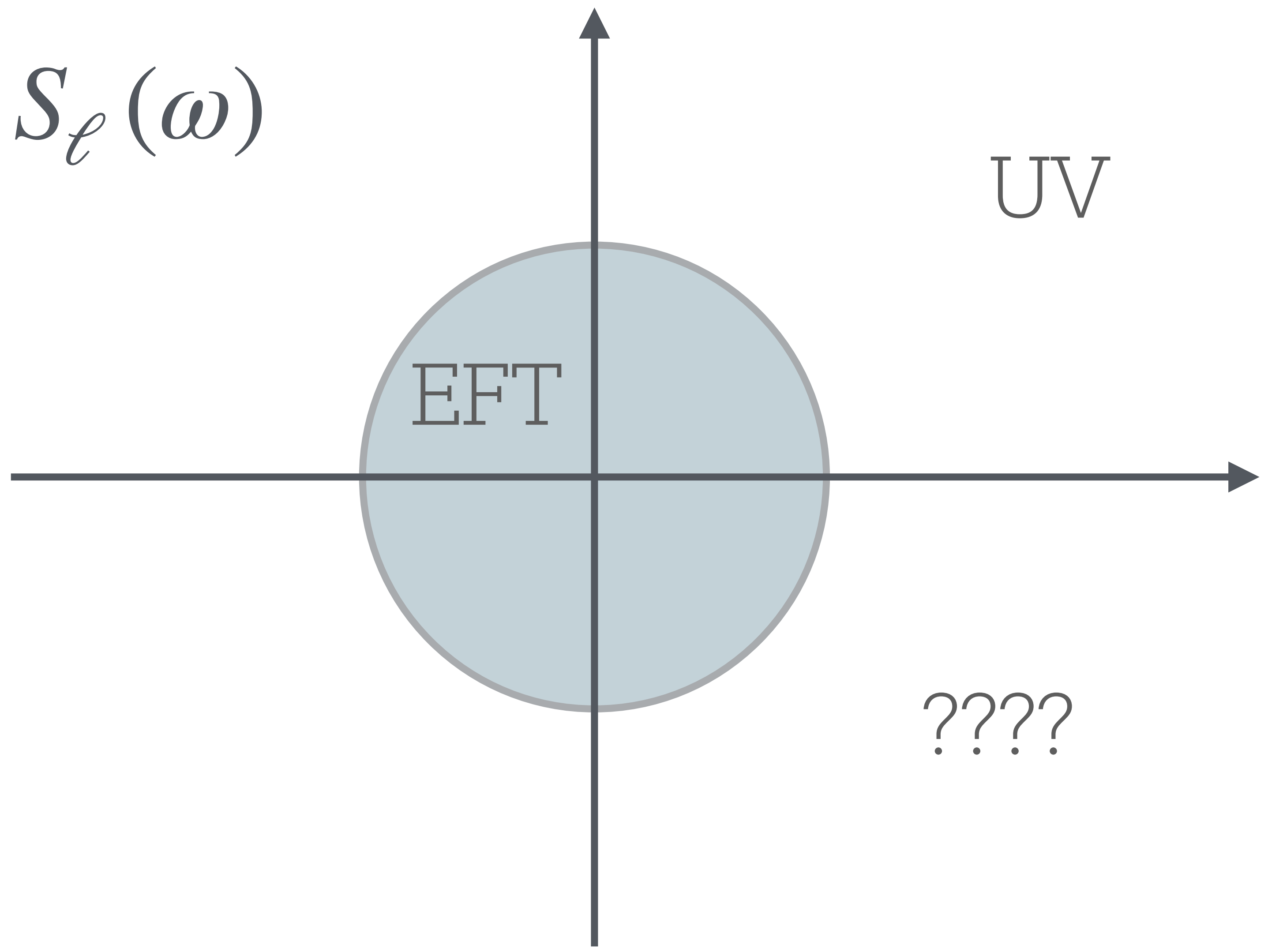


What is the analytic structure of $S_\ell(\omega)$?

Motivation

Applications

- Positivity bounds on Love numbers



Theoretical

- Poles/Branch cuts capture the response of the black hole

- S-matrix in an open system/ background breaking translations

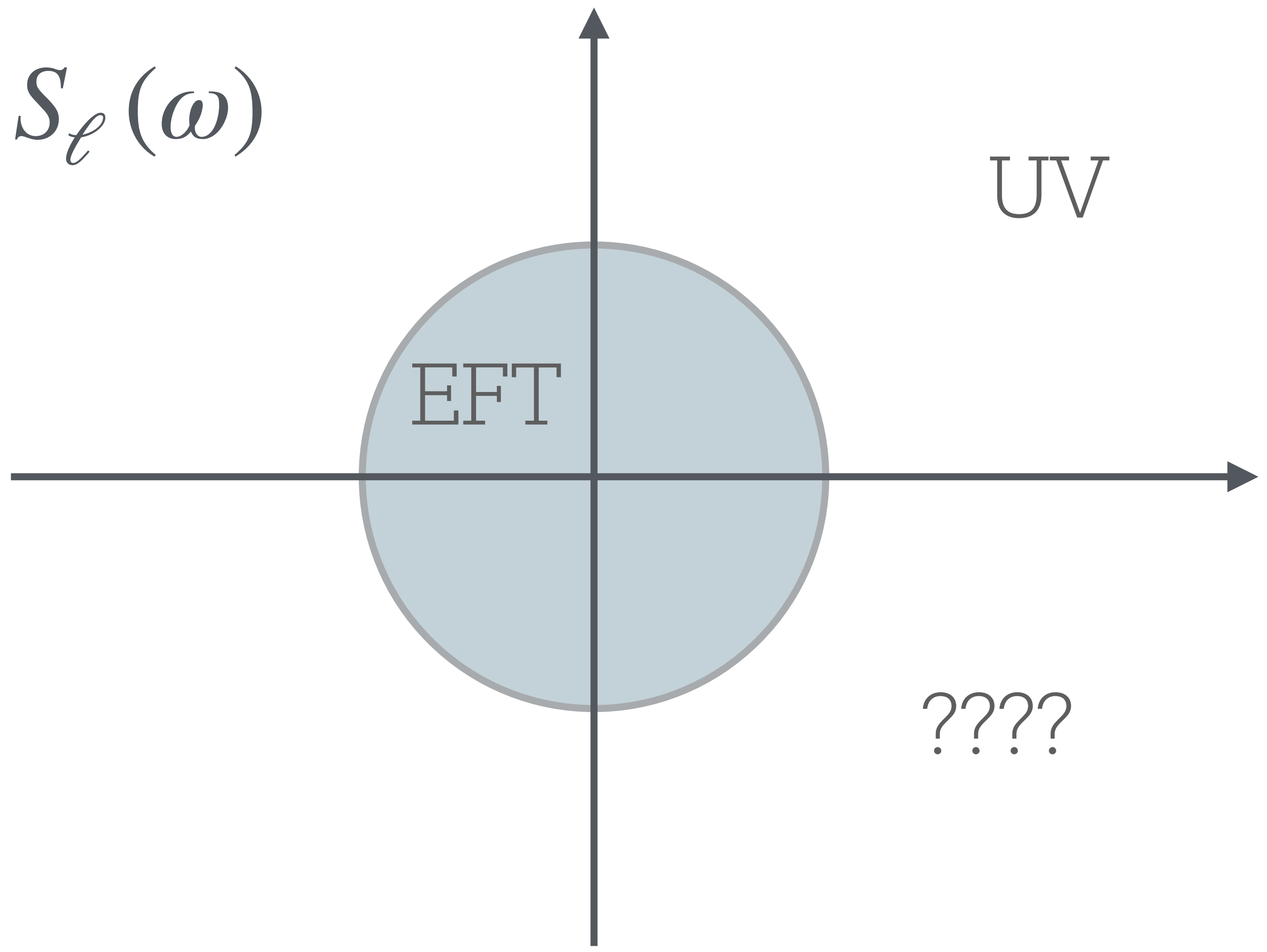
Causality/Unitarity?

What is the analytic structure of $S_\ell(\omega)$?

Motivation

Applications

- Positivity bounds on Love numbers



Theoretical

- Poles/Branch cuts capture the response of the black hole

- S-matrix in an open system/ background breaking translations

Causality/Unitarity?

What is the analytic structure of $S_\ell(\omega)$?

Setup

Regge-Wheeler : $\partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \phi \right) = 0$

Tortoise coordinates: $x = r + R_s \log \left(\frac{r}{R_s} - 1 \right)$

Setup

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$$\left[\frac{d^2}{dx^2} + \omega^2 - V(x) \right] \phi(\omega, x) = 0$$

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$x \to -\infty$

$x \to +\infty$

Horizon

Spatial infinity

$\phi_L \sim e^{-i\omega x}$

$\phi_R \sim e^{i\omega x}$

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$x \to +\infty$

Horizon

Spatial infinity

$\phi_L \sim e^{-i\omega x}$

1+1d problem

$\phi_R \sim e^{i\omega x}$

Relation S-matrix and retarded Green's function

$\phi_L(\omega, x \to -\infty) = e^{-i\omega x}$

$\phi_R(\omega, x \to -\infty) = e^{i\omega x}$

Relation S-matrix and retarded Green's function

S-matrix

(or reflection coefficient)

$$\phi_L(\omega, x \to +\infty) = A_{\text{in}}(\omega)\, e^{-i\omega x} + A_{\text{out}}(\omega)\, e^{i\omega x}$$

$$S_\ell(\omega) = -\frac{A_{\text{out}}(\omega)}{A_{\text{in}}(\omega)}$$

$$\phi_L(\omega, x \to -\infty) = e^{-i\omega x}$$

$$\phi_R(\omega, x \to -\infty) = e^{i\omega x}$$

Relation S-matrix and retarded Green's function

S-matrix
(or reflection coefficient)

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$$S_\ell(\omega) = -\frac{A_{\text{out}}(\omega)}{A_{\text{in}}(\omega)}$$

Retarded Green's function

$$\left[\frac{d^2}{dx^2} + \omega^2 - V(x)\right] G_R(\omega, x) = \delta(x - x')$$

$$G_R(\omega, x, x') = \frac{1}{2i\omega A_{\text{in}}}\left[\phi_L(x)\,\phi_R(x')\,\theta(x' - x) + \phi_L(x')\,\phi_R(x)\,\theta(x - x')\right]$$

$$\phi_L(\omega, x \to -\infty) = e^{-i\omega x}$$

$$\phi_R(\omega, x \to -\infty) = e^{i\omega x}$$

Relation S-matrix and retarded Green's function

S-matrix
(or reflection coefficient)

$$\phi_L(\omega, x \to +\infty) = A_{\text{in}}(\omega)\, e^{-i\omega x} + A_{\text{out}}(\omega)\, e^{i\omega x}$$

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Retarded Green's function

$$\left[\frac{d^2}{dx^2} + \omega^2 - V(x)\right] G_R(\omega, x) = \delta(x - x')$$

$$G_R(\omega, x, x') = \frac{1}{2i\omega A_{\text{in}}}\left[\phi_L(x)\, \phi_R(x')\, \theta(x' - x) + \phi_L(x')\, \phi_R(x)\, \theta(x - x')\right]$$

$$G_R(\omega, x, x') \xrightarrow{x, x' \to \infty} -\frac{1}{2i\omega}\left[e^{i\omega(x - x')} - S_\ell(\omega)\, e^{i\omega(x + x')}\right]$$

Analyticity for generic potentials

J.R. Taylor (1983)
R.G. Newton (2013)

Warmup: Volterra equations for the wavefunction

Integral solutions

$$\phi_L(\omega, x) = e^{-i\omega x} + \frac{1}{2i\omega}\int_{-\infty}^{x} \left[e^{i\omega(x-x')} - e^{i\omega(x'-x)}\right] \, V(x')\, \phi_L(\omega, x')\, dx'$$

$$\phi_R(\omega, x) = e^{i\omega x} + \frac{1}{2i\omega}\int_{x}^{\infty} \left[e^{i\omega(x-x')} - e^{i\omega(x'-x)}\right] \, V(x')\, \phi_R(\omega, x')\, dx'$$

- Right BCs
- Solutions of wave equation

Warmup: Volterra equations for the wavefunction

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$$\phi_R(\omega,x) = e^{i\omega x} + \frac{1}{2i\omega}\int_{x}^{\infty} \left[e^{i\omega(x-x')} - e^{i\omega(x'-x)}\right]\, V(x')\,\phi_R(\omega,x')\,dx'$$

- Right BCs
- Solutions of wave equation

$$\phi_L(\omega,x) = e^{-i\omega x}\chi_L(\omega,x)$$

$$\phi_R(\omega,x) = e^{i\omega x}\chi_R(\omega,x)$$

Warmup: Volterra equations for the wavefunction

Integral solutions

$$\chi_L(\omega, x) = 1 + \int_{-\infty}^{x} G_0(\omega, x - x')\, V(x')\, \chi_L(\omega, x')\, dx'$$

$$\chi_R(\omega, x) = 1 + \int_{x}^{\infty} G_0(\omega, x - x')\, V(x')\, \chi_R(\omega, x')\, dx'$$

$$G_0(\omega, x - x') = \frac{e^{2i\omega(x - x')} - 1}{2i\omega}$$

- Right BCs
- Solutions of wave equation

$$\phi_L(\omega, x) = e^{-i\omega x} \chi_L(\omega, x)$$

$$\phi_R(\omega, x) = e^{i\omega x} \chi_R(\omega, x)$$

Warmup: Volterra equations for the wavefunction

Integral solutions

$$\chi_L(\omega,x) = 1 + \int_{-\infty}^{x} G_0(\omega, x-x')\, V(x')\, \chi_L(\omega,x')\, dx'$$

$$\chi_R(\omega,x) = 1 + \int_{x}^{\infty} G_0(\omega, x-x')\, V(x')\, \chi_R(\omega,x')\, dx'$$

$$G_0(\omega, x-x') = \frac{e^{2i\omega(x-x')} - 1}{2i\omega}$$

- Right BCs
- Solutions of wave equation

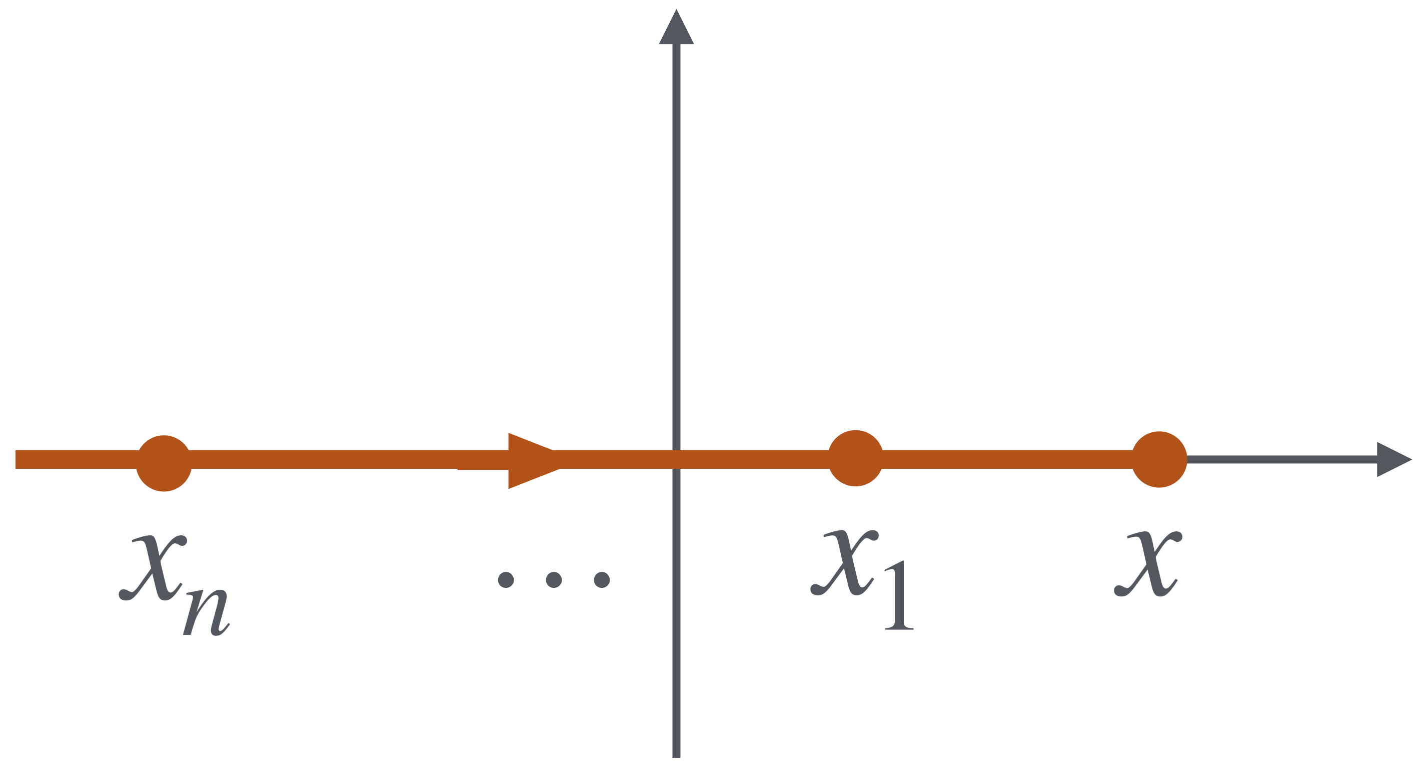
$$\phi_L(\omega,x) = e^{-i\omega x}\chi_L(\omega,x)$$

$$\phi_R(\omega,x) = e^{i\omega x}\chi_R(\omega,x)$$

Solve the Volterra equations iteratively

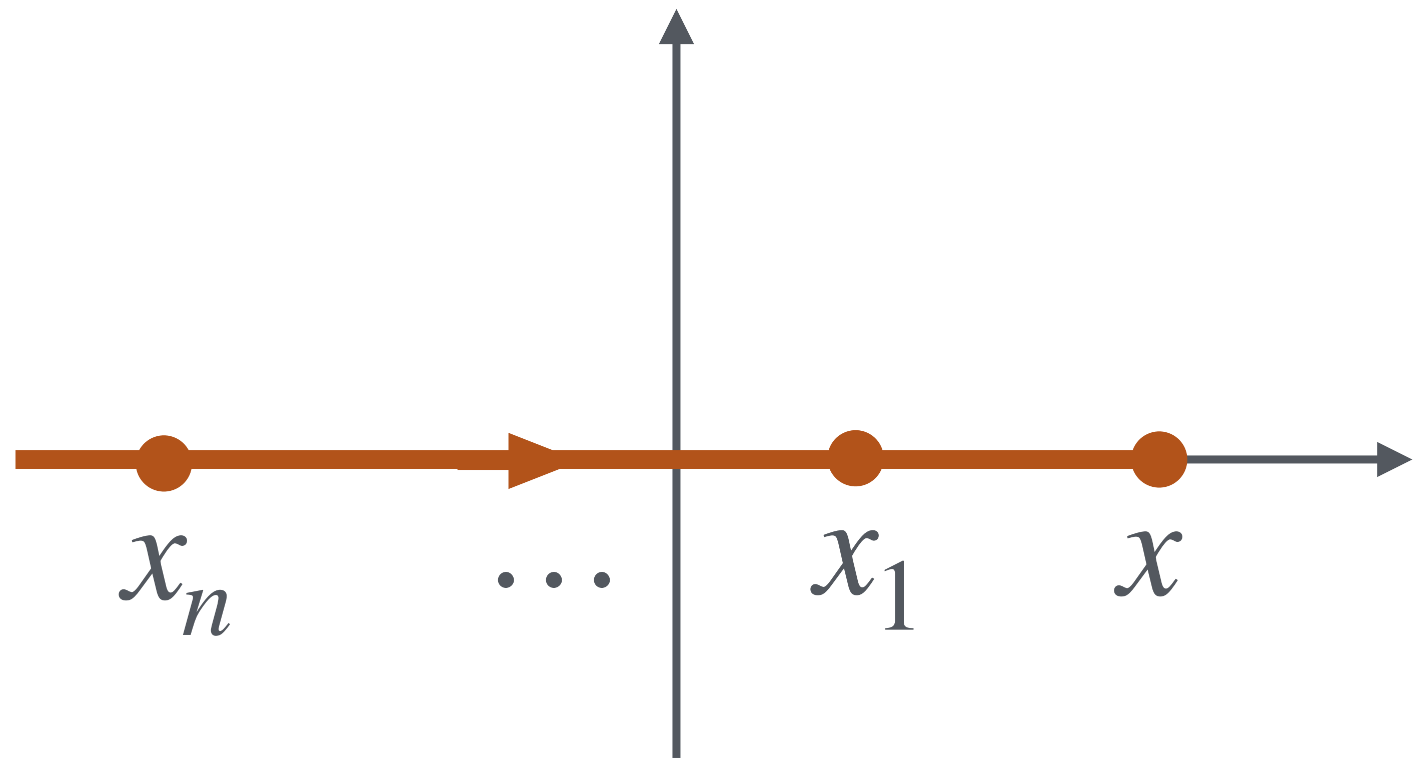
$$\chi_L(\omega,x) = \sum_n \chi_L^{(n)}(\omega,x)$$

Warmup: Volterra equations for the wavefunction



$$\chi_L^{(n)}(\omega, x) = \int_{x > x_1 > \ldots > x_n > -\infty} G_0(\omega, x - x_1) \cdots G_0(\omega, x_{n-1} - x_n)\, V(x_1) \cdots V(x_n)\, dx_1 \cdots dx_n$$

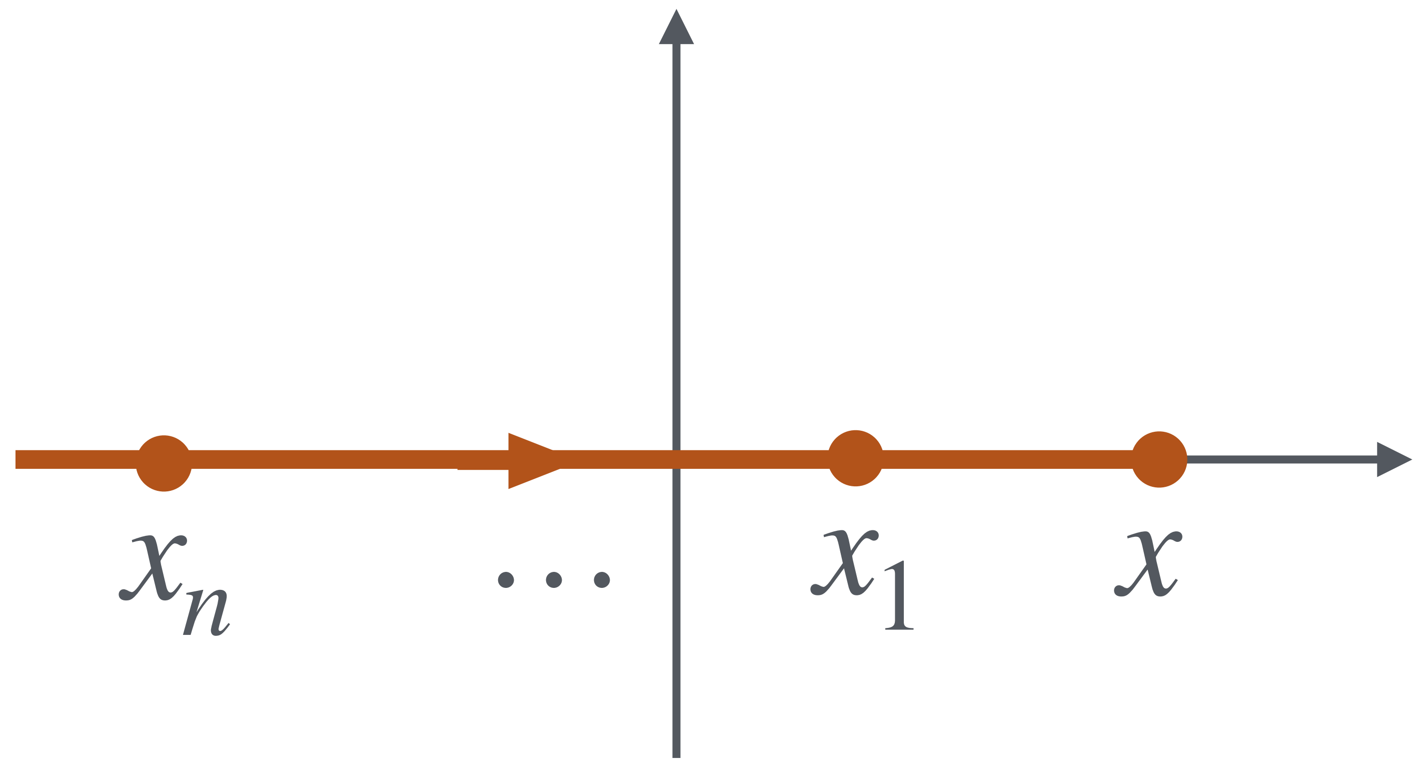
Warmup: Volterra equations for the wavefunction



$$|\chi_L^{(n)}(\omega,x)| \leq \int\limits_{x > x_1 > \ldots > x_n > -\infty} |G_0(\omega,x-x_1)| \cdots |G_0(\omega,x_{n-1}-x_n)|\,|V(x_1)| \cdots |V(x_n)|\,dx_1 \cdots dx_n$$

$$|G_0(\omega,x-x')| = \left|\frac{e^{2i\omega\overbrace{(x-x')}^{<0}}-1}{2i\omega}\right|$$

Warmup: Volterra equations for the wavefunction

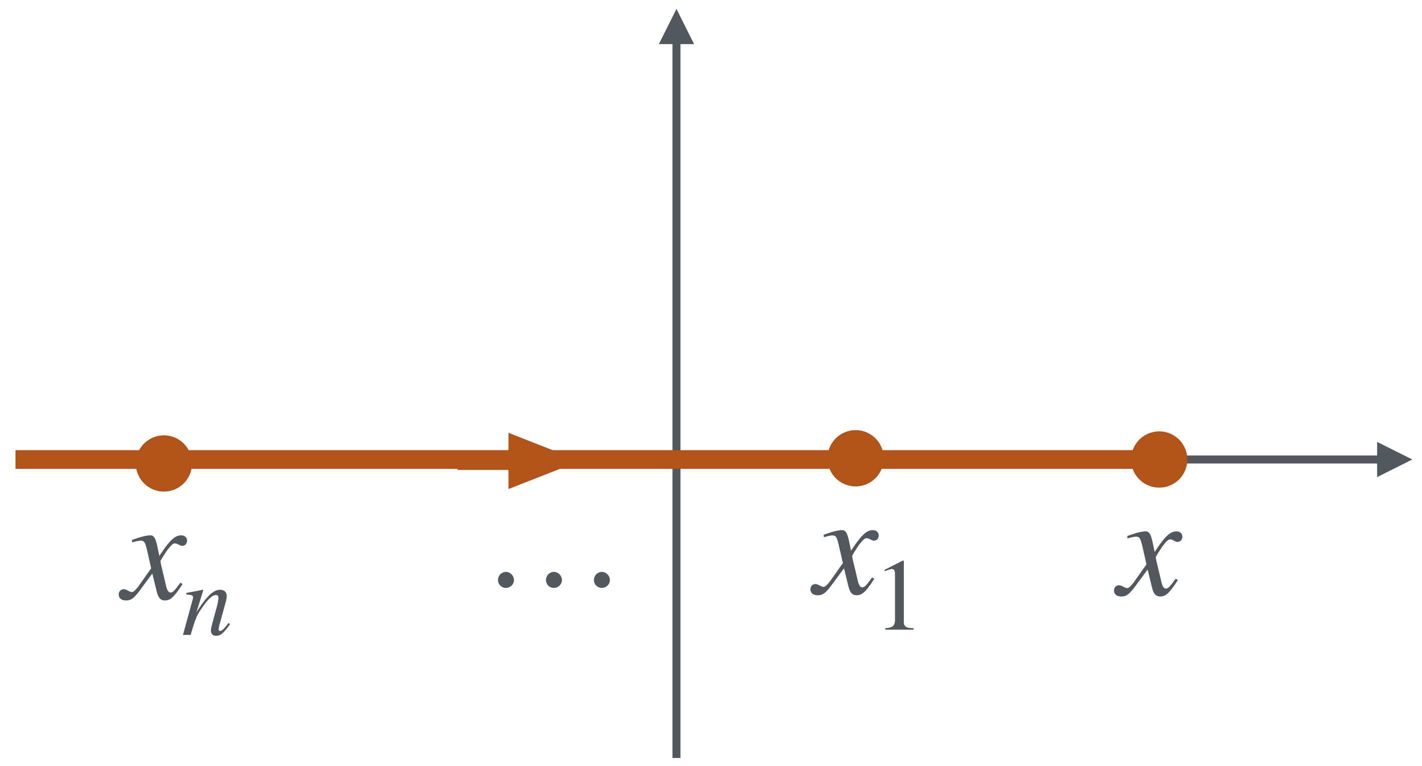


$$|\chi_L^{(n)}(\omega, x)| \leq \int_{x > x_1 > \ldots > x_n > -\infty} |G_0(\omega, x - x_1)| \cdots |G_0(\omega, x_{n-1} - x_n)| \, |V(x_1)| \cdots |V(x_n)| \, dx_1 \cdots dx_n$$

$$|G_0(\omega, x - x')| = \left| \frac{e^{2i\omega\overbrace{(x-x')}^{<0}} - 1}{2i\omega} \right| < \frac{1}{|\omega|}$$

$\mathrm{Im}\, \omega > 0$

Warmup: Volterra equations for the wavefunction

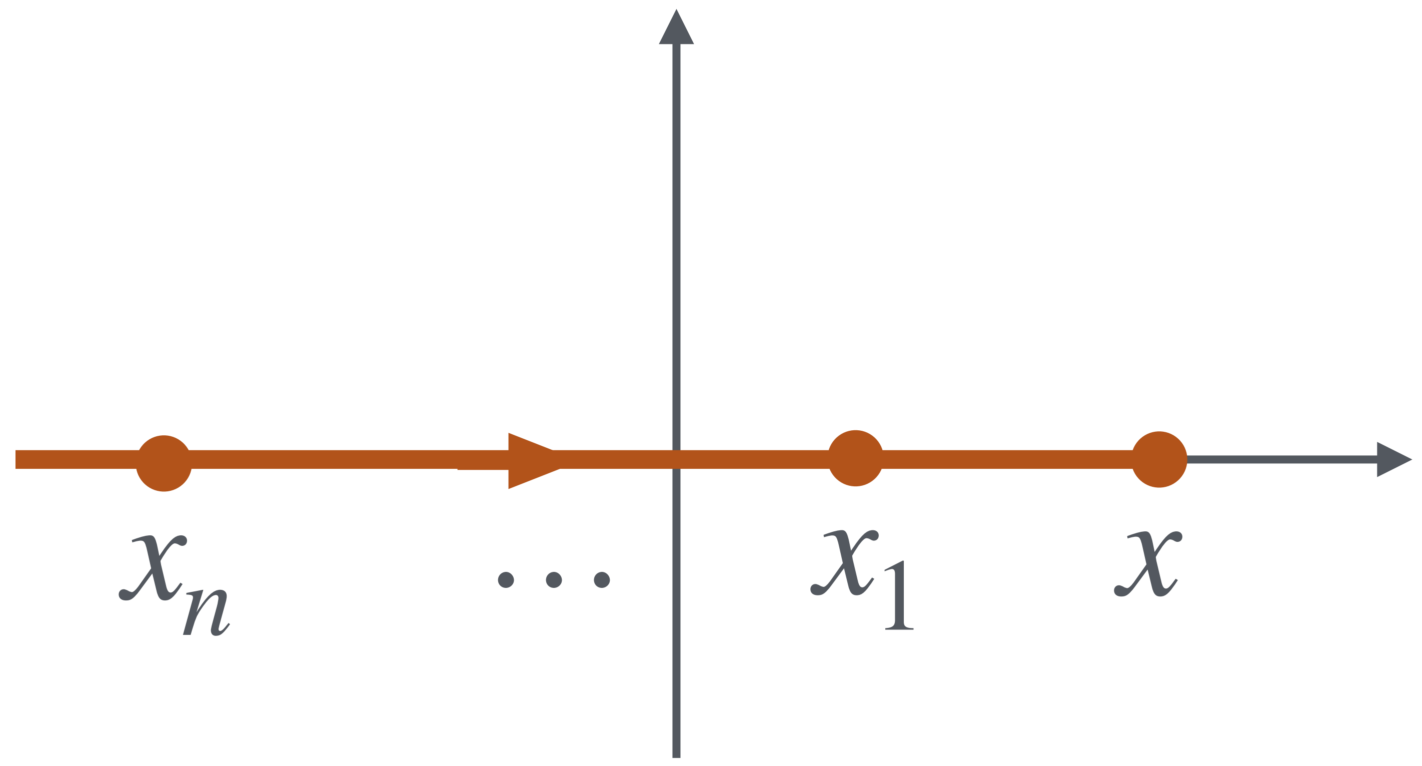


$$|\chi_L^{(n)}(\omega, x)| \leq \int_{x > x_1 > \ldots > x_n > -\infty} |G_0(\omega, x - x_1)| \cdots |G_0(\omega, x_{n-1} - x_n)| \, |V(x_1)| \cdots |V(x_n)| \, dx_1 \cdots dx_n$$

$$\leq \frac{1}{|\omega|^n} \int_{x > x_1 > \ldots > x_n > -\infty} |V(x_1)| \cdots |V(x_n)| \, dx_1 \cdots dx_n$$

$\operatorname{Im}\omega > 0$

Warmup: Volterra equations for the wavefunction

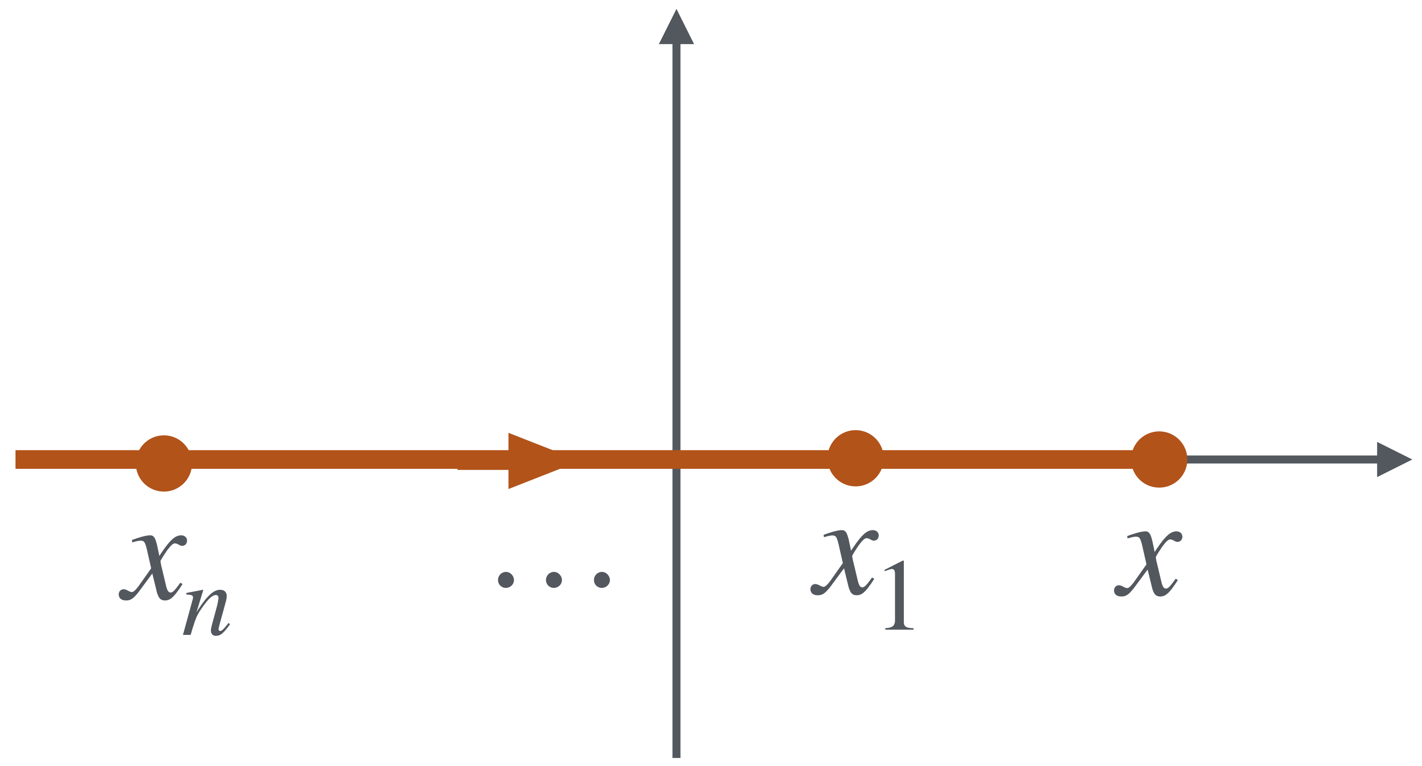


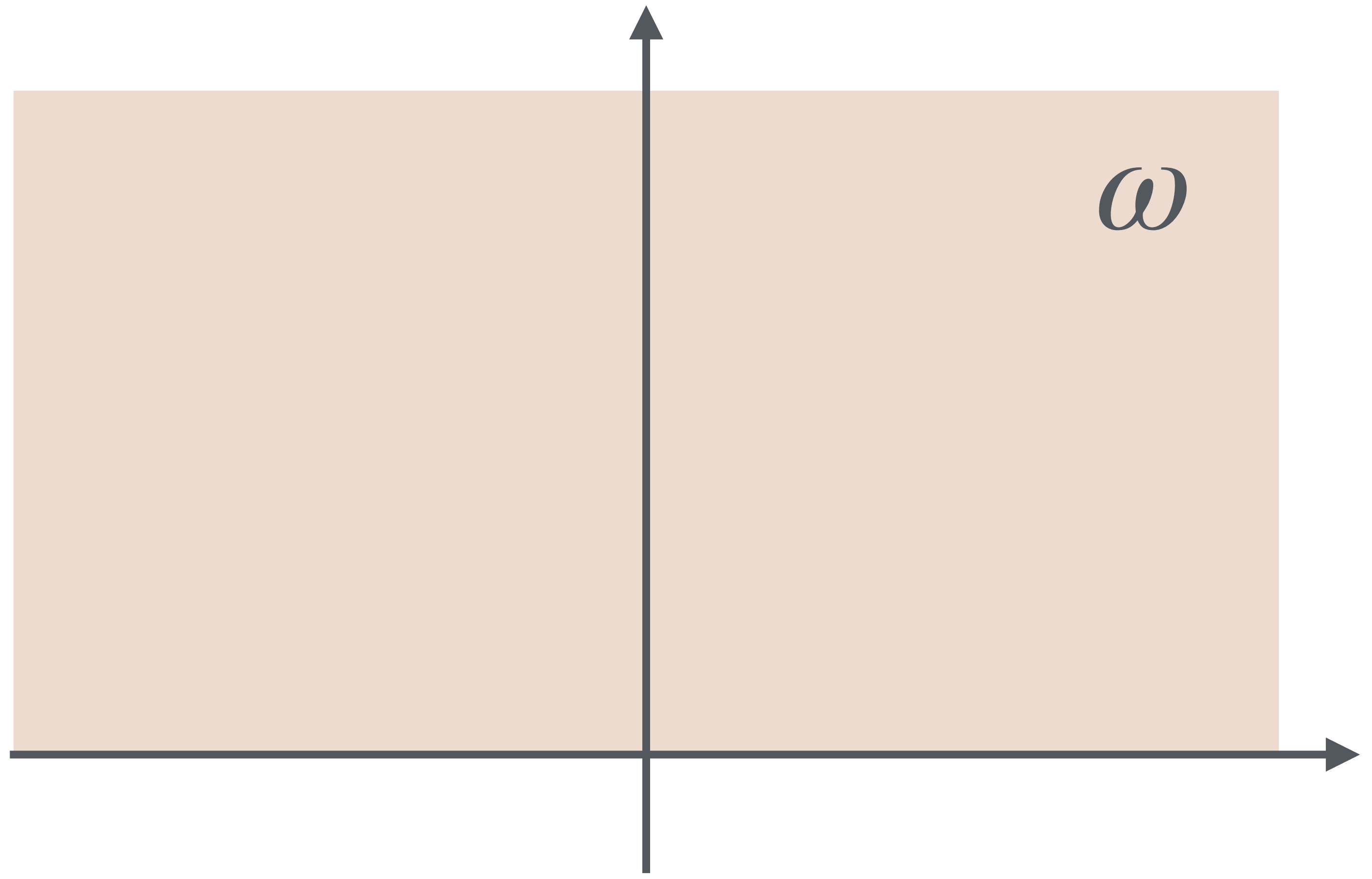
$$|\chi_L^{(n)}(\omega,x)| \le \int_{x > x_1 > \ldots > x_n > -\infty} |G_0(\omega, x - x_1)| \cdots |G_0(\omega, x_{n-1} - x_n)|\, |V(x_1)| \cdots |V(x_n)|\, dx_1 \cdots dx_n$$

$$\le \frac{1}{|\omega|^n} \int_{x > x_1 > \ldots > x_n > -\infty} |V(x_1)| \cdots |V(x_n)|\, dx_1 \cdots dx_n = \frac{1}{n!} \left[\frac{1}{|\omega|} \int_{-\infty}^{x} |V(x_1)|\, dx_1 \right]^n$$

$\mathrm{Im}\, \omega > 0$

Warmup: Volterra equations for the wavefunction



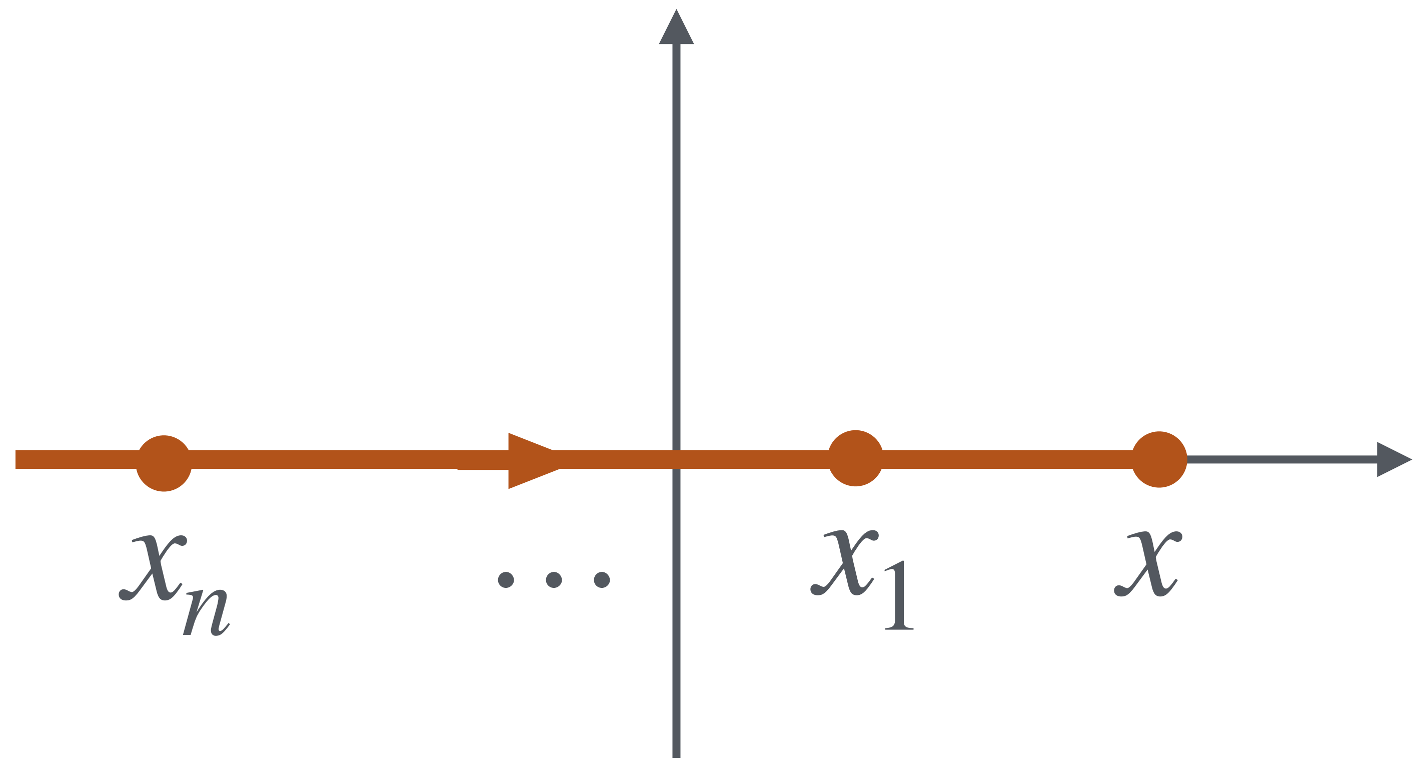


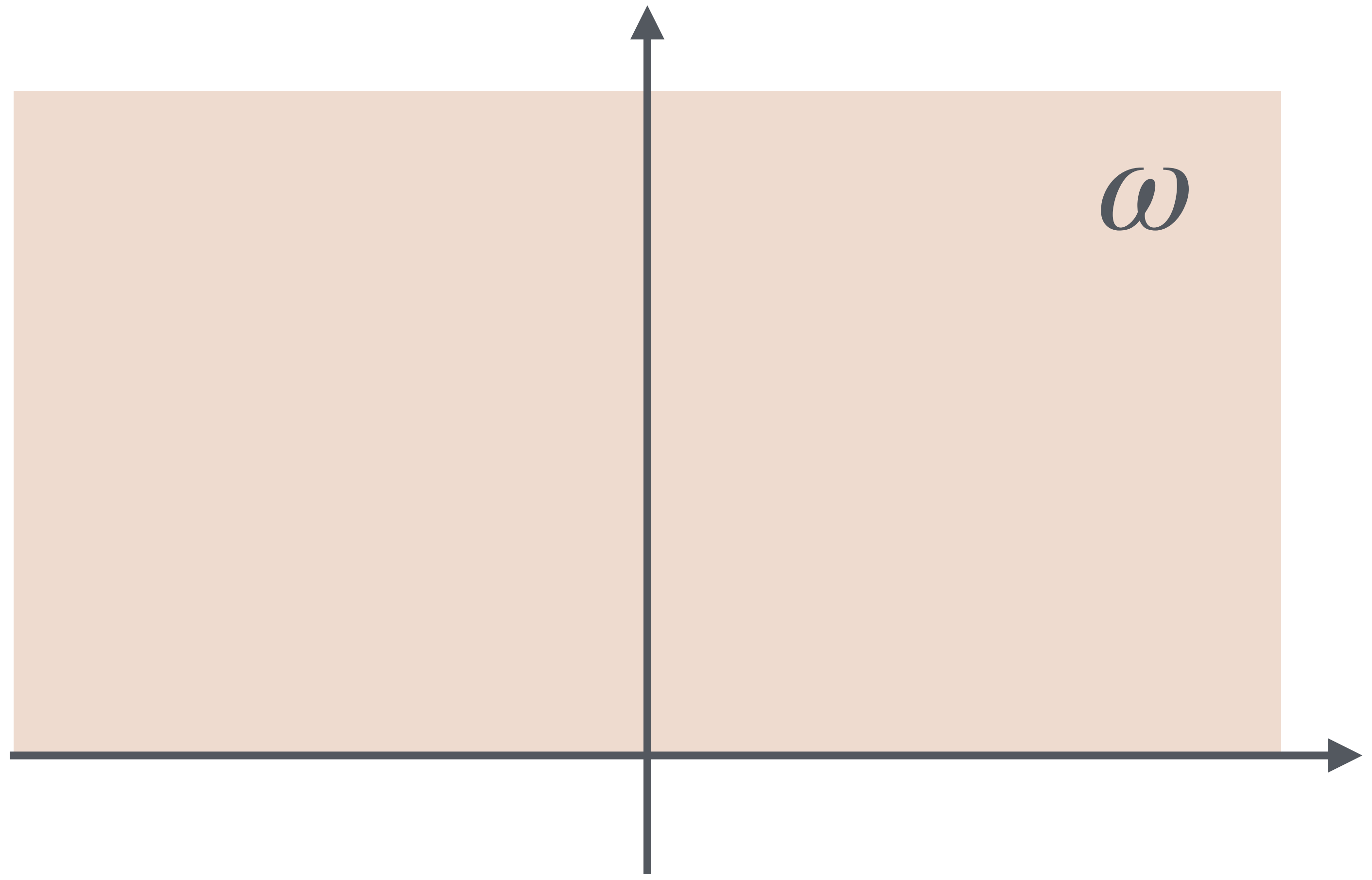
uniform convergence

$$|\chi_L(\omega,x)| = |\sum_n \chi_L^{(n)}(\omega,x)| \leq e^{\frac{1}{|\omega|}\int_{-\infty}^{x}|V(x')|dx'}$$

$\chi_L(\omega,x)$ is analytic in the UHP

Warmup: Volterra equations for the wavefunction





$$|\chi_L(\omega, x)| = |\sum_n \chi_L^{(n)}(\omega, x)| \leq e^{\frac{1}{|\omega|}\int_{-\infty}^{x} |V(x')| dx'} \xrightarrow{\text{uniform convergence}}$$

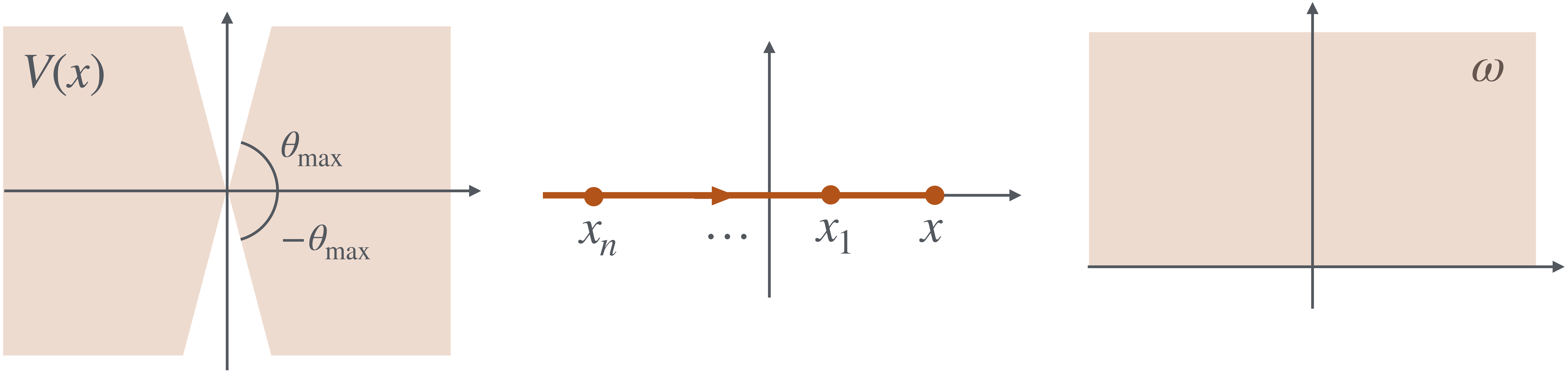
$\chi_L(\omega, x)$ is analytic in the UHP

$G_R(\omega, x, x')$ is analytic in the UHP

Simplest incarnation of causality

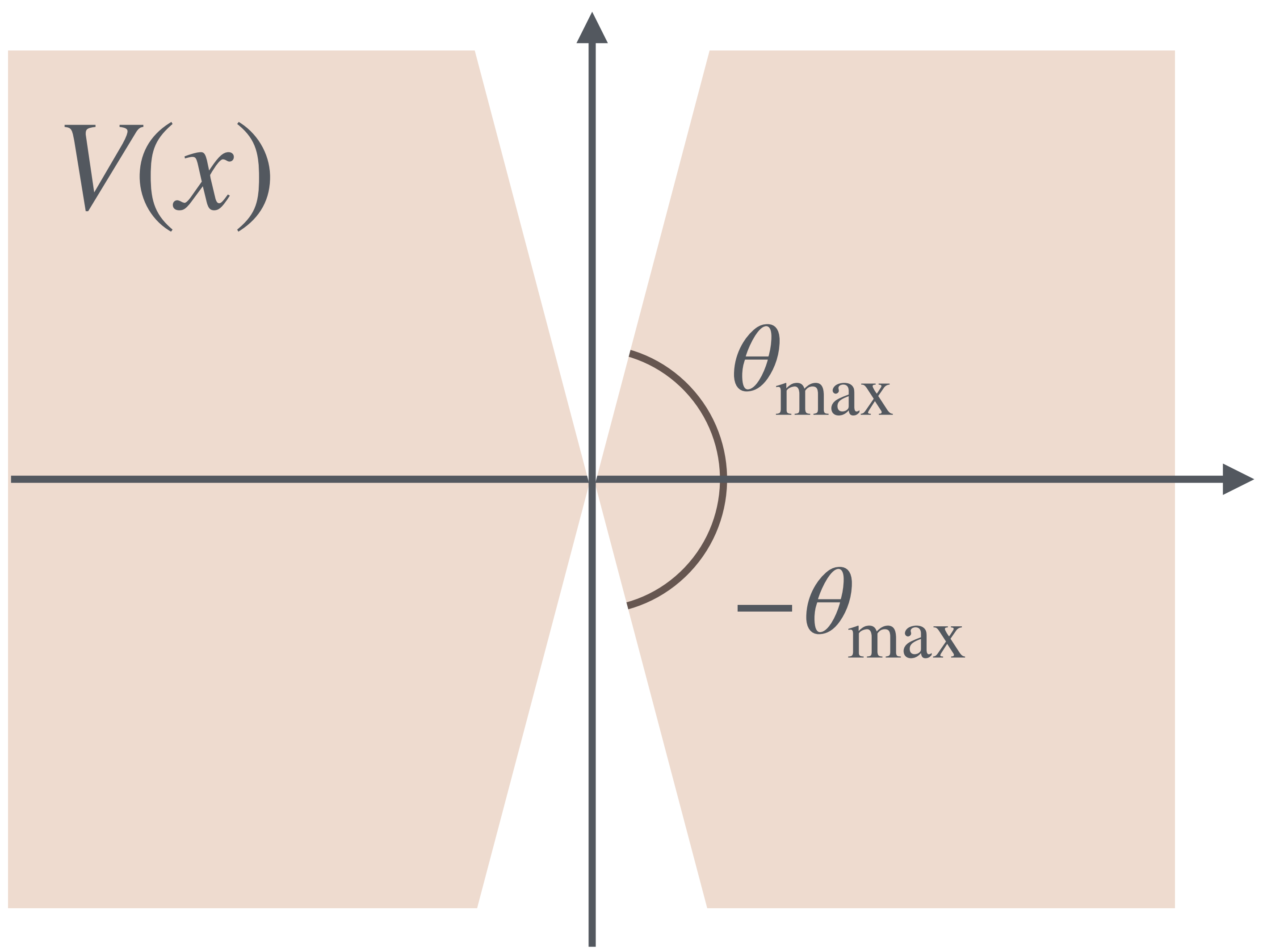
$$G_R(t - t', x) = 0\,, \qquad t - t' < 0$$

More analyticity from the potential



$$V(x \to \infty) \lesssim \frac{1}{|x|}$$

More analyticity from the potential



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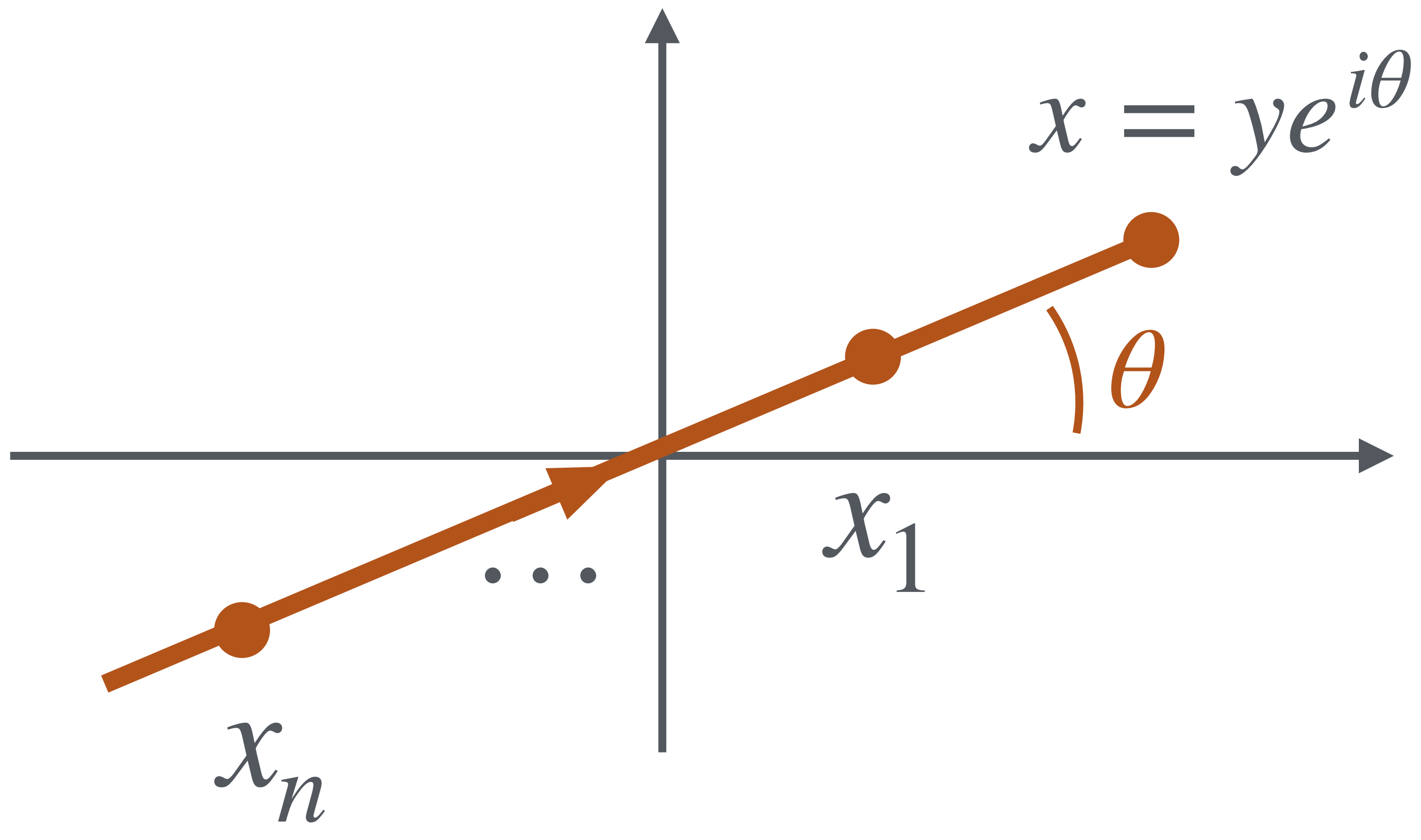
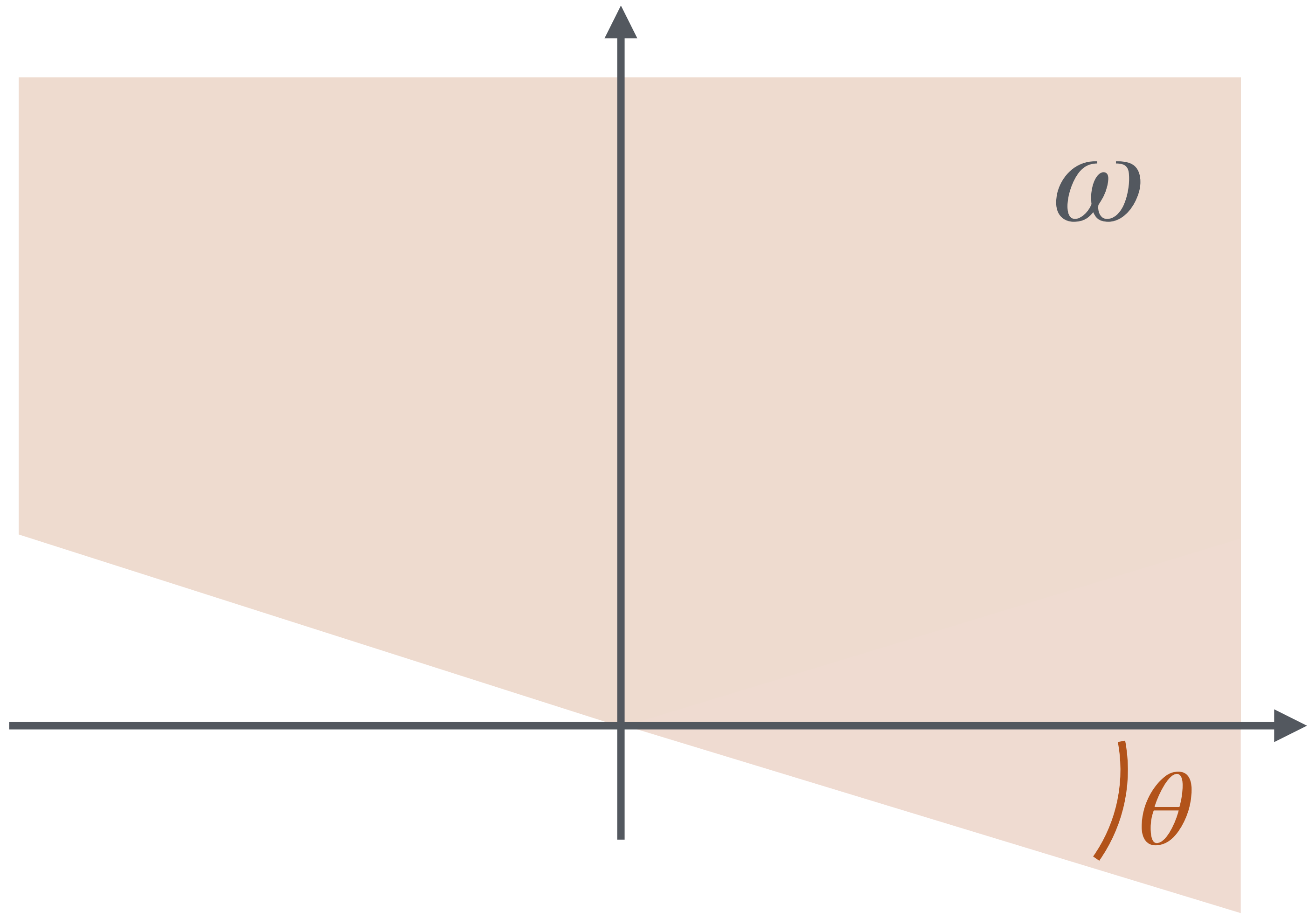
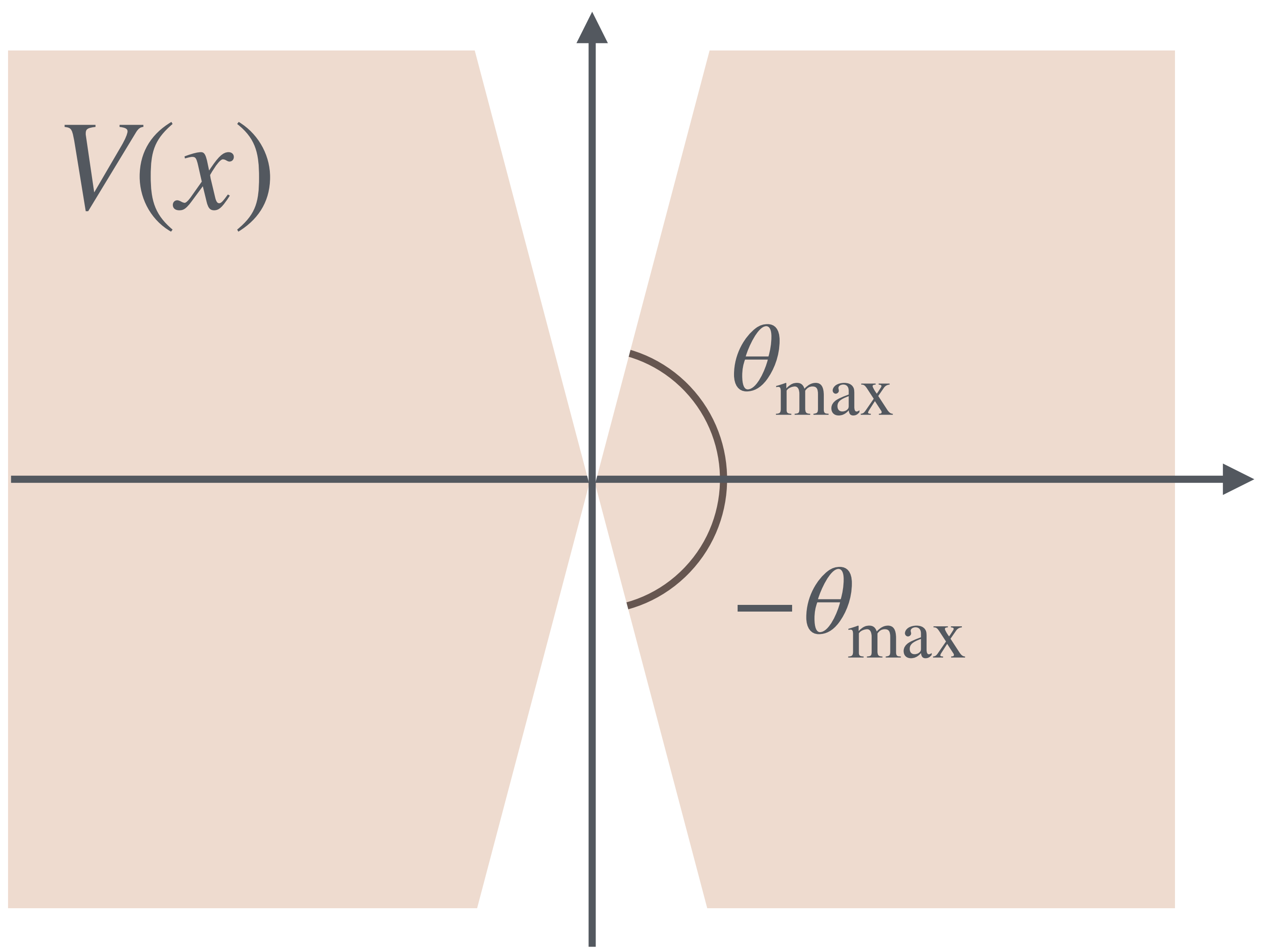




$$|\chi_L(\omega, ye^{i\theta})| = |\sum_n \chi_L^{(n)}(\omega, ye^{i\theta})| \leq e^{\frac{1}{|\omega|}\int_{-\infty}^{y}|V(e^{i\theta}y')|dy'}$$

$$\mathrm{Im}\, e^{i\theta}\omega > 0$$

More analyticity from the potential



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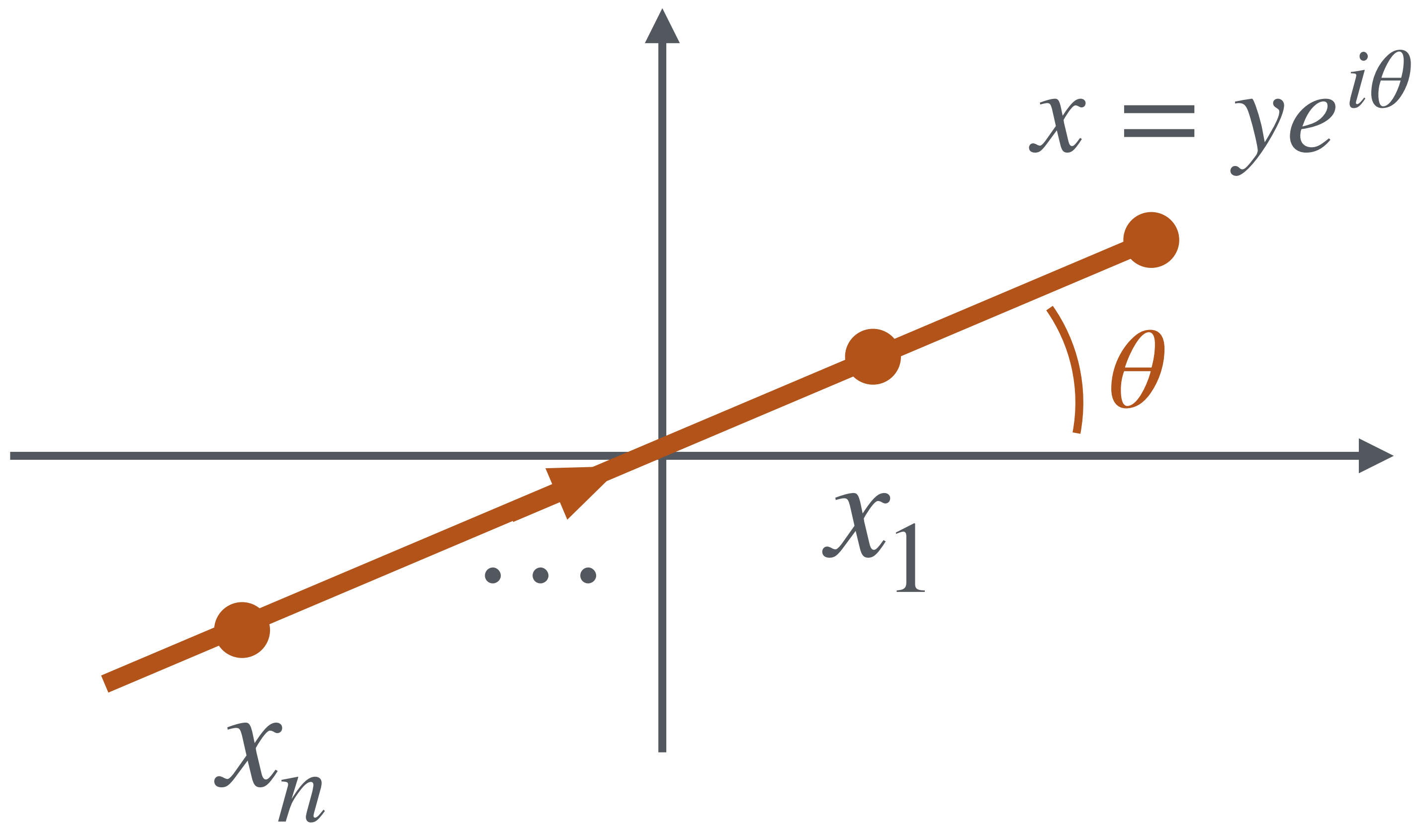

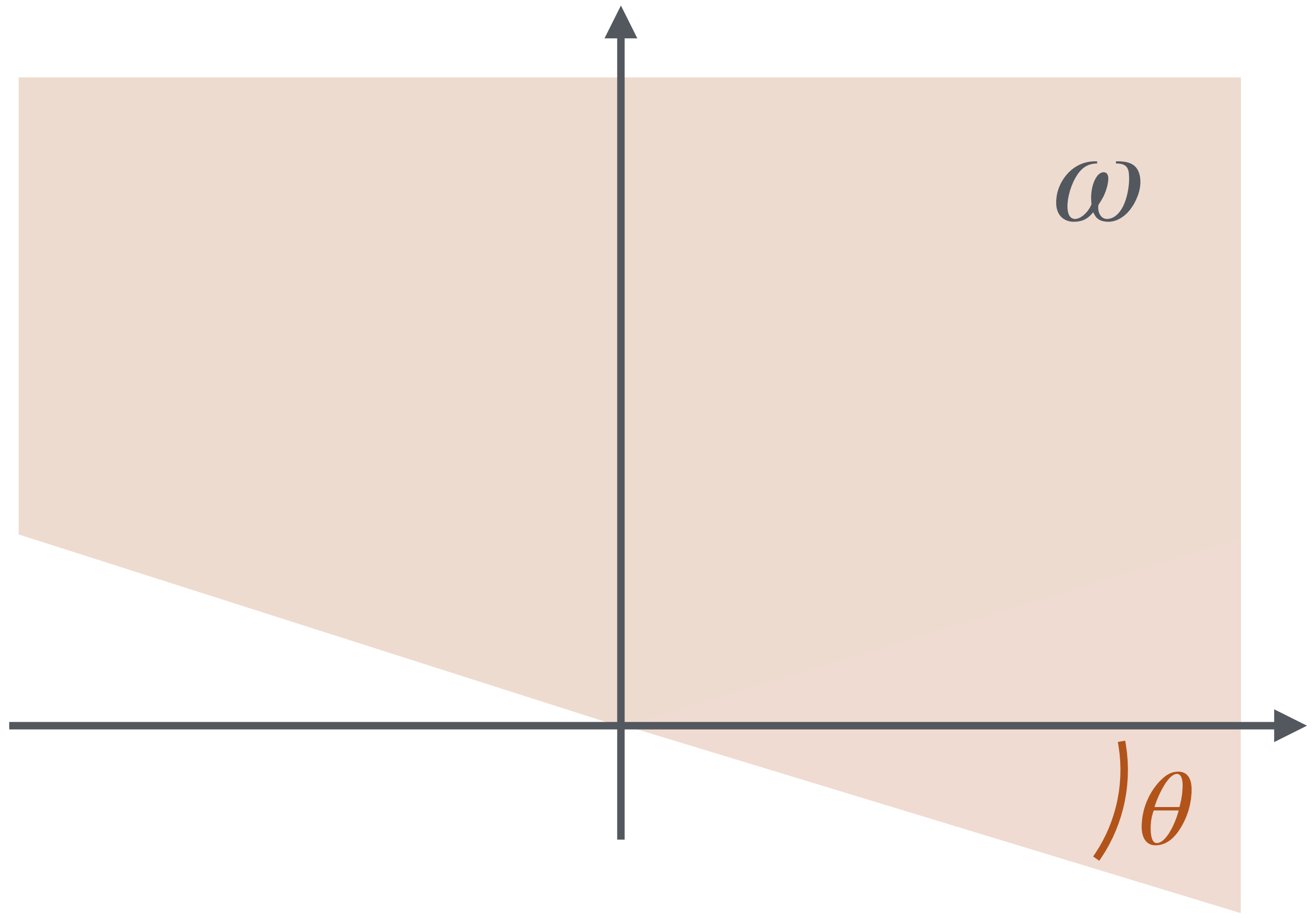

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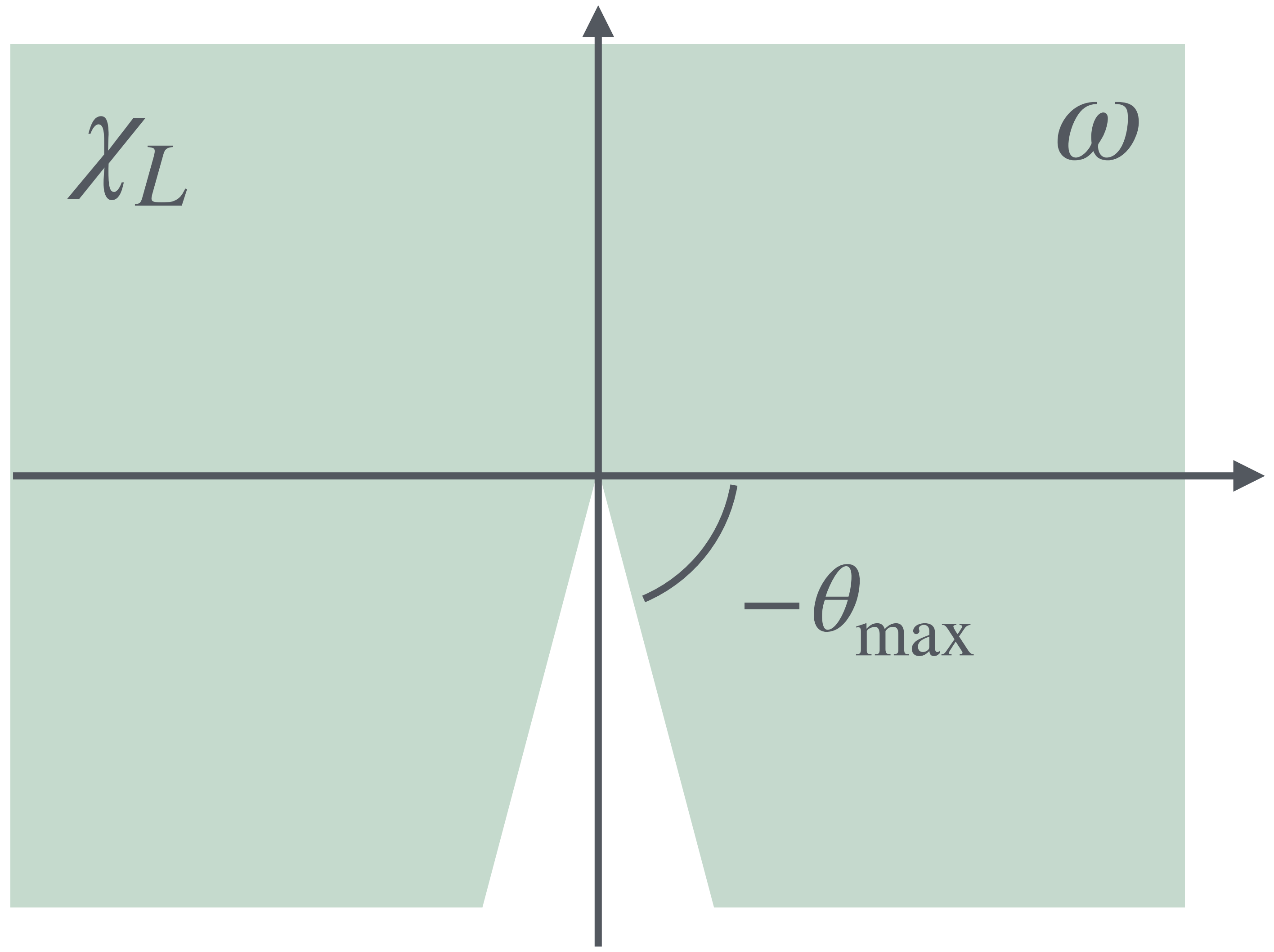

Analyticity of connection coefficients

$$\phi_L(\omega, x) = e^{-i\omega x} + \frac{1}{2i\omega}\int_{-\infty}^{x} \left[e^{i\omega(x-x')} - e^{i\omega(x'-x)}\right] V(x')\,\phi_L(\omega, x')\,dx'$$

$$x \to \infty$$

$$A_{\rm in}(\omega) = 1 - \frac{1}{2i\omega}\int_{-\infty}^{\infty} V(x')\,\chi_L(\omega, x')\,dx'$$

$$A_{\rm out}(\omega) = \frac{1}{2i\omega}\int_{-\infty}^{\infty} e^{-2i\omega x'} V(x')\,\chi_L(\omega, x')\,dx'$$

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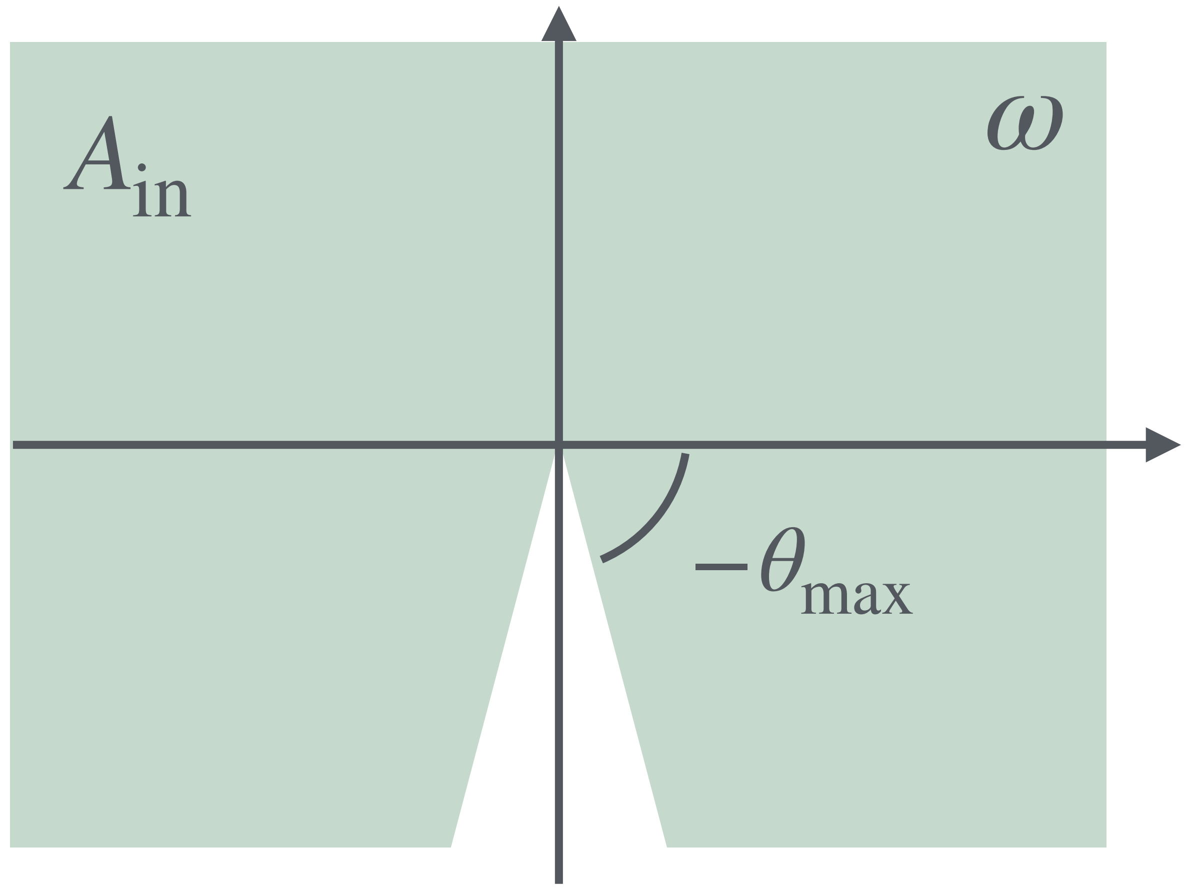

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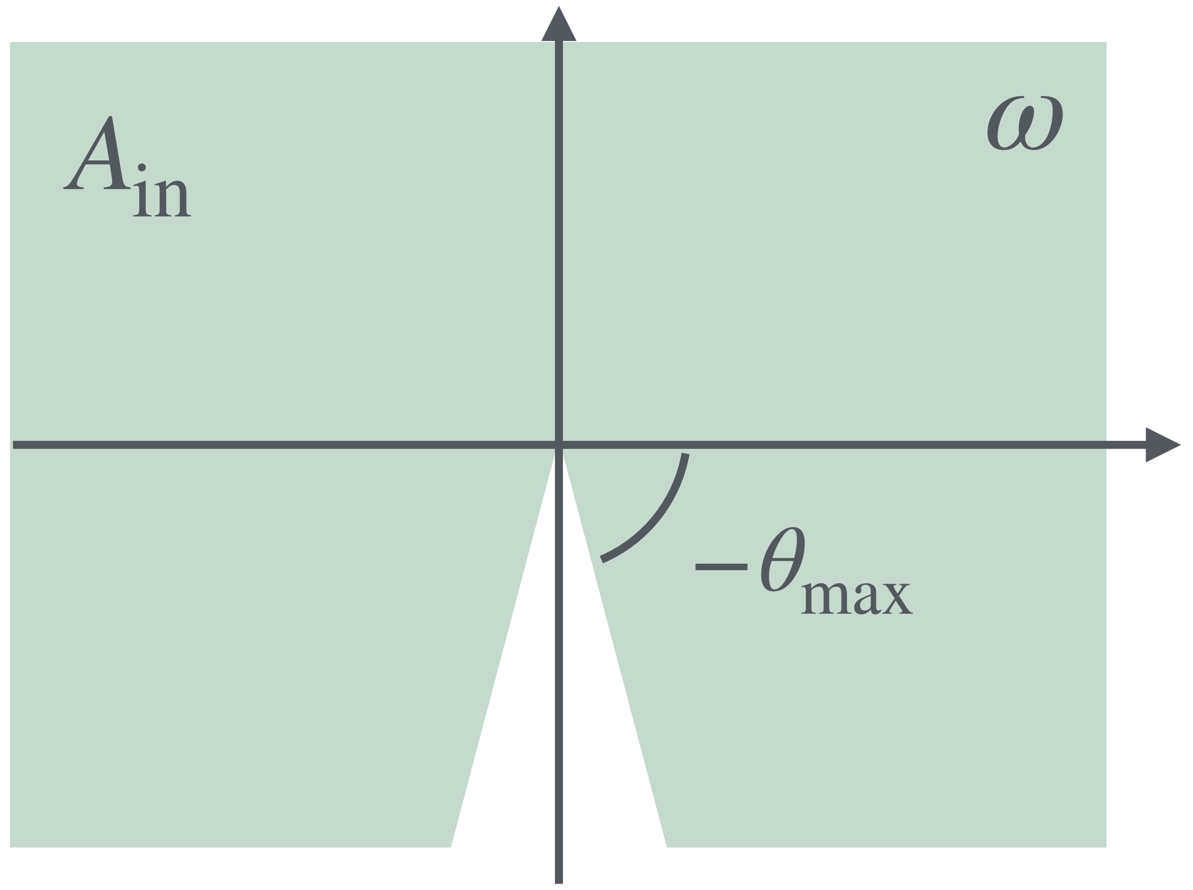

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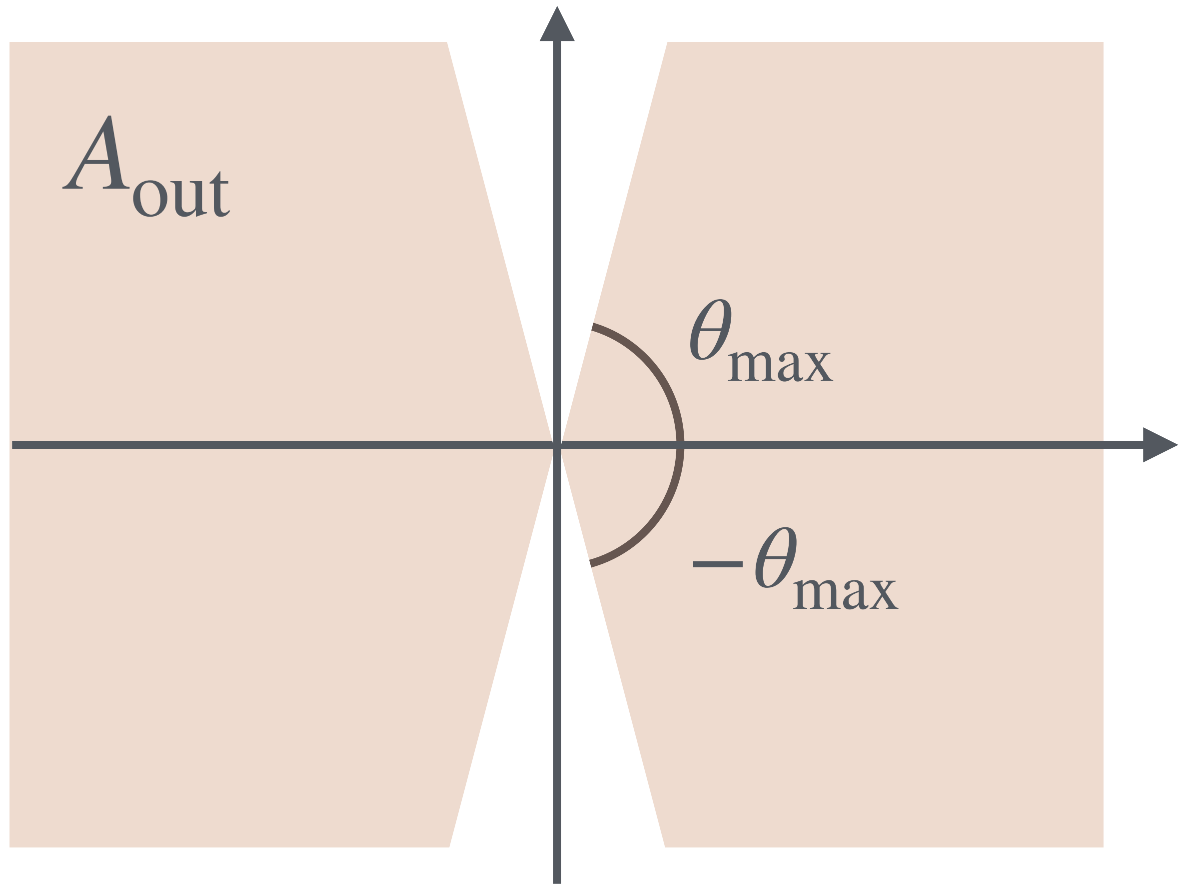
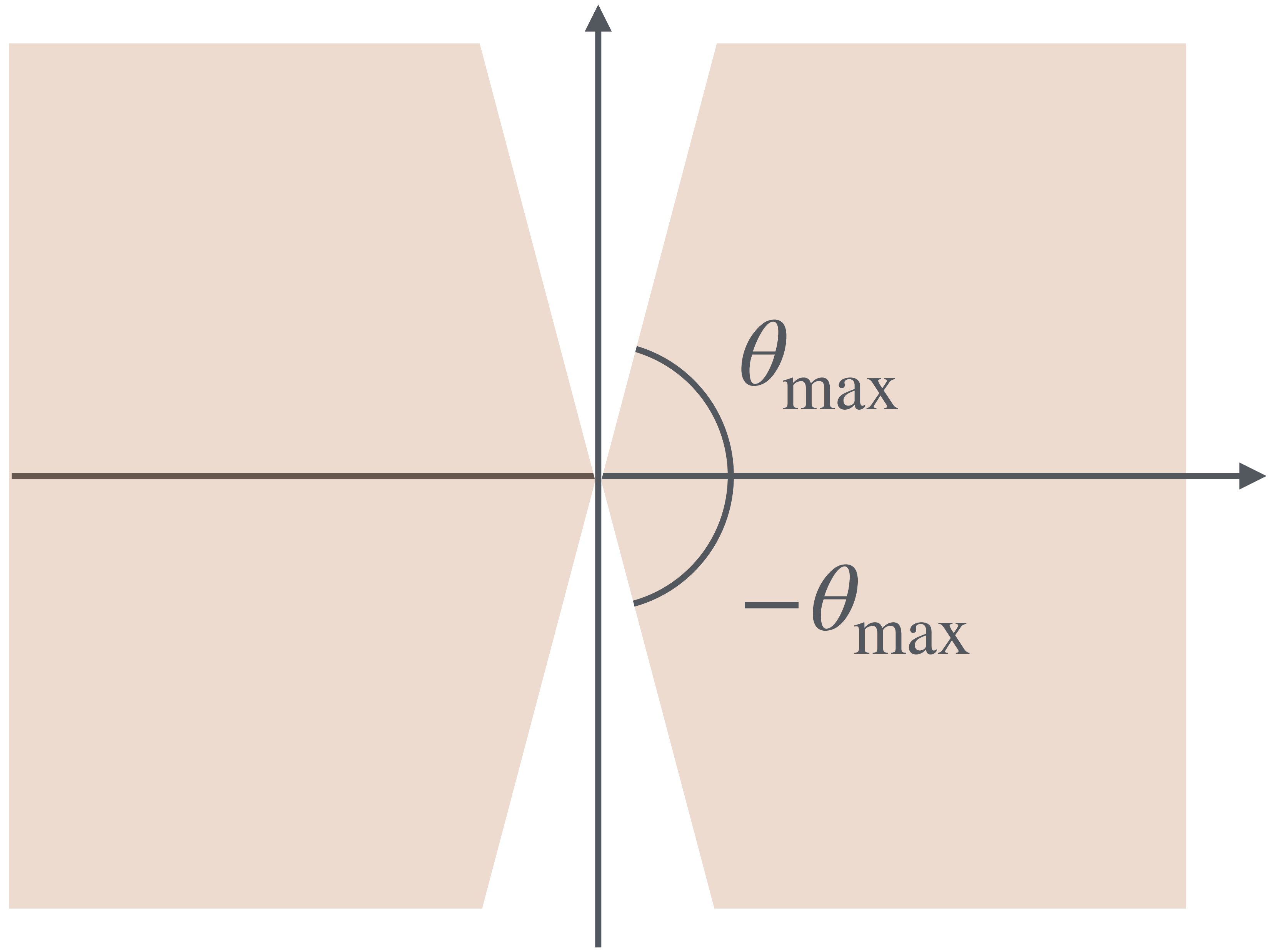
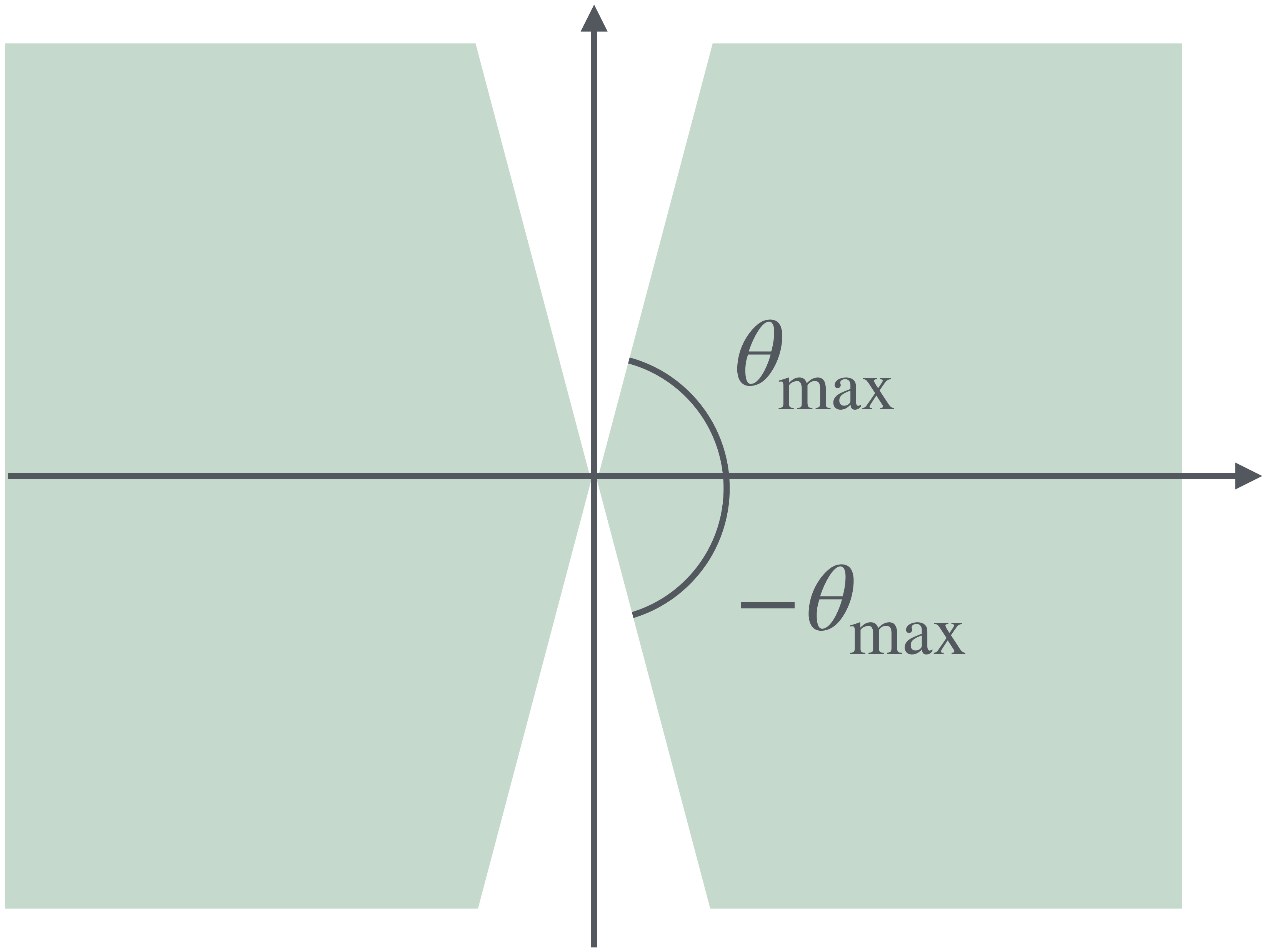


Analyticity of the S-matrix

$$V(x \to \infty) \lesssim \frac{1}{|x|}$$

$$S_\ell(\omega) = -\frac{A_{\rm out}(\omega)}{A_{\rm in}(\omega)}$$





Meromorphic

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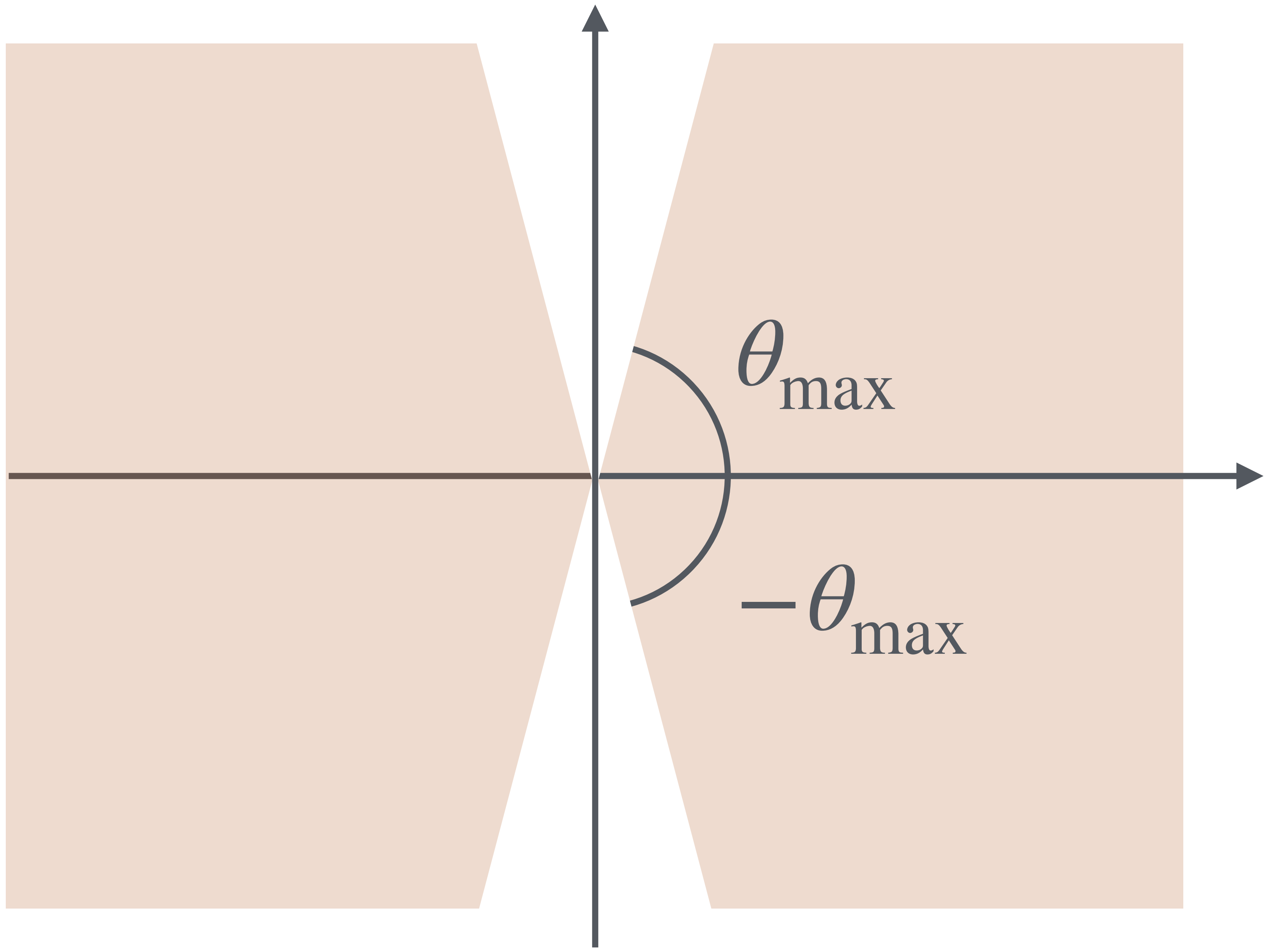

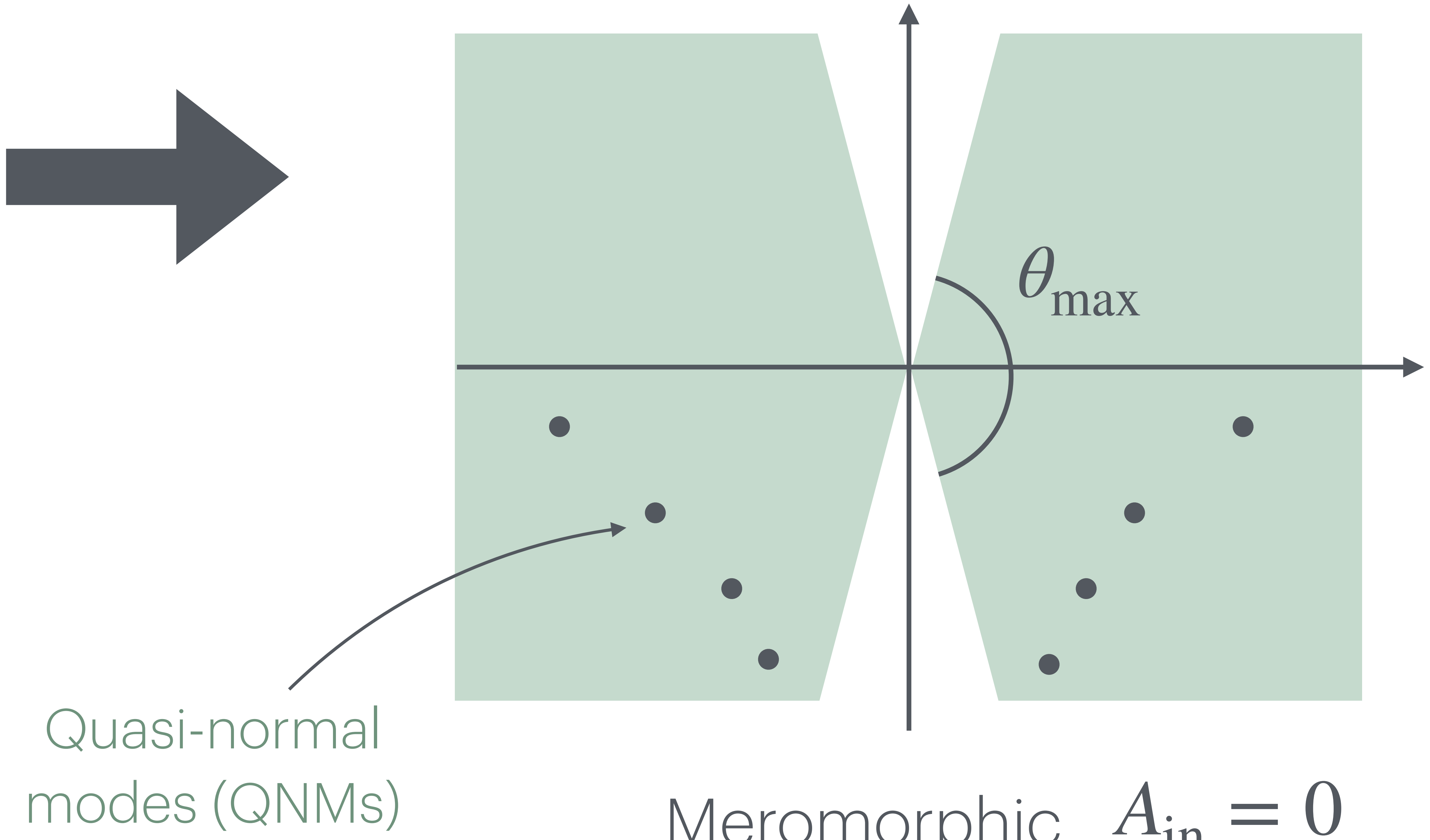

Meromorphic $A_{\mathrm{in}} = 0$

Analyticity of the Black Hole S-matrix

The Schwarzschild potential

Regge-Wheeler : $\partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \phi \right) = 0$

Tortoise coordinates: $x = r + R_s \log\left(\frac{r}{R_s} - 1 \right)$

$$\left[\frac{d^2}{dx^2} + \omega^2 - V(x) \right] \phi(\omega, x) = 0$$

Angular momentum

$$V\left(r(x)\right) = \left(1 - \frac{R_s}{r}\right) \left(\frac{\ell(\ell+1)}{r^2} + \frac{R_s}{r^3}(1 - s^2) \right)$$

Spin of the scattered wave

The Schwarzschild potential

$$V\left(r(x)\right) = \left(1 - \frac{R_s}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + \frac{R_s}{r^3}(1-s^2)\right)$$

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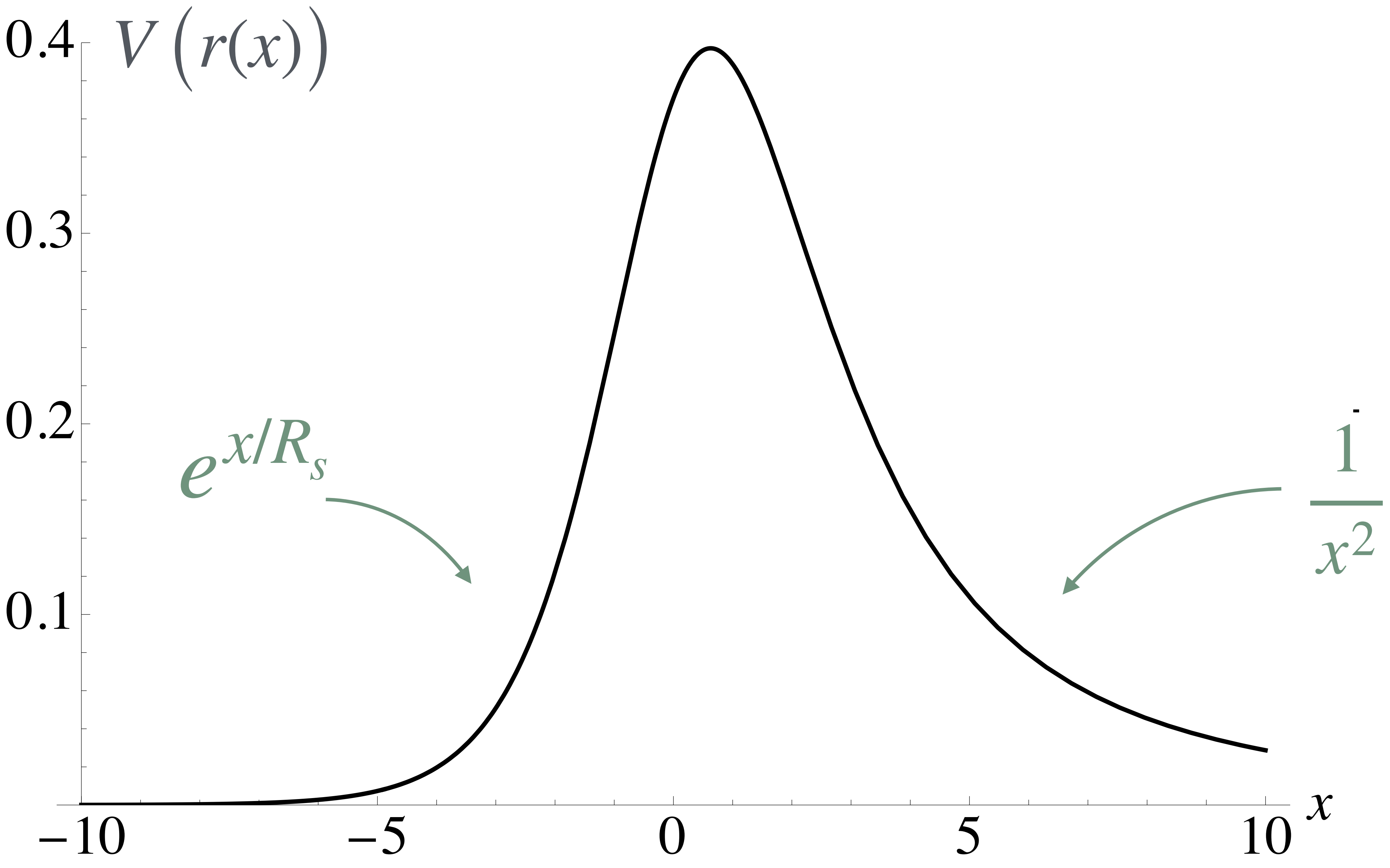

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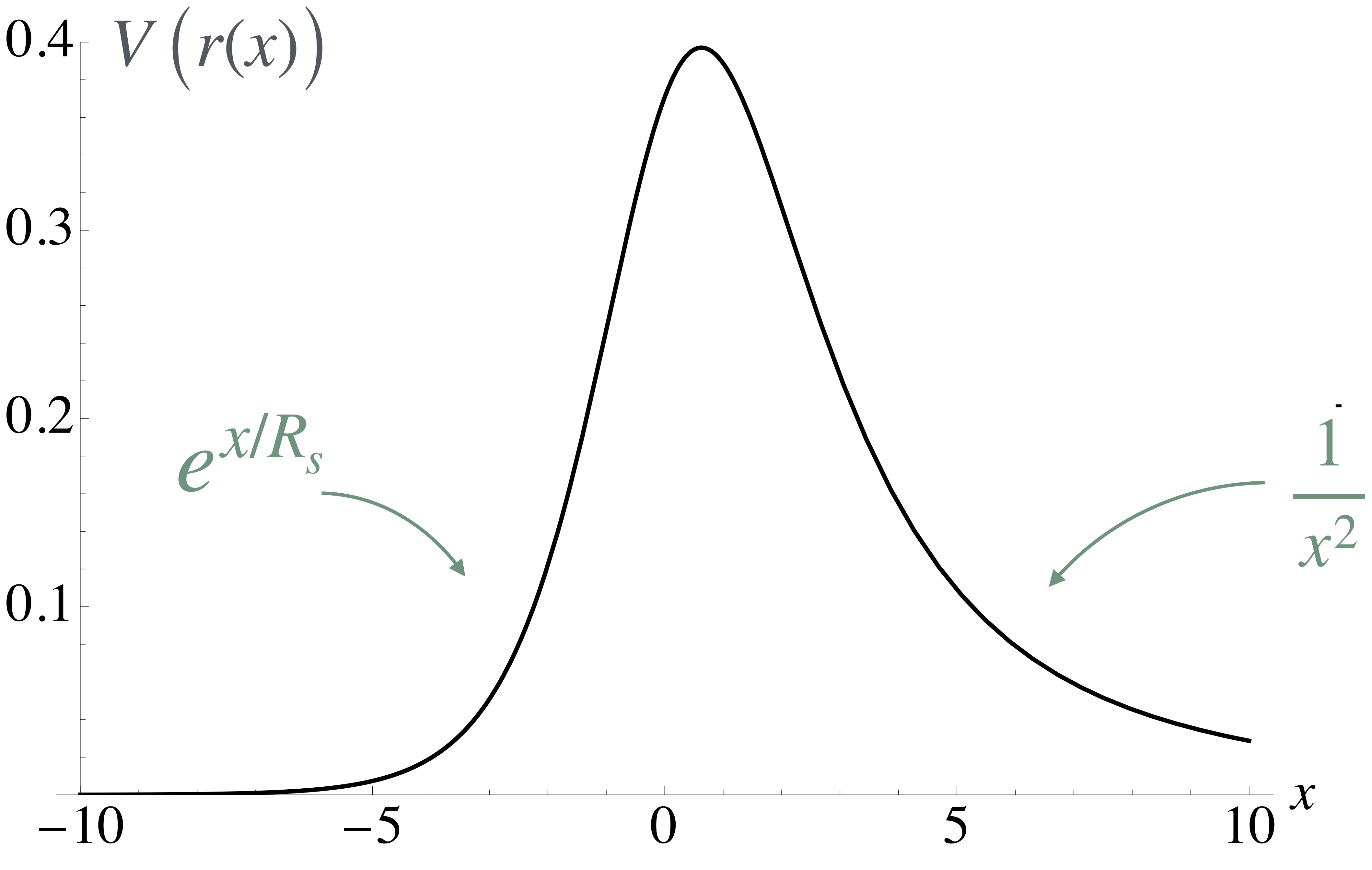

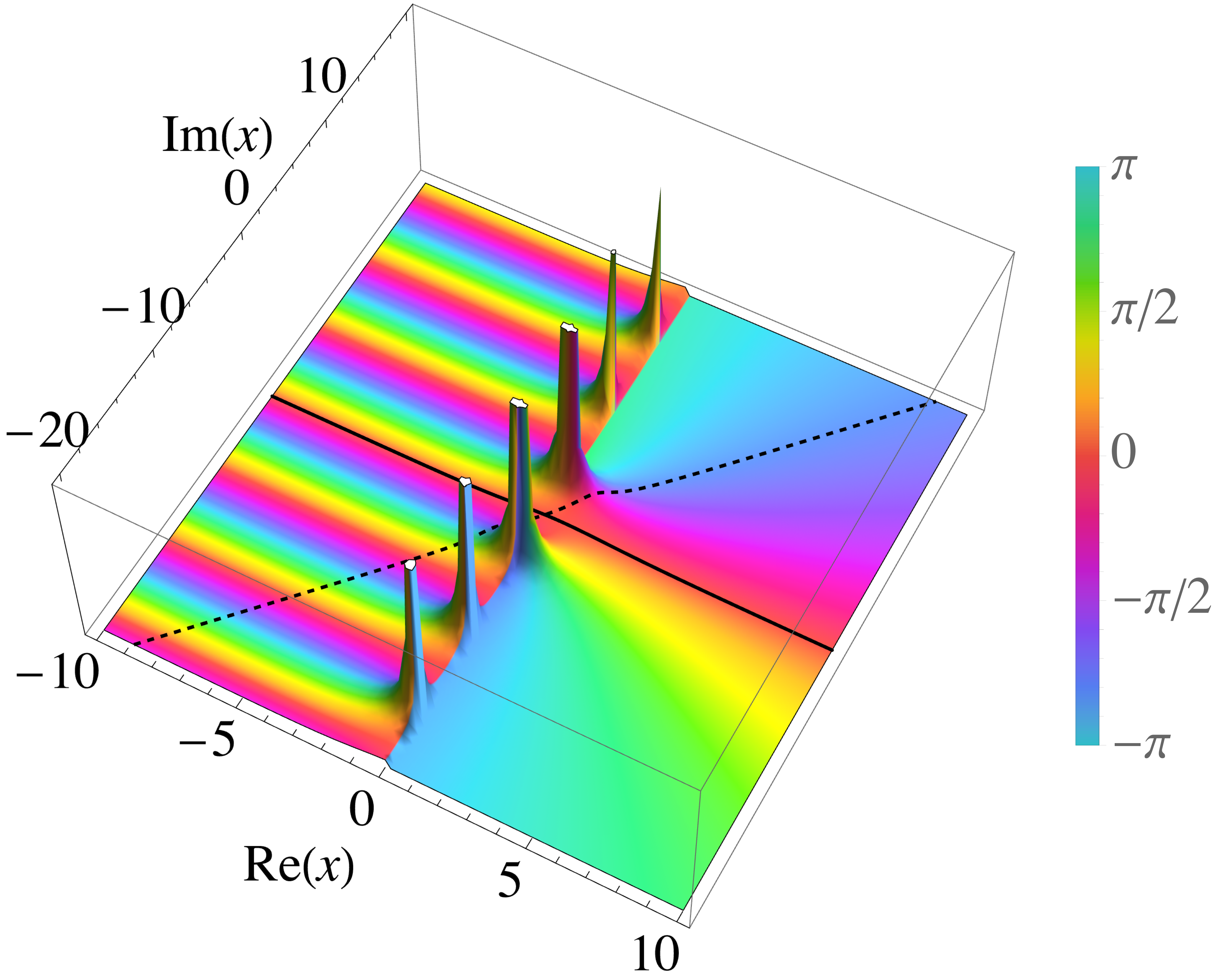

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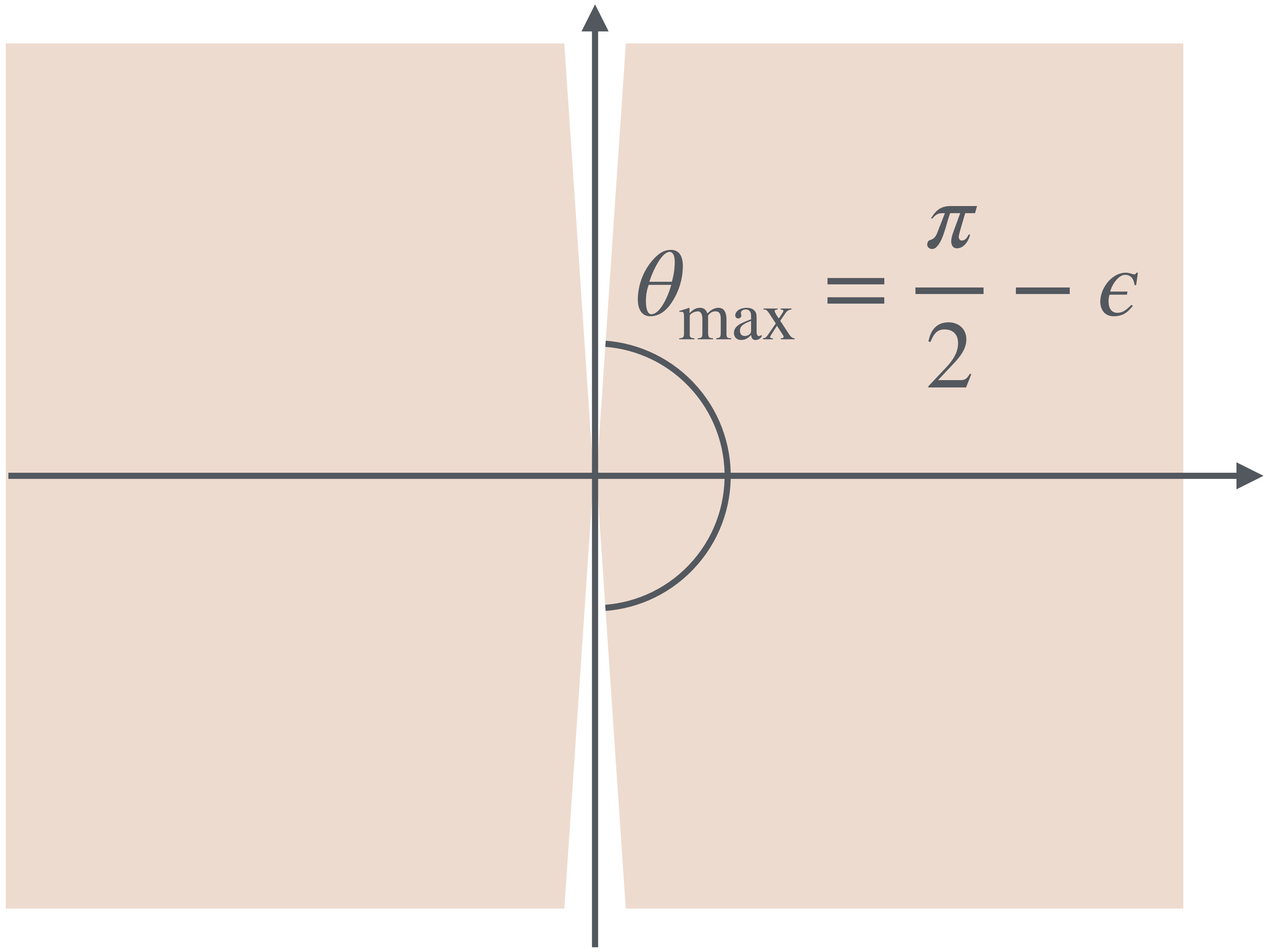
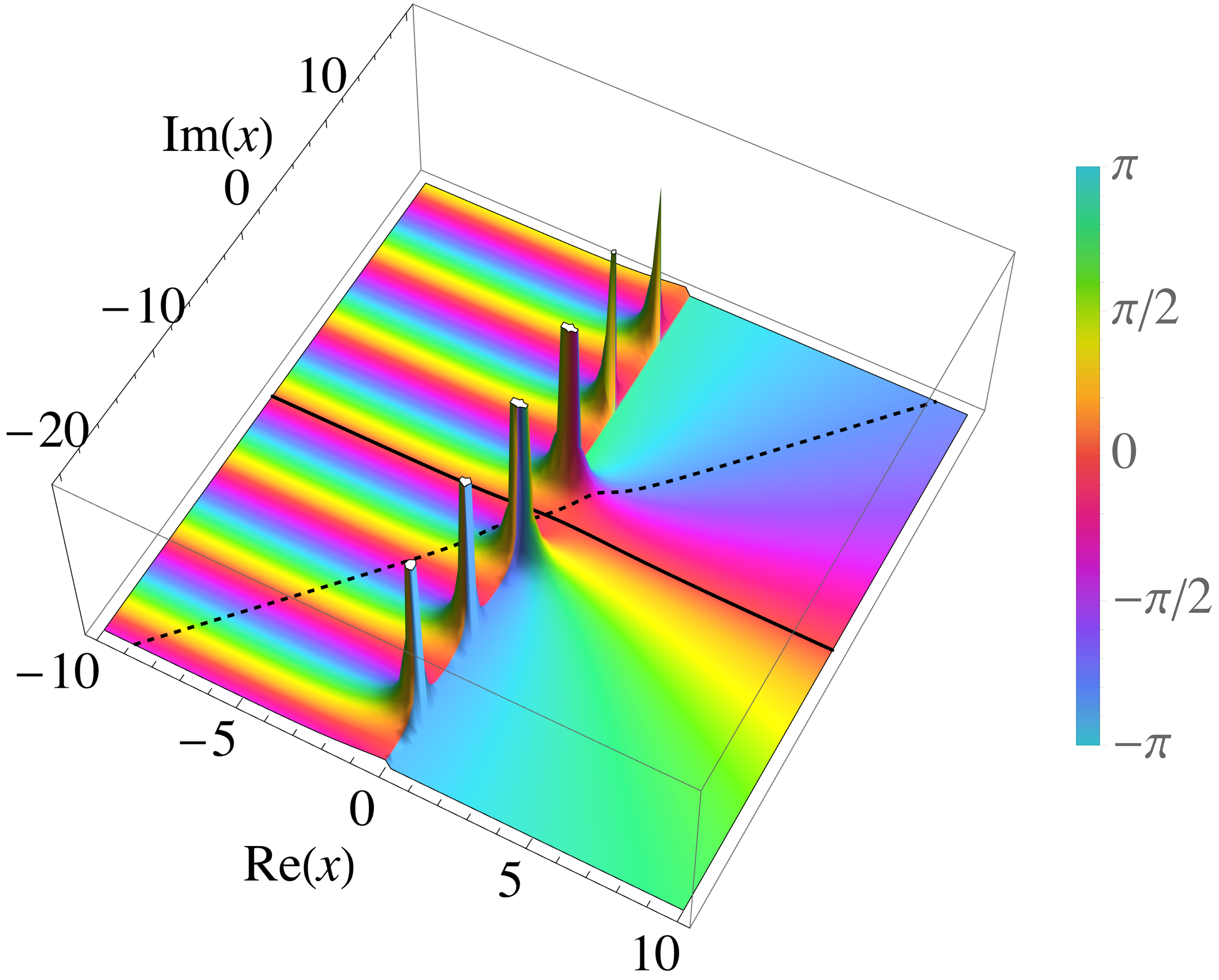
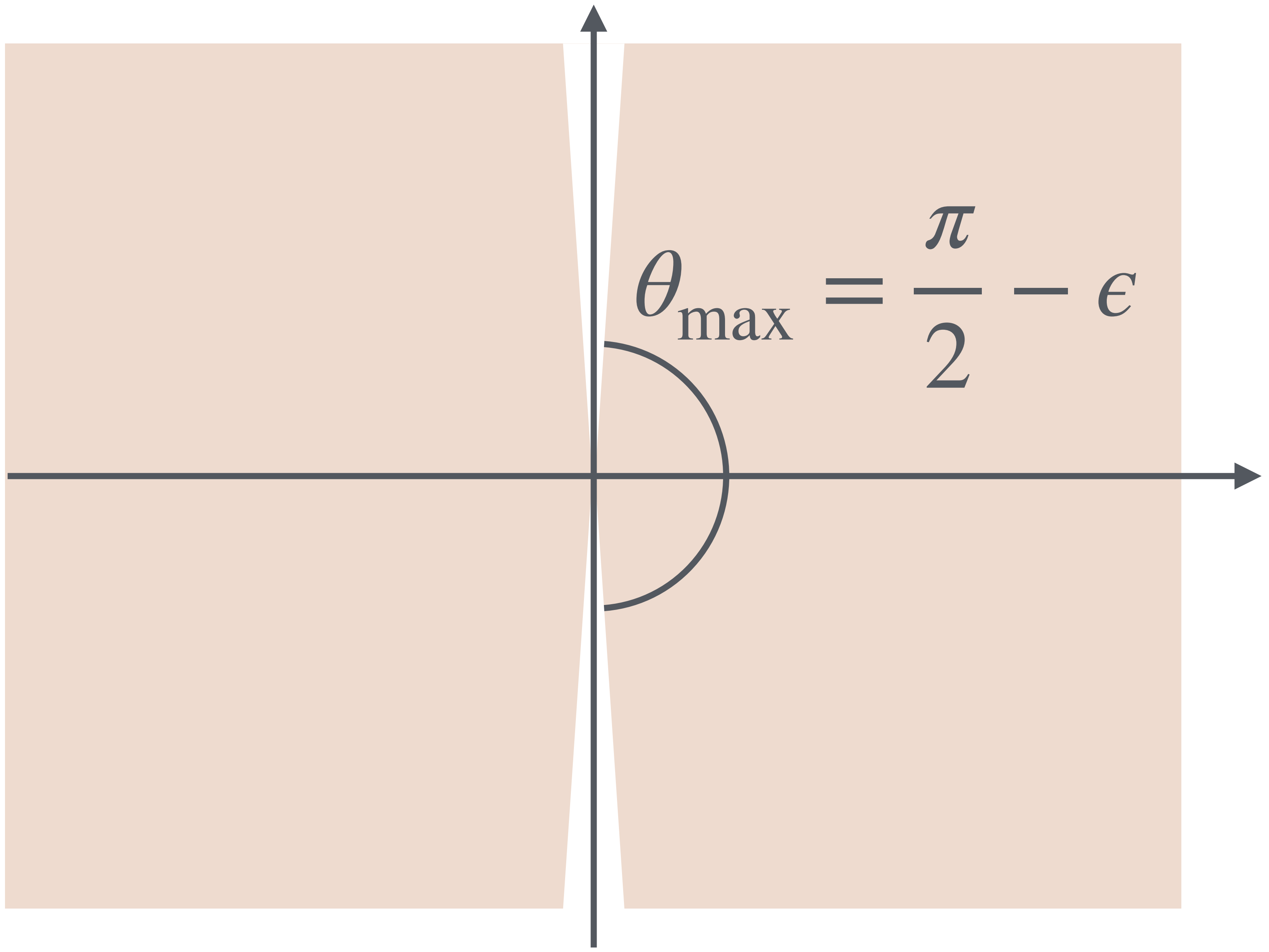
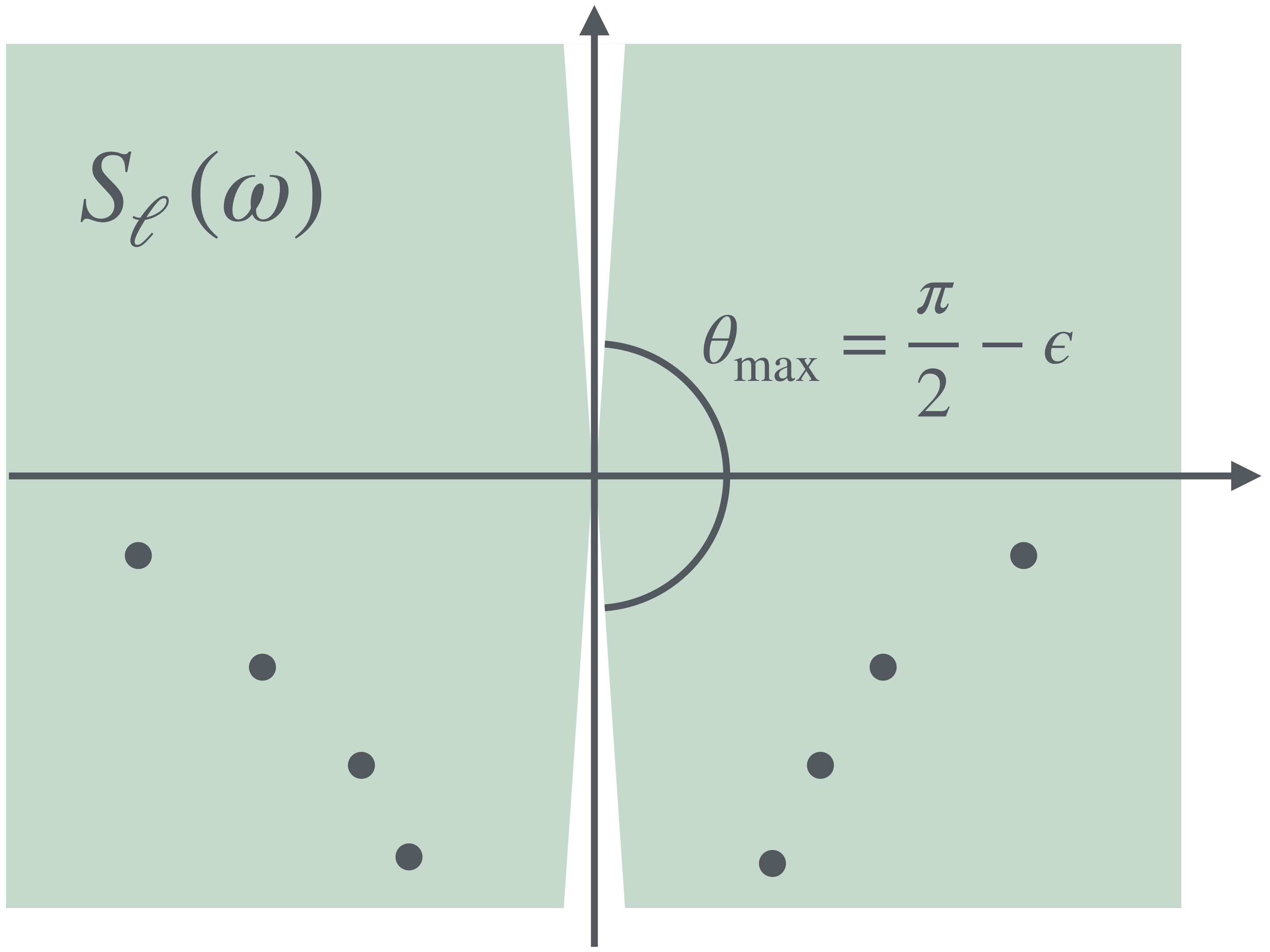


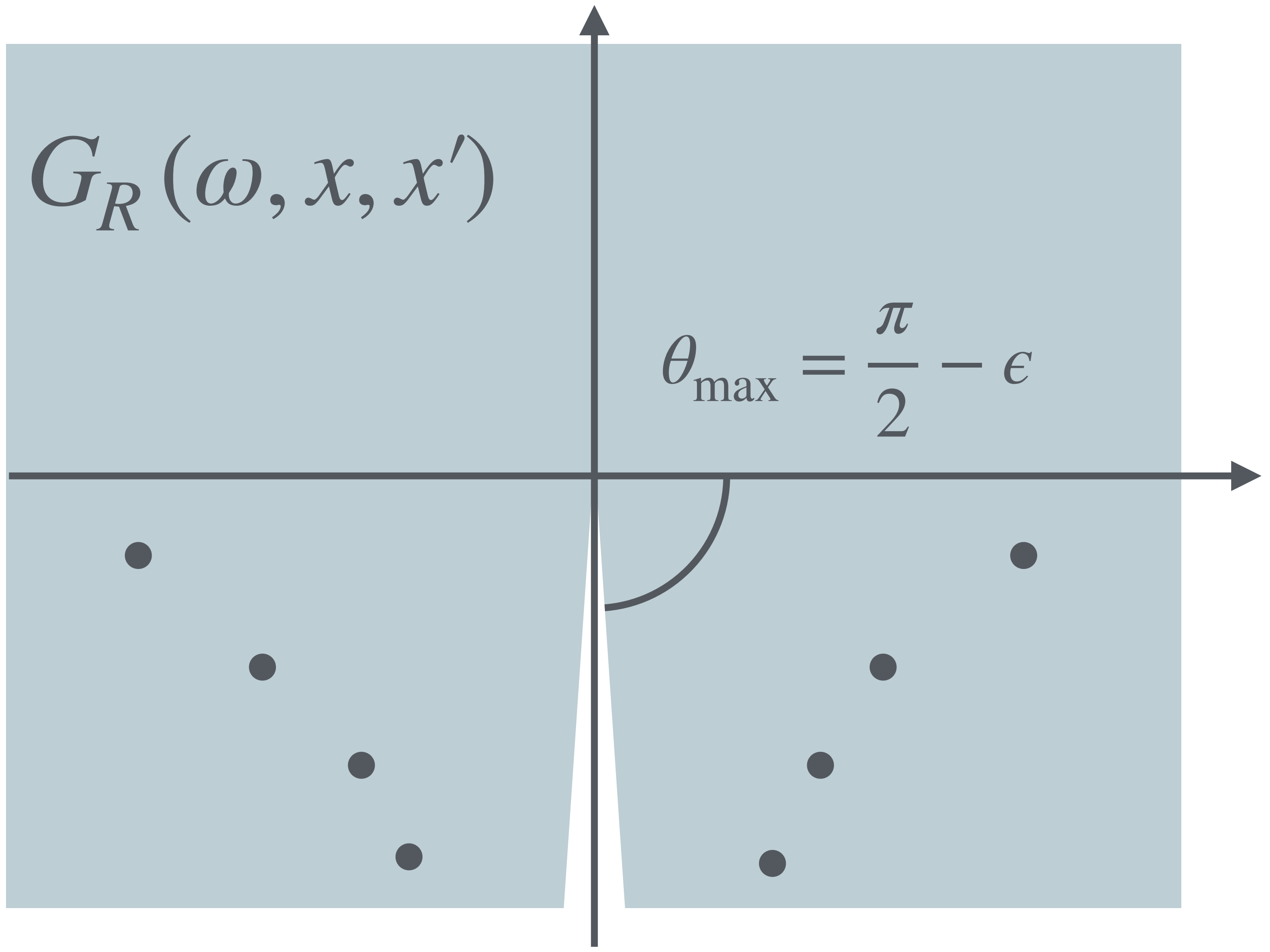


Proven analyticity of the Black Hole amplitude

$$V(x \to \infty) \lesssim \frac{1}{|x|}$$





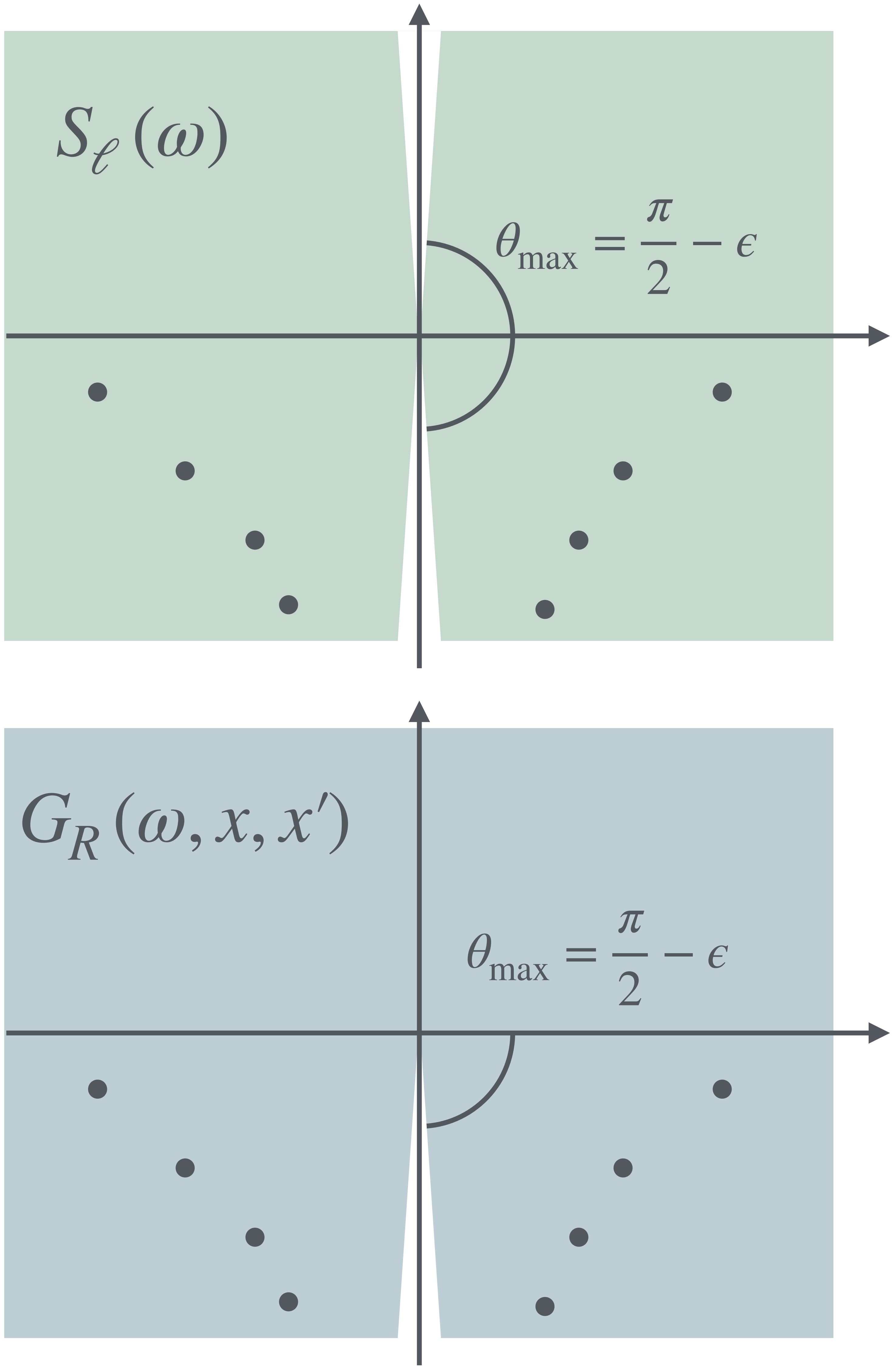


Proven analyticity of the Black Hole amplitude

- Explore solvable regimes
  - Small frequency expansion
  - Coulomb potential
  - High frequencies limit

- Consistency with

$$G_R(\omega, x, x') \xrightarrow{x, x' \to \infty} -\frac{1}{2i\omega}\left[e^{i\omega(x-x')} - S_\ell(\omega)\, e^{i\omega(x+x')}\right]$$



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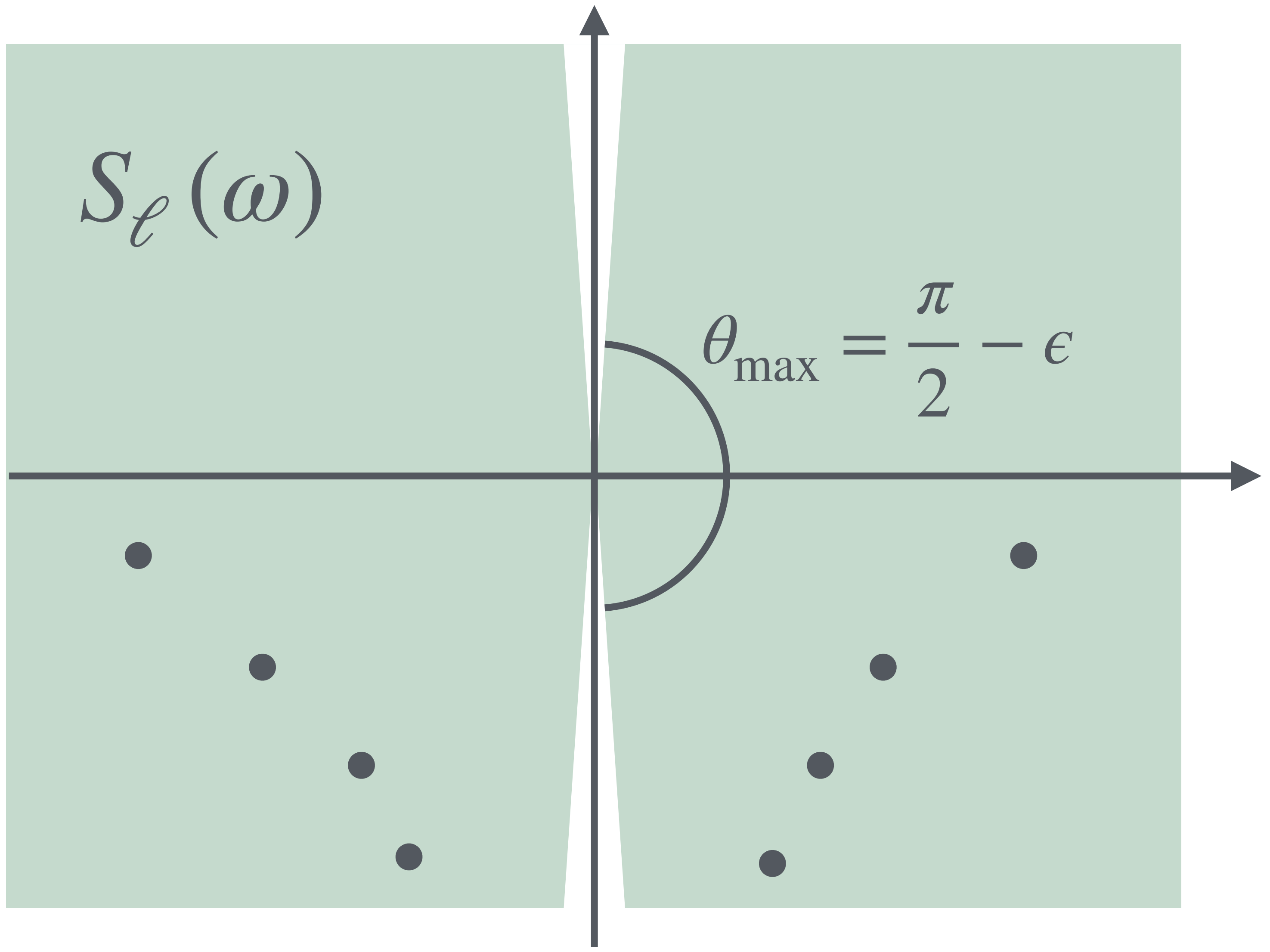
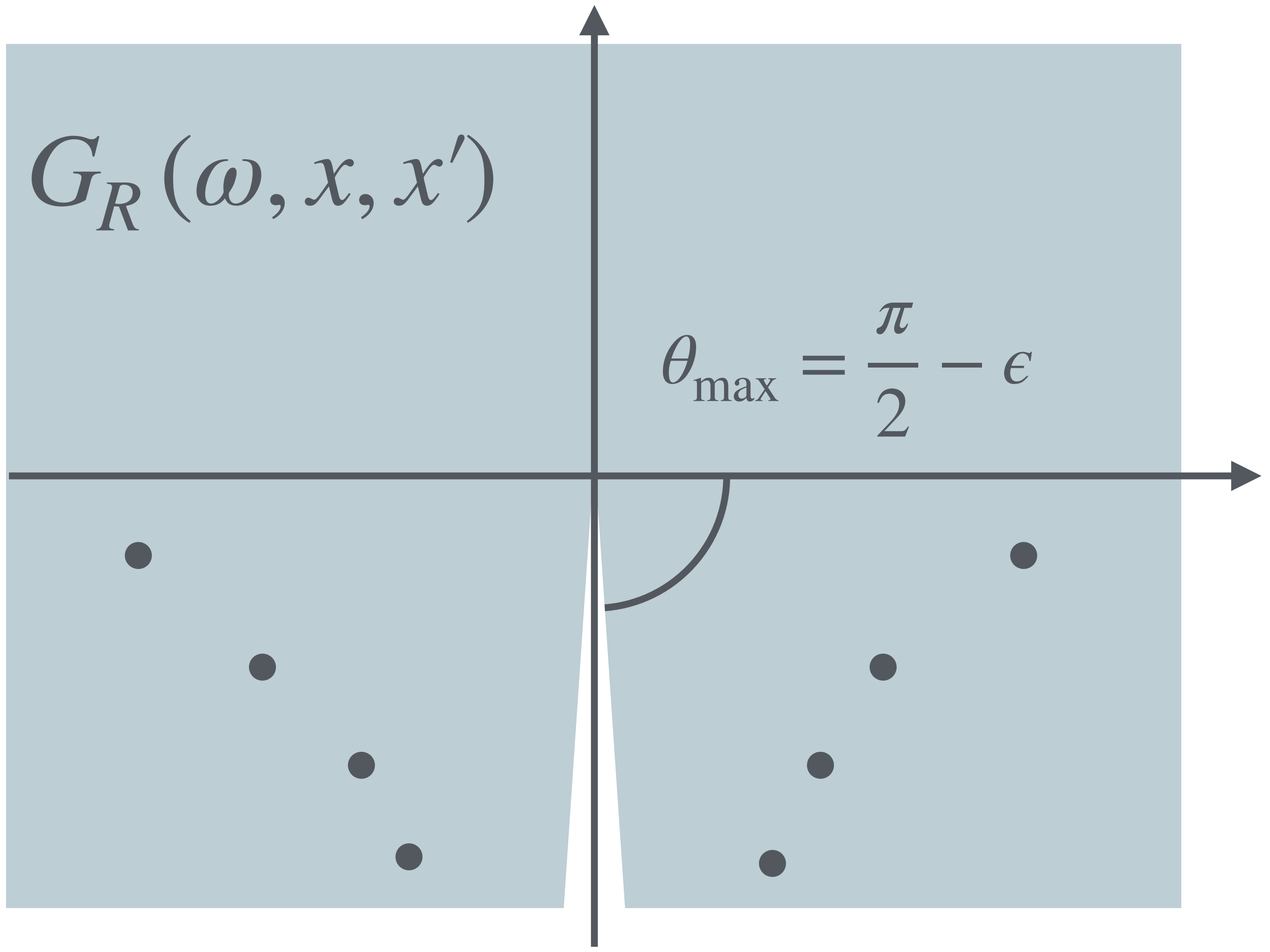
Do not miss the answers to these questions during Anna's Gong Show!

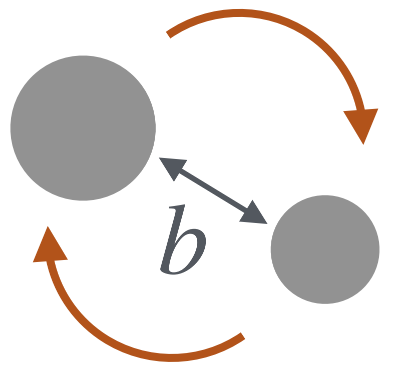




Conclusion

Eikonal regime

$b \gg \lambda_i$



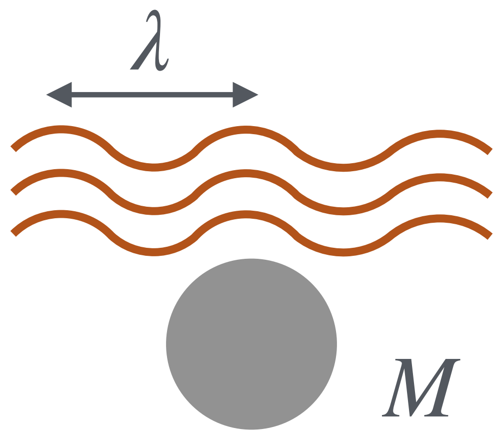
$$S(s,b) \sim e^{\frac{i}{\hbar} I(s,b)}$$

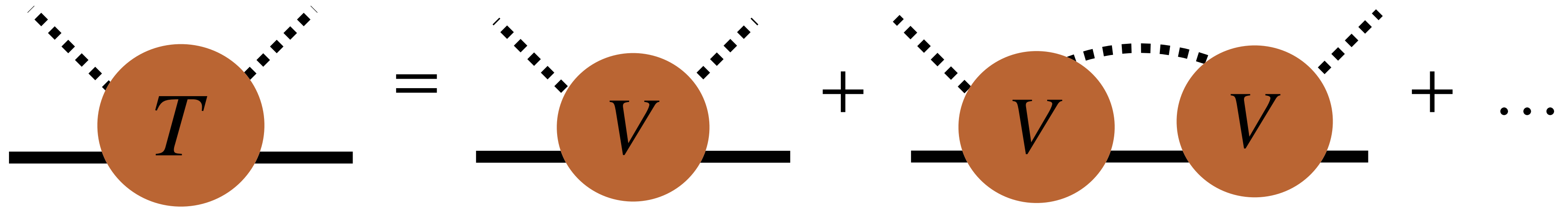
Extracting observables

Computational simplification

Born regime

$\boxed{\omega \ll M}$ $\quad \lambda \gg \lambda_i$





Effective Wave equation

Computational simplification