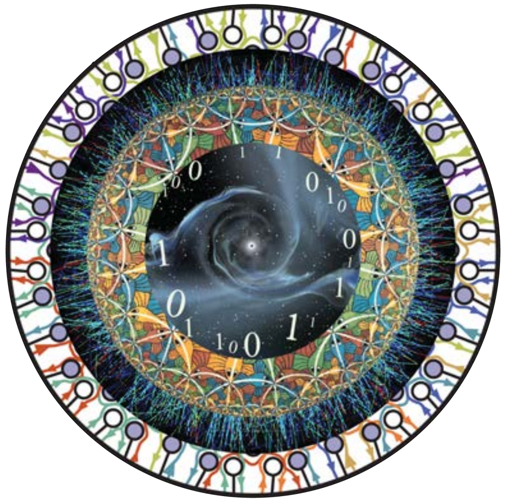


Negative geometry expansion of scattering amplitudes

Jaroslav Trnka

Center for Quantum Mathematics and Physics (QMAP),
University of California, Davis, USA

support by

based on work to appear with
Lance Dixon, Umut Oktem, Shruti Paranjape, Yongqun Xu, Shun-Qing Zhang

and related work with
Nima Arkani-Hamed, Taro Brown, Dmitry Chicherin, Johannes Henn, Elia Mazzucchelli, Umut Oktem, Shruti Paranjape, Qinglin Yang, Shun-Qing Zhang, 2112.06956, 2312.17736, 2410.11456, 2503.17185, 2508.05443

CA Amplitudes Meeting, UCLA, November 8, 2025

MOTIVATION

Planar N=4 SYM: playground for new ideas in scattering amplitudes

Four-point amplitudes: restricted kinematics, powerful symmetries

Bern-Dixon-Smirnov (BDS) ansatz

(Bern, Dixon, Smirnov 2005)

$$\mathcal{M}_n \equiv 1 + \sum_{L=1}^{\infty} a^L M_n^{(L)}(\epsilon) = \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\right)\right]$$

- Kinematical part fully fixed, **leading IR divergence** predicted by integrability

(Beisert, Eden, Staudacher 2006)

Higher-point amplitudes: non-trivial IR finite kinematical part

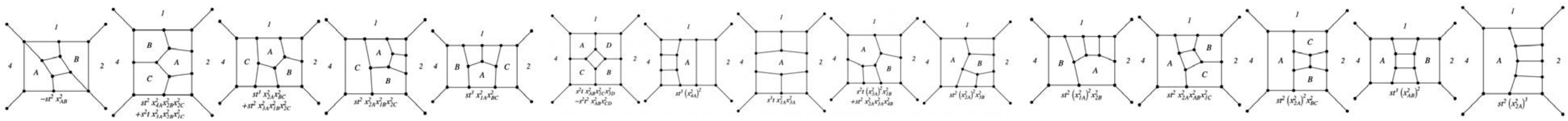
- Remainder function, ratio function -- start at 6pt
- Powerful techniques: symbols, hexagon bootstrap, flux-tube S-matrix,.....

(Goncharov, Spradlin, Vergu, Volovich) (Dixon, McLeod, von Hippel, Caron-Huot, Drummond, Henn, Dulat, Papathanasiou, Gurdogan, Wilhelm) (Basso, Sever, Vieira)

MOTIVATION

Loop integrand - rational function of loop and external momenta

- Very complex even at 4pt if we go to high loop order



- In D=4: presence of IR divergencies - regularization
- We will work with **IR finite object (Wilson loop)**: negative geometries

Perturbation theory: finite radius of convergence

$$g^2 = g_{\rm YM}^2 N \qquad g_{\rm YM} \to 0,\ N \to \infty \quad \text{t'Hooft limit}$$

- Can we determine some quantities to all loop orders in g ? g does not run

  -> re-derive integrability predictions from amplitudes + more

- We need to control the integrand to all loops + find a way to integrate + resum

MOTIVATION

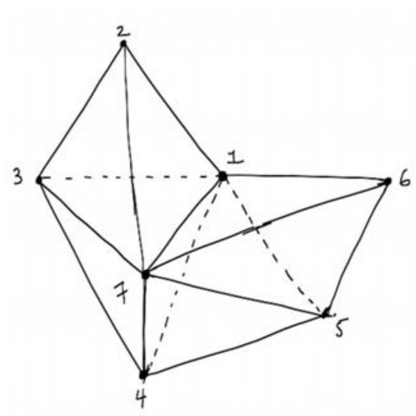
The integrand must be complicated because it contains a lot of "data", infinite number of cuts that must be satisfied

Are these data lost after integration (how do they transform into numbers)?

Can we calculate the IR divergence from amplitudes? Where is integrability there?

Is the full IR finite object (Wilson loop) exactly calculable?

Amplituhedron: new geometric definition for the all-loop integrand



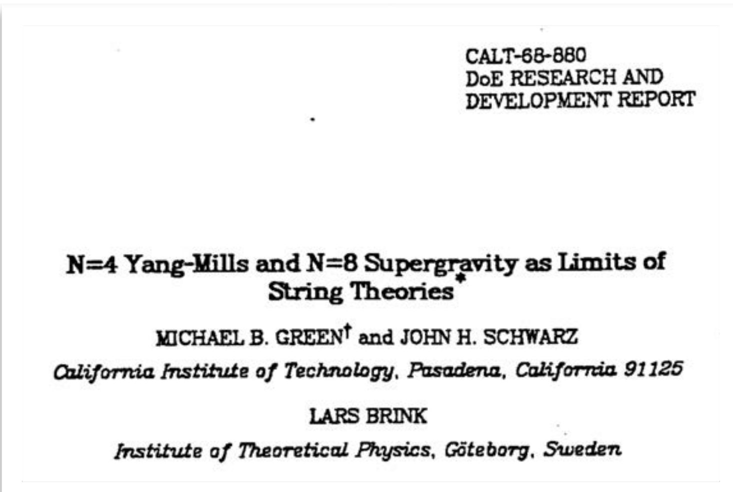
Can we use it to calculate the integrand to all loops?

If yes, can we integrate, resum and explore strong coupling?

INTRODUCTION

EARLY RESULTS

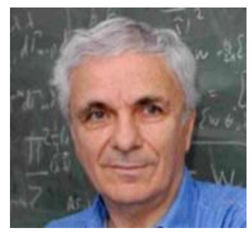
One-loop amplitude calculated in 1982, (full) two-loop in 1997

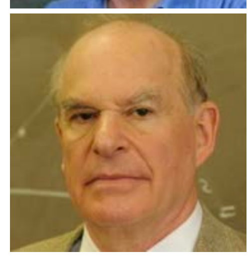

CALT-68-880
DoE RESEARCH AND
DEVELOPMENT REPORT

N=4 Yang-Mills and N=8 Supergravity as Limits of String Theories*

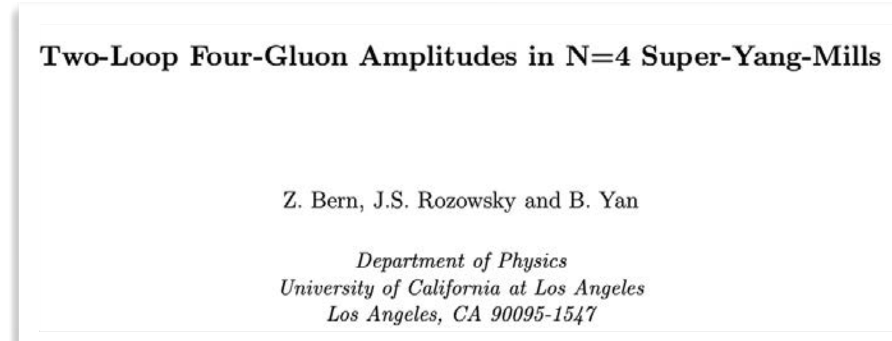
MICHAEL B. GREEN† and JOHN H. SCHWARZ
California Institute of Technology, Pasadena, California 91125

LARS BRINK
Institute of Theoretical Physics, Göteborg, Sweden



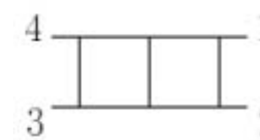


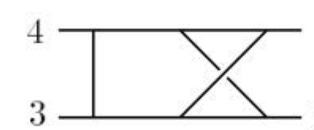
Lars Brink
1943-2022


Two-Loop Four-Gluon Amplitudes in N=4 Super-Yang-Mills

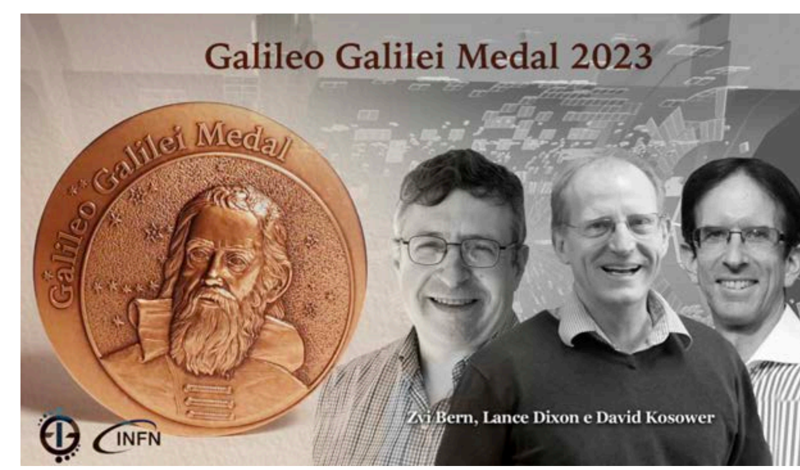
Z. Bern, J.S. Rozowsky and B. Yan

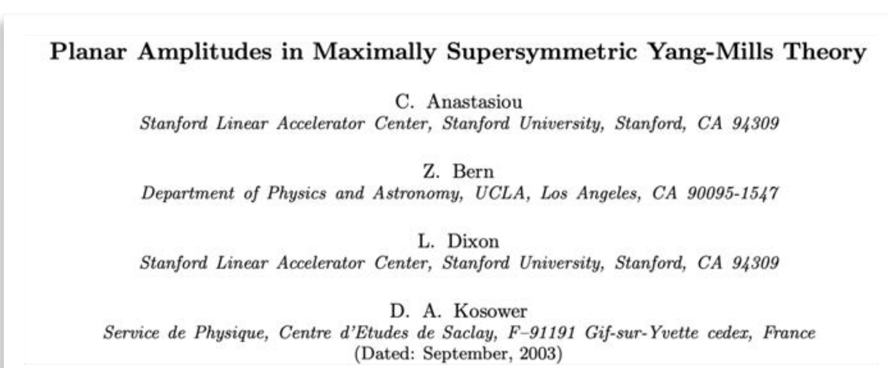
Department of Physics
University of California at Los Angeles
Los Angeles, CA 90095-1547





In 2003 Anastasiou, Bern, Dixon and Kosower:
planar sector (large N limit) of the amplitude




Planar Amplitudes in Maximally Supersymmetric Yang-Mills Theory

C. Anastasiou
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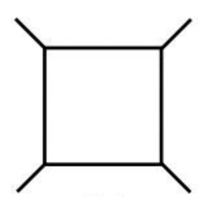
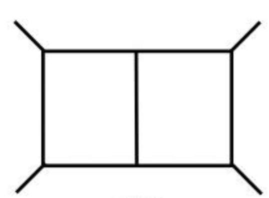
Z. Bern
Department of Physics and Astronomy, UCLA, Los Angeles, CA 90095-1547

L. Dixon
Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309

D. A. Kosower
Service de Physique, Centre d'Etudes de Saclay, F-91191 Gif-sur-Yvette cedex, France
(Dated: September, 2003)



(a)



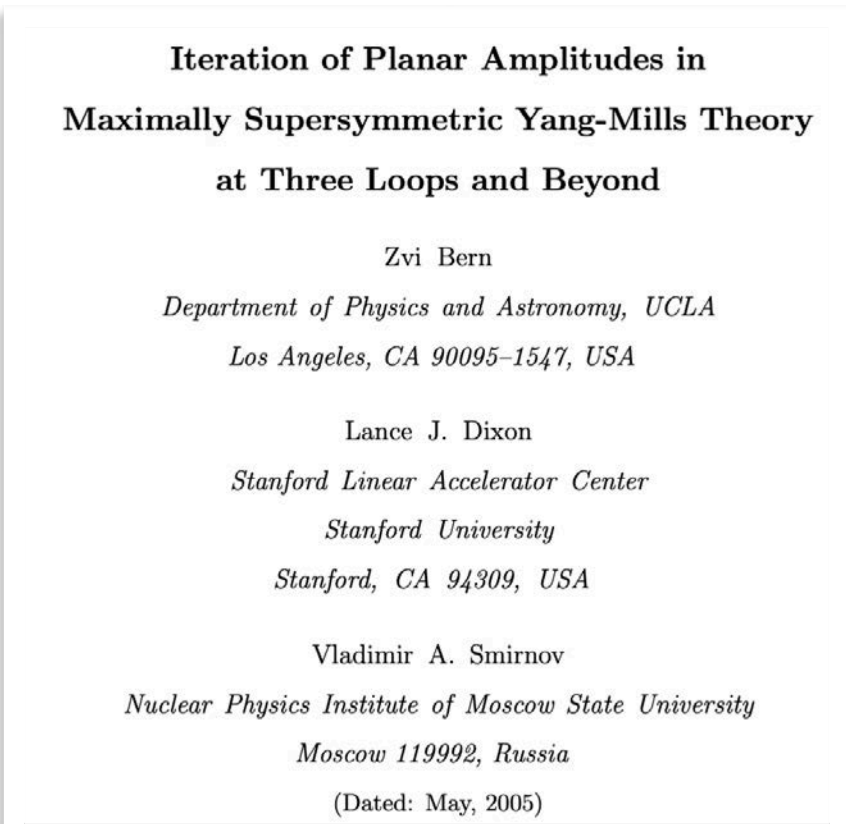
(b)

Observed **relation** between two-loop and one-loop in dimensional regularization

$$M_n^{(2)}(\epsilon) = \frac{1}{2}\left(M_n^{(1)}(\epsilon)\right)^2 + f(\epsilon)\, M_n^{(1)}(2\epsilon) - \frac{5}{4}\zeta_4$$

BDS ANSATZ

In 2005, Bern, Dixon and Smirnov calculated 3-loop amplitude


Iteration of Planar Amplitudes in Maximally Supersymmetric Yang-Mills Theory at Three Loops and Beyond

Zvi Bern
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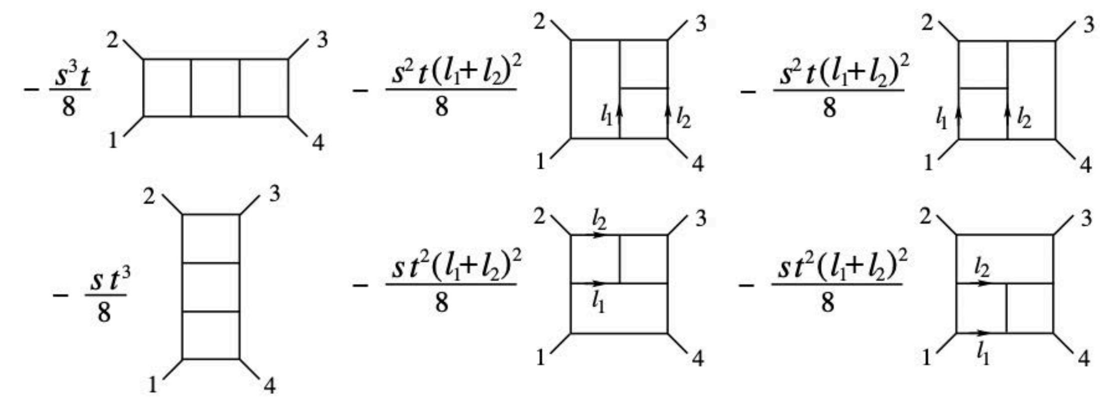
Los Angeles, CA 90095-1547, USA

Lance J. Dixon
Stanford Linear Accelerator Center
Stanford University
Stanford, CA 94309, USA

Vladimir A. Smirnov
Nuclear Physics Institute of Moscow State University
Moscow 119992, Russia

(Dated: May, 2005)

$$M_4^{(3)}(\epsilon) = -\frac{1}{8} st \left(s^2\, I_4^{(3)a}(s,t) + 2s\, I_4^{(3)b}(t,s) + t^2\, I_4^{(3)a}(t,s) + 2t\, I_4^{(3)b}(s,t) \right)$$



integrand already given in 1997 paper

The integrand obtained using **unitarity methods**, after integration they found the same iterative structure

$$M_4^{(3)}(\epsilon) \;=\; -\frac{1}{3}\Big[M_4^{(1)}(\epsilon)\Big]^3 + M_4^{(1)}(\epsilon)\, M_4^{(2)}(\epsilon) + f^{(3)}(\epsilon)\, M_4^{(1)}(3\,\epsilon) + C^{(3)} + \mathcal{O}(\epsilon)$$

Conjecture: $$\boxed{\mathcal{M}_n \equiv 1 + \sum_{L=1}^{\infty} a^L M_n^{(L)}(\epsilon) = \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon) \right)\right]}$$

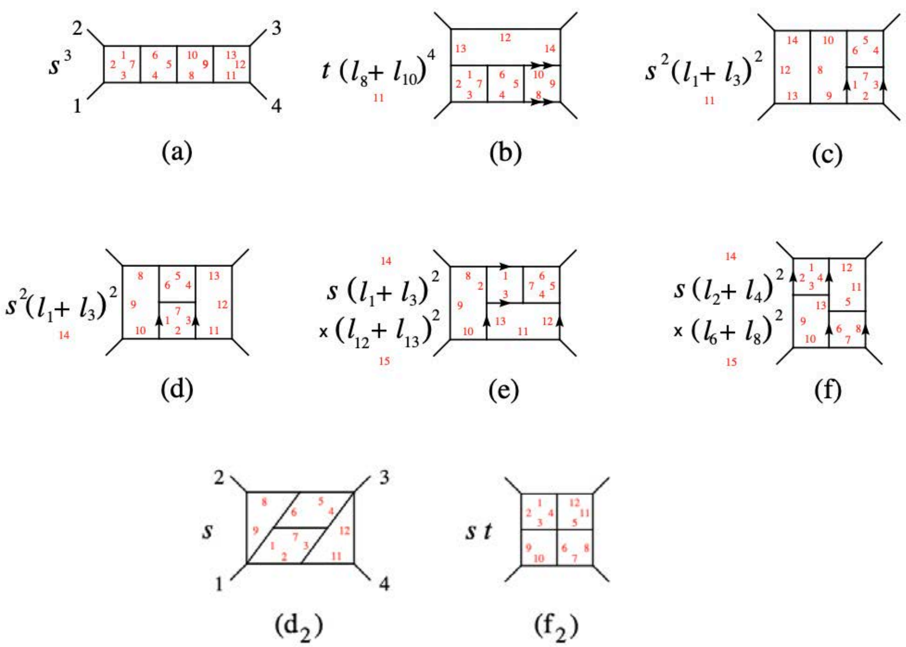
cusp-anomalous dimension calculated in 2006 by Beisert, Eden and Staudacher from integrability

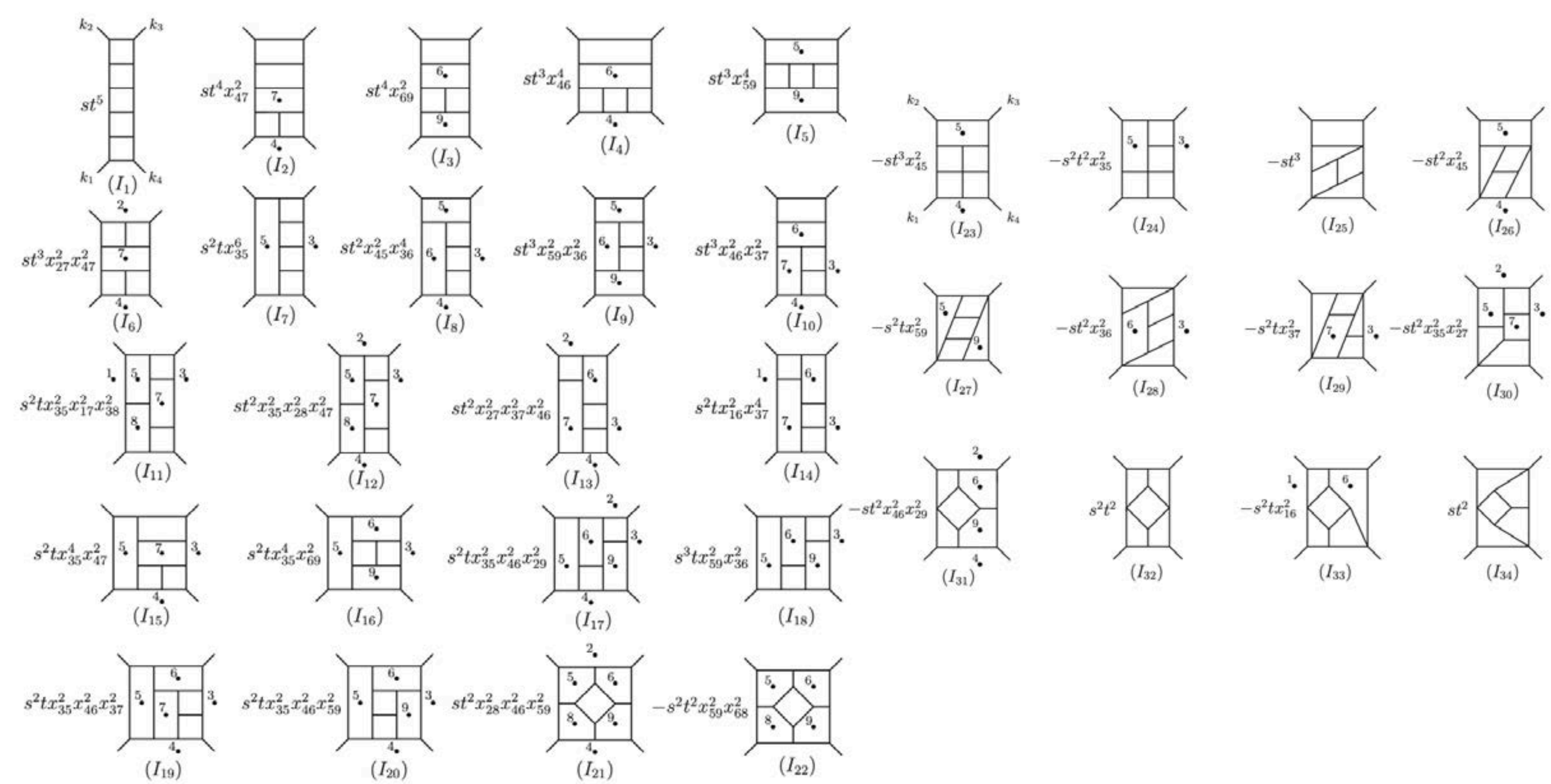
$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}$$

HIGHER LOOPS

In next two years, 4-loop and 5-loop integrands were constructed

(Bern, Czakon, Dixon, Kosower, Smirnov, 2006) (Bern, Carrasco, Johansson, Kosower, 2007)





- analytic results not known, leading IR divergence verified at 4-loops numerically
- numerators can be chosen to be invariant under dual conformal symmetry

(Drummond, Henn, Korchemsky, Sokatchev, 2008)

Today calculated up to 12-loops using f-graphs

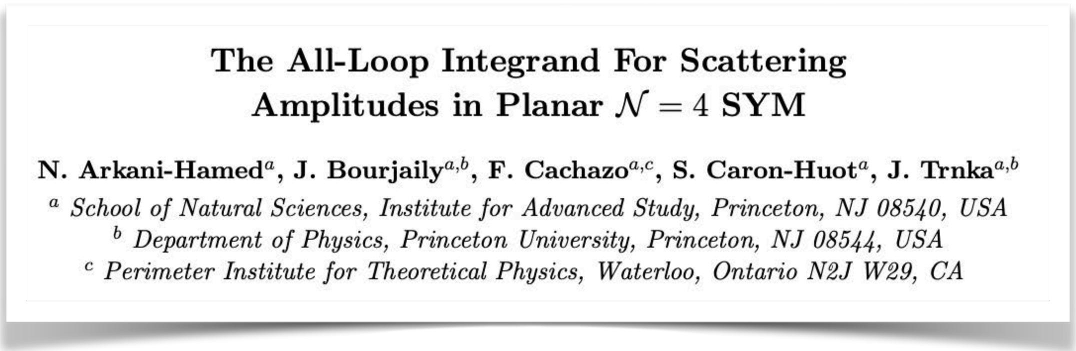
(Bourjaily, Heslop, Tran, 2016) (He, Shi, Tang, Zhang, 2024) (Bourjaily, He, Shi, Tang, 2025)

LOOP INTEGRAND

In 2010, we took the planar integrand seriously and formulated recursion relations for N=4 SYM in momentum twistor space (Hodges, 2009)

- the integrand is a **unique rational function**

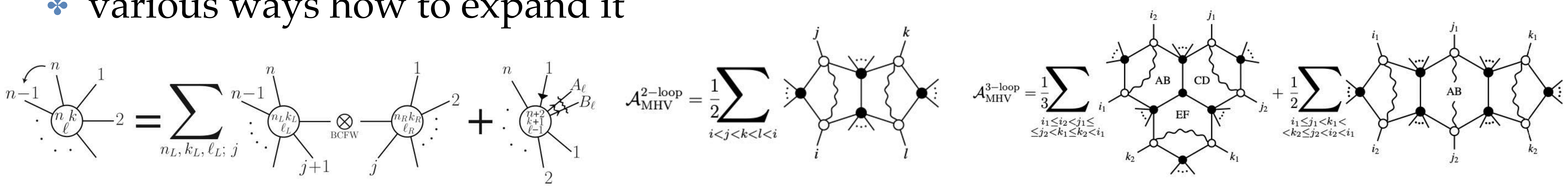
$$\boxed{\mathcal{I}^{\ell-\text{loop}}_{n,k}(AB_1, AB_2, \ldots, AB_\ell, Z_1, Z_2, \ldots, Z_n)}$$


The All-Loop Integrand For Scattering Amplitudes in Planar $\mathcal{N}=4$ SYM

N. Arkani-Hamed[a], J. Bourjaily[a,b], F. Cachazo[a,c], S. Caron-Huot[a], J. Trnka[a,b]

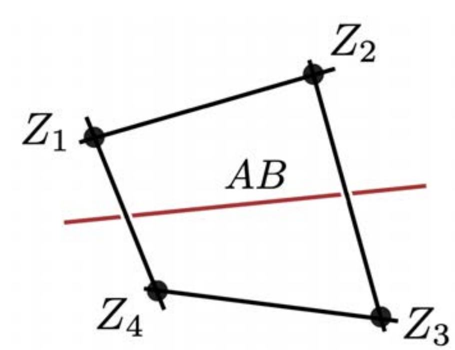
[a] School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA
[b] Department of Physics, Princeton University, Princeton, NJ 08544, USA
[c] Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2J W29, CA

- various ways how to expand it



Properties of the loop integrand

- symmetric function of all loop lines AB_i
- the only poles are: $\langle AB_i\, j\, j{+}1\rangle$ or $\langle AB_i AB_j\rangle$
- cuts in momentum twistor space: localizing AB_i - intersection with other lines



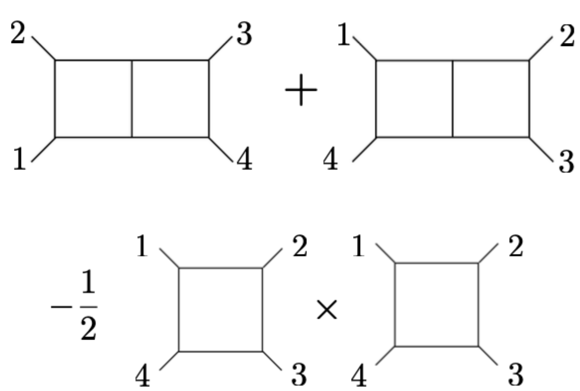
AMPLITUDE LOGARITHM

As we learnt from BDS ansatz **logarithm of the amplitude** is a special function with **mild IR divergence**

$$\ln \mathcal{M}_n = \sum_{l=1}^{\infty} a^l f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon) = \frac{\gamma_{\rm cusp}}{\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right)$$

It makes sense to construct the integrand for the logarithm from products of amplitudes, which makes this property manifest

two-loop 4pt example: $\widetilde{\mathcal{I}}_4^{(2)}(AB_i, Z_j) = \mathcal{I}_4^{(2)}(AB, CD) - \mathcal{I}_4^{(1)}(AB) \times \mathcal{I}_4^{(1)}(CD)$



not a planar object!

$$\widetilde{\mathcal{I}}_4^{(2)} = \frac{\langle 1234\rangle^3(\langle AB13\rangle\langle CD24\rangle + \langle AB24\rangle\langle CD13\rangle)}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle\langle ABCD\rangle\langle CD12\rangle\langle CD23\rangle\langle CD34\rangle\langle CD41\rangle}$$

(Arkani-Hamed, Bourjaily, Cachazo, Trnka, 2010) (Drummond, Henn, 2010)

IR property: vanishing in collinear regions $Z_A \to Z_2, \quad Z_B \to Z_1 + \alpha Z_3$

in fact, only non-zero residue when we move all loop lines in the collinear region

AMPLITUDE LOGARITHM

We need to integrate over all loops to get IR divergence

Can we extract an **IR finite function** from the integrand?

$\widetilde{\mathcal{I}}_L(\ell_1,\ldots,\ell_L)$ ⟵ rational function

$\downarrow$ integrate over ℓ_L

$\mathcal{I}_L(\ell_1,\ldots,\ell_{L-1})$ ⟵ weight 2 function

$\downarrow$ integrate over ℓ_{L-1}

$\vdots$ $\vdots$

$\downarrow$ integrate over ℓ_2

all are IR finite

$\mathcal{I}_L(\ell_1)$

$\downarrow$ integrate over ℓ_1

$\ln M_L$ ⟵ IR divergent, BDS ansatz

weight (2L-2) function

4pt: single cross ratio

$\mathcal{F}_{L-1}(z)$

$$z=\frac{\langle AB12\rangle\langle AB34\rangle}{\langle AB23\rangle\langle AB14\rangle}$$

GAMMA CUSP

But the leading IR divergence is predicted to be known to all loop orders -- exact formula $\gamma^{\text{cusp}}(g)$ (Beisert, Eden, Staudacher 2005)

Integrability of the planar N=4 SYM theory

- It is an input into a number of computational methods
- Fascinating function of coupling: predicted by BES equation

$$\gamma^{\text{cusp}}(g) = 8g^2 - \frac{8\pi^2}{3}g^4 - \frac{88\pi^4}{45}g^6 - 16\left(\frac{73\pi^6}{630} + 4\zeta_3^2\right)g^8 + \ldots = 2g - \frac{3\ln 2}{2\pi} + \ldots$$

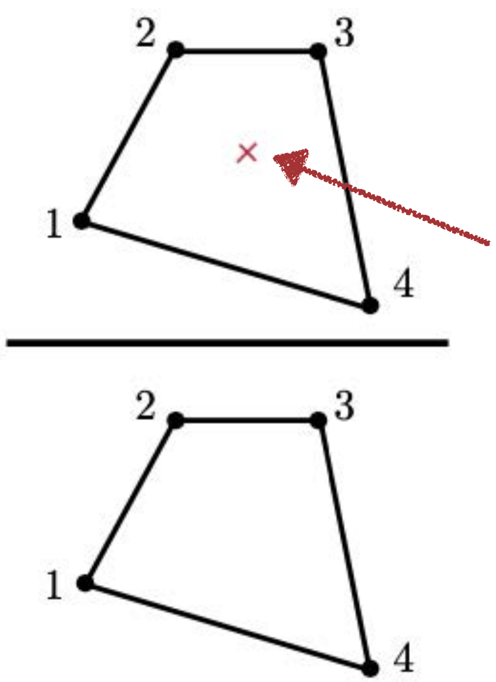
(Basso, Korchemsky, Kotanski 2007)

very special property:
only **single zeta values**
at weak coupling

$$\zeta_k = \sum_{n=1}^{\infty} \frac{1}{n^k}$$

WILSON LOOP DUALITY

Same object appeared in the study of Wilson loops



Lagrangian insertion

$$\frac{\langle W_F(x_1,x_2,x_3,x_4)\mathcal{L}(x_0)\rangle}{\langle W_F(x_1,x_2,x_3,x_4)\rangle} = \frac{g^2}{\pi^2}\frac{\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle}\mathcal{F}(g,z)$$

- **weak coupling**: expansion in g^2 (known up to 3-loops)
  (Alday, Henn, Sikorowski, 2013) (Henn, Korchemsky, Mistlberger 2019)
- **strong coupling**: expansion in $1/g$
  (Alday, Buchbinder, Tseytlin, 2011) (Engelund, Roiban, 2011, 2012)

$$\mathcal{F}(g,z) = g\frac{z}{(z-1)^3}\left[2(1-z)+(z+1)\log z\right]+\cdots$$

subleading term impossibly hard to calculate

We can extract $\Gamma_{\rm cusp}(g)$ from this function without hitting a divergence

(Alday, Henn, Sikorowski, 2013) (Henn, Korchemsky, Mistlberger, 2019) (Arkani-Hamed, Henn, JT, 2021)

$$g\frac{\partial}{\partial g}\Gamma_{\rm cusp}(g) = -\frac{1}{\pi}\int_{-\pi}^{\pi} d\phi\, F(g,z=e^{i\phi})$$

QUESTIONS

Big Question 1: Can we derive the gamma cusp formula from amplitudes? How is it related to the structures in the loop integrand/combinatorics of the Amplituhedron?

Big Question 2: Is it possible to find the exact solution for full amplitudes, non-trivial functions of kinematics?

- Resummation of perturbation theory?
- Positive geometry, cluster algebras etc. at strong coupling?

Our method: different organization of perturbation theory using negative (Amplituhedron-like) geometries

AMPLITUHEDRON

(Arkani-Hamed, JT 2013)

(Arkani-Hamed, Thomas, JT 2017)

AMPLITUHEDRON

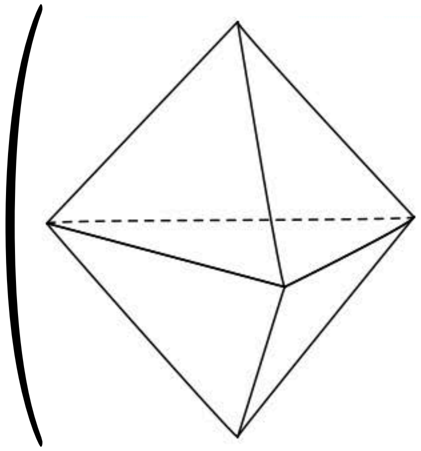
In 2013 together with Nima we found a new **geometric construction** for planar integrands in N=4 SYM

(Arkani-Hamed, JT, 2013) (Arkani-Hamed, Thomas, JT, 2017) (Ferro, Lukowski, 2022)

This is a generalization of our earlier work on the on-shell diagrams and positive Grassmannian and Hodges' polytopes

(Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, JT, 2012) (Hodges 2009) (Arkani-Hamed, Bourjaily, Cachazo, Hodges, JT, 2010)

$$\mathcal{A}_{n,k}^{\ell-\text{loop}} = \Omega\left(\right.$$



$$\left. \right)$$

Tree-level amplitudes and loop integrands

"**Volume**": differential form with logarithmic singularities on the boundaries of the space

Amplituhedron defined by a set of inequalities in the kinematical space

General Amplituhedron is curvy: complicated boundary structure

(Franco, Galloni, Mariotti, JT, 2014)
(Lukowski, Moerman, 2020)
(Dian, Heslop, Stewart, 2022)

How to calculate Ω ?

AMPLITUHEDRON

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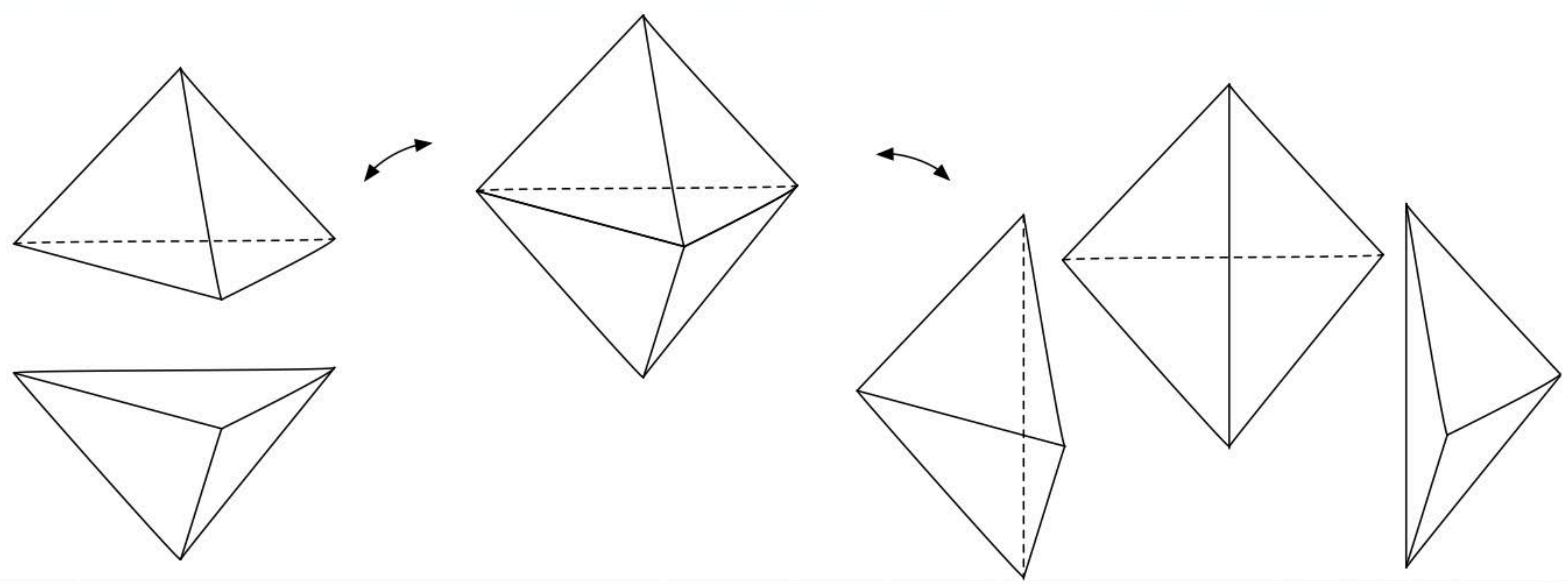


Triangulation in terms of "simplices": difficult to do in general

AMPLITUHEDRON DEFINITON

(Arkani-Hamed, JT 2013) (Arkani-Hamed, Thomas, JT 2017)

Amplituhedron geometry is labeled by 4 labels:

$$\mathcal{A}_{n,k,\ell,m}$$

number of external legs

helicity number,
we restrict to k=0
(MHV amplitudes)

number of loops

"dimensionality" of kinematics,
we assume m=4

Geometry = set of inequalities for kinematical variables
which define a geometric region

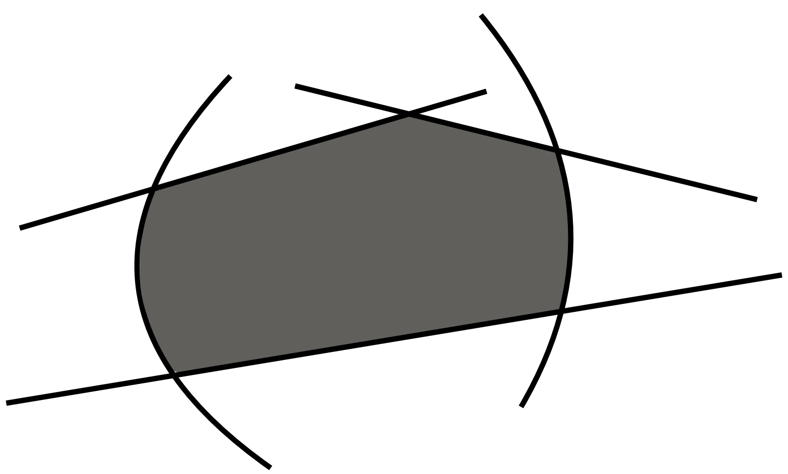
Amplitude (tree-level, loop integrand) = canonical form
calculates a volume of the region

POSITIVE GEOMETRY

Space of dimension D given by polynomial inequalities

(kinematics) -> (parameters x_i)

$$F_k(x_i) \geq 0$$



Canonical form Ω
with logarithmic singularities
on boundaries

$$\Omega_D \xrightarrow[x=0]{} \frac{dx}{x}\Omega_{D-1}$$

form on the boundary

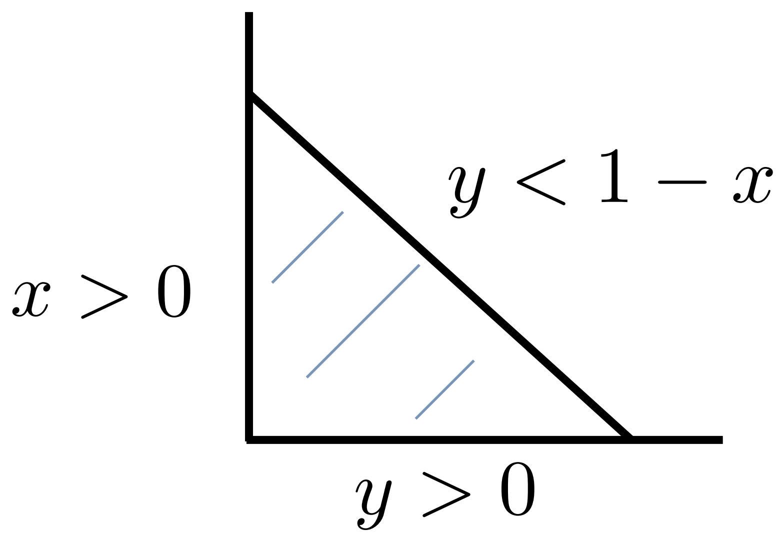
Example:
line segment

$$F_1(x) = x - x_1 > 0$$
$$F_2(x) = x_2 - x > 0$$

$$\Omega = \frac{dx\,(x_1 - x_2)}{(x - x_1)(x - x_2)}$$

POSITIVE GEOMETRY

Triangle



$$y = (0, \infty)$$
$$x = (0, 1 - y)$$

$$\boxed{\Omega = \frac{dx\, dy}{xy(1 - x - y)}}$$

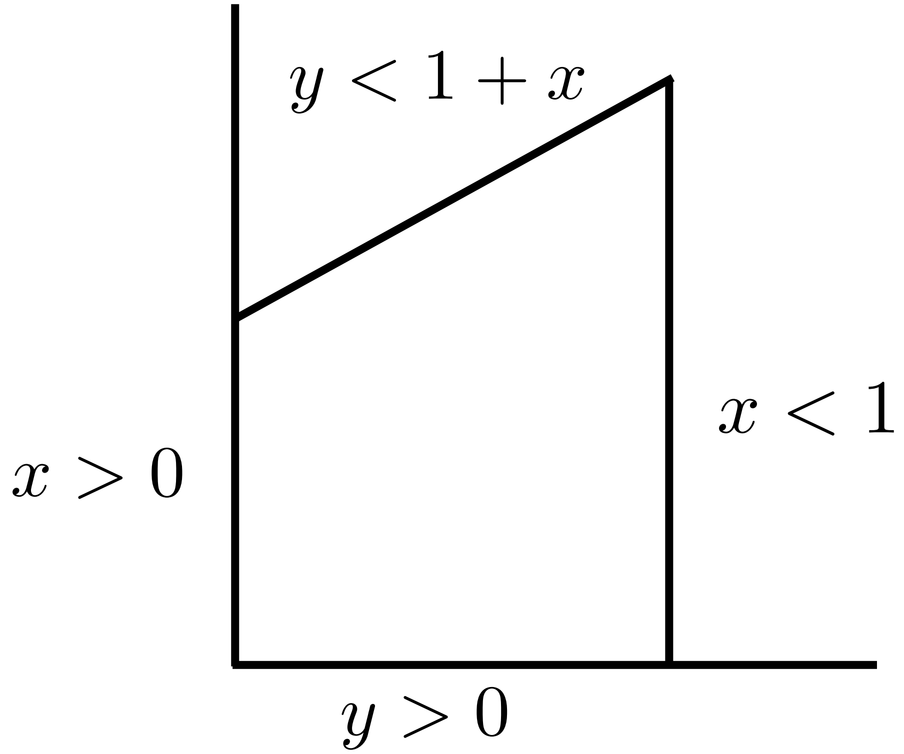
canonical form
on a line x=(0,1)

check we get lower
dimensional form correct:

$$\mathrm{Res}_{y=0}\Omega = \frac{dx}{x(1-x)}$$

POSITIVE GEOMETRY

Polygon



$$x = (0, 1)$$
$$y = (0, 1 + x)$$

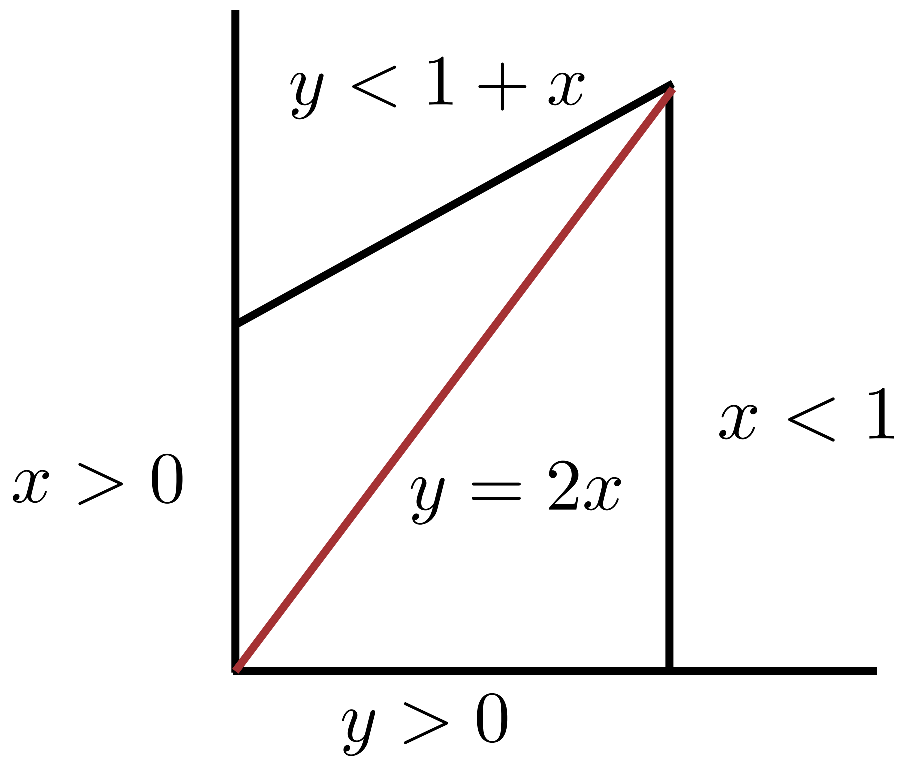
we can write

$$\Omega = \frac{n(x,y)}{xy(x-1)(y-x-1)}$$

Two ways to fix the numerator

POSITIVE GEOMETRY

Triangulation

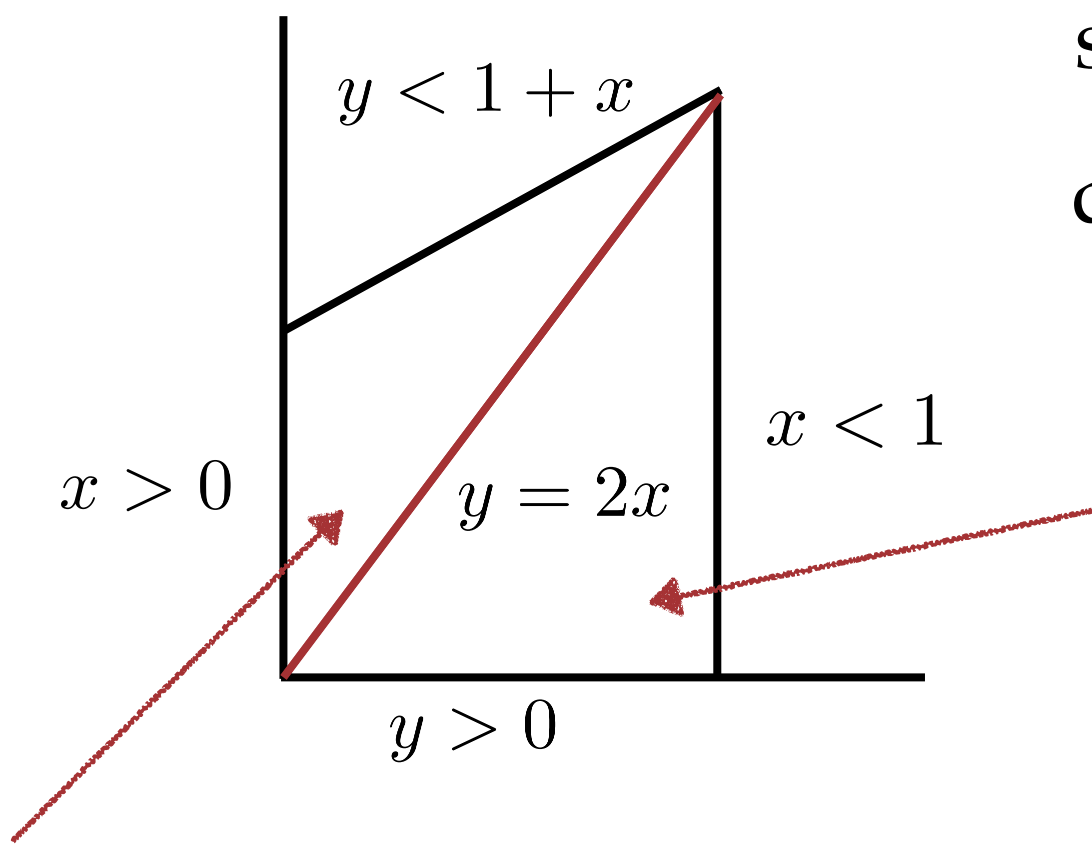


spurious line: y=2x

divides the space into two triangles

POSITIVE GEOMETRY

Triangulation



spurious line: y=2x

divides the space into two triangles

$$\Omega_2 = -\frac{2dx\,dy}{(x-1)y(y-2x)}$$

$$\Omega_1 = \frac{dx\,dy}{x(y-x-1)(y-2x)}$$

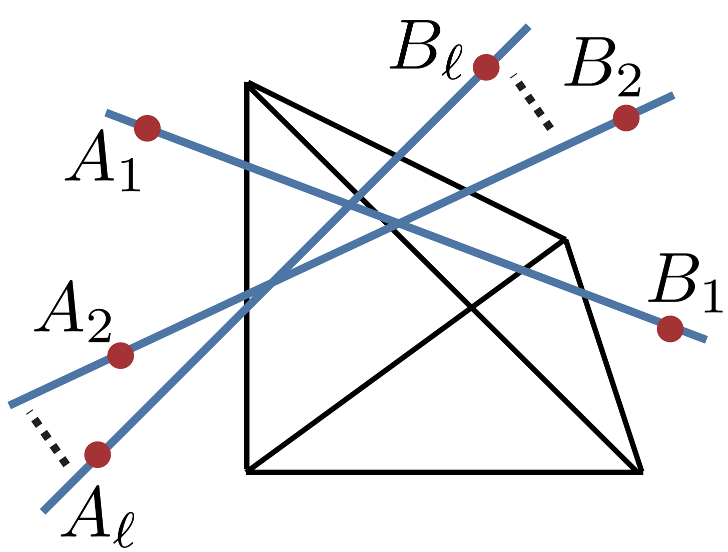
$$\boxed{\Omega = \Omega_1 + \Omega_2 = \frac{(1+x)dx\,dy}{xy(x-1)(y-x-1)}}$$

spurious pole cancels

FOUR-POINT AMPLITUHEDRON

(Arkani-Hamed, JT 2013) (Arkani-Hamed, Thomas, JT 2017)

Geometry of ℓ lines in $\mathbb{P}^3$



each line $D_j = \begin{pmatrix} 1 & x_j & 0 & -y_j \\ 0 & w_j & 1 & z_j \end{pmatrix}$

Definition of the L-loop space:

$$x_j, y_j, w_j, z_j > 0$$

$$(x_j - x_k)(z_j - z_k) + (w_j - w_k)(y_j - y_k) < 0$$

Triangulation: divide the geometry into "simplices" -- very hard

Known triangulation: loop BCFW recursion relations

(Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, JT 2010) (Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, JT 2012)

(Galashin, in progress)

A… ON

Geo…

$-y_j$

z_j

…e:

$y_k) < 0$

Tria…

Kno…

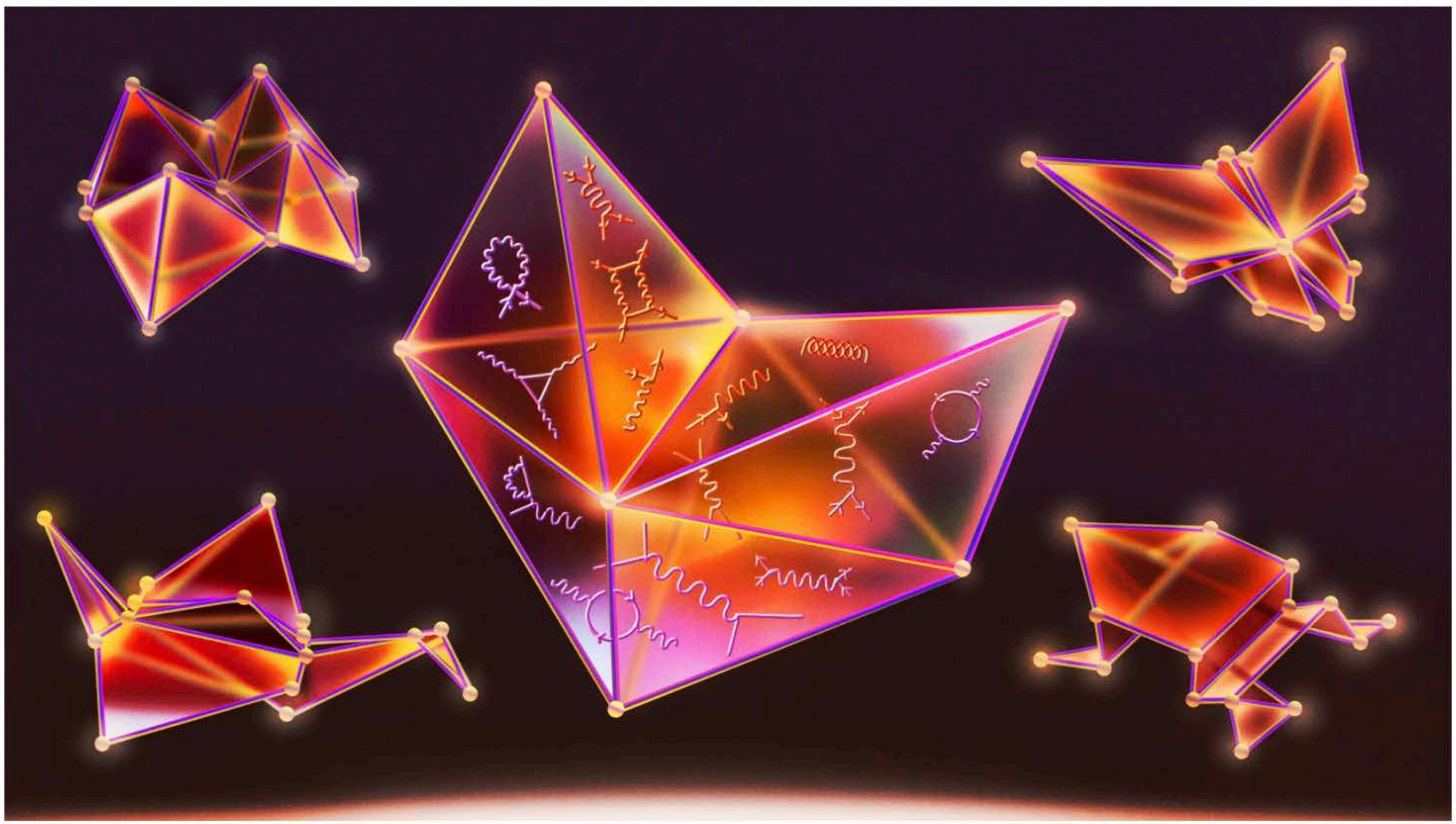
(Arkani-… …ikov, JT 2012)

MATHEMATICAL PHYSICS

Origami Patterns Solve a Major Physics Riddle

4

The amplituhedron, a shape at the heart of particle physics, appears to be deeply connected to the mathematics of paper folding.



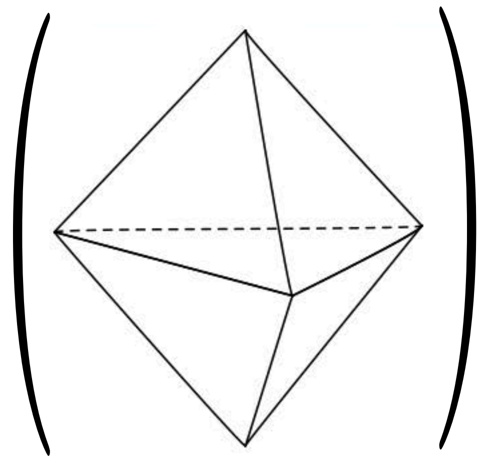
NEGATIVE GEOMETRIES

(Arkani-Hamed, Henn, JT 2021)

AMPLITUHEDRON

General story:

$$\mathcal{I}_n^{\ell-\mathrm{loop}} = \Omega$$



Integrand always well-defined:
nice rational function

Then later we would like to integrate:

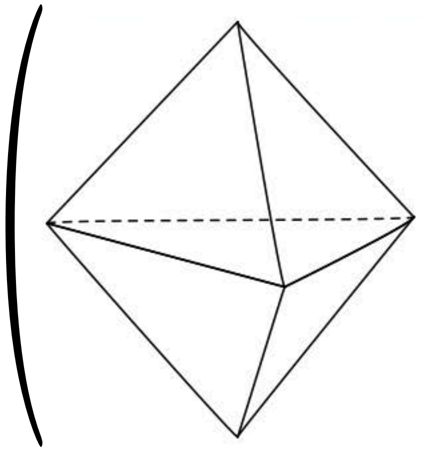
$$A_n^{\ell-\mathrm{loop}} = \int \mathcal{I}_n^{\ell-\mathrm{loop}} = \infty$$

negative geometries

this talk

AMPLITUHEDRON

General story:

$\mathcal{I}_n^{\ell-\mathrm{loop}} = \Omega\Big($  $\Big)$

Integrand always well-defined:
nice rational function

Then later we would like to integrate:

$$A_n^{\ell-\mathrm{loop}} = \int \mathcal{I}_n^{\ell-\mathrm{loop}} = \infty$$

negative geometries

this talk

deformed Amplituhedron

not in this talk: work with Nima Arkani-Hamed, Johannes Henn, Wojciech Flieger, Anders Schreiber, Melvyn Nabavi

AMPLITUHEDRON

General story:

$$\mathcal{I}_n^{\ell-\text{loop}} = \Omega\Bigg($$

always well-defined:

ational function

Then later we would

$$A_n^{\ell-\text{loop}} = \int \mathcal{I}_n^{\ell-}$$

See Melvyn's Gong show

tive geometries

this talk

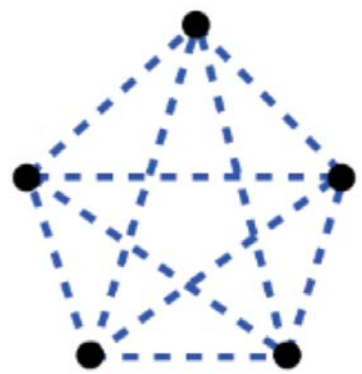
med Amplituhedron

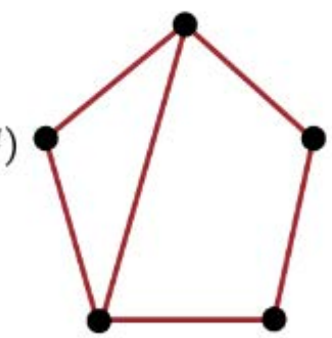
: work with Nima Arkani-Hamed, Johannes Henn, Wojciech Flieger, Anders Schreiber, Melvyn Nabavi

NEGATIVE GEOMETRIES

(Arkani-Hamed, Henn, JT, 2021)

Cousins of the Amplituhedron: **negative geometries**

$\Omega_L =$ 

$\widetilde{\Omega}_L = \sum_G (-1)^{E(G)}$ 

single Amplituhedron geometry for the **amplitude**

sum over geometries for the **logarithm of the amplitude**

- **Each graph**: positive geometry given by inequalities
- **Vertex** = loop momentum: "one-loop Amplituhedron"

$$\langle AB_i 12\rangle, \langle AB_i 23\rangle, \langle AB_i 34\rangle, \langle AB_i 14\rangle > 0, \quad \langle AB_i 13\rangle, \langle AB_i 24\rangle < 0$$

- **Edges** = "quadratic" constraints

  blue: $\langle AB_i AB_j\rangle > 0$ red: $\langle AB_i AB_j\rangle < 0$

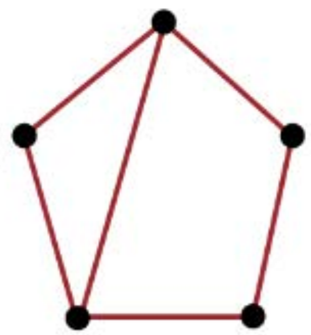
fewer edges = simpler geometry

- Differential form for each geometry = integrand

NEGATIVE GEOMETRIES

(Arkani-Hamed, Henn, JT, 2021)

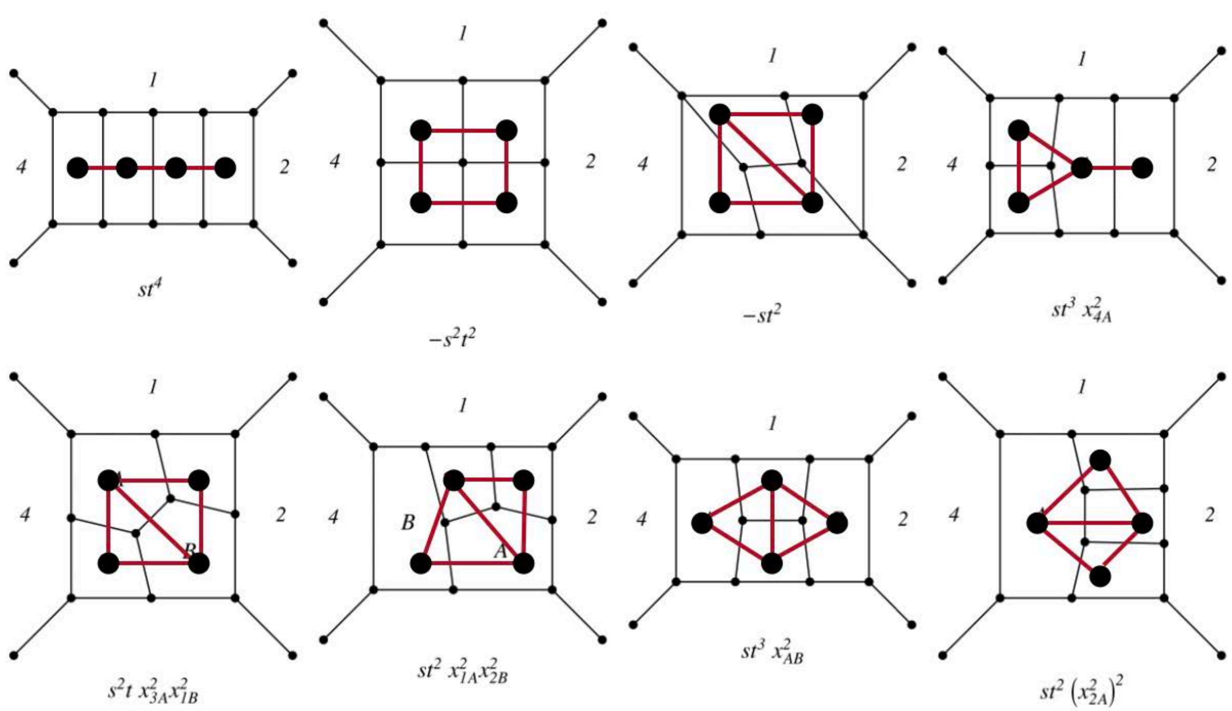
Individual negative geometries: no physical meaning



- "Basis" for the logarithm of the amplitude
- Designed to have correct IR properties

integration gives $\dfrac{1}{\epsilon^2}$ divergence

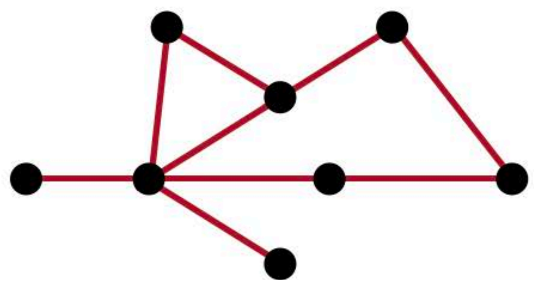
never true with standard diagrams



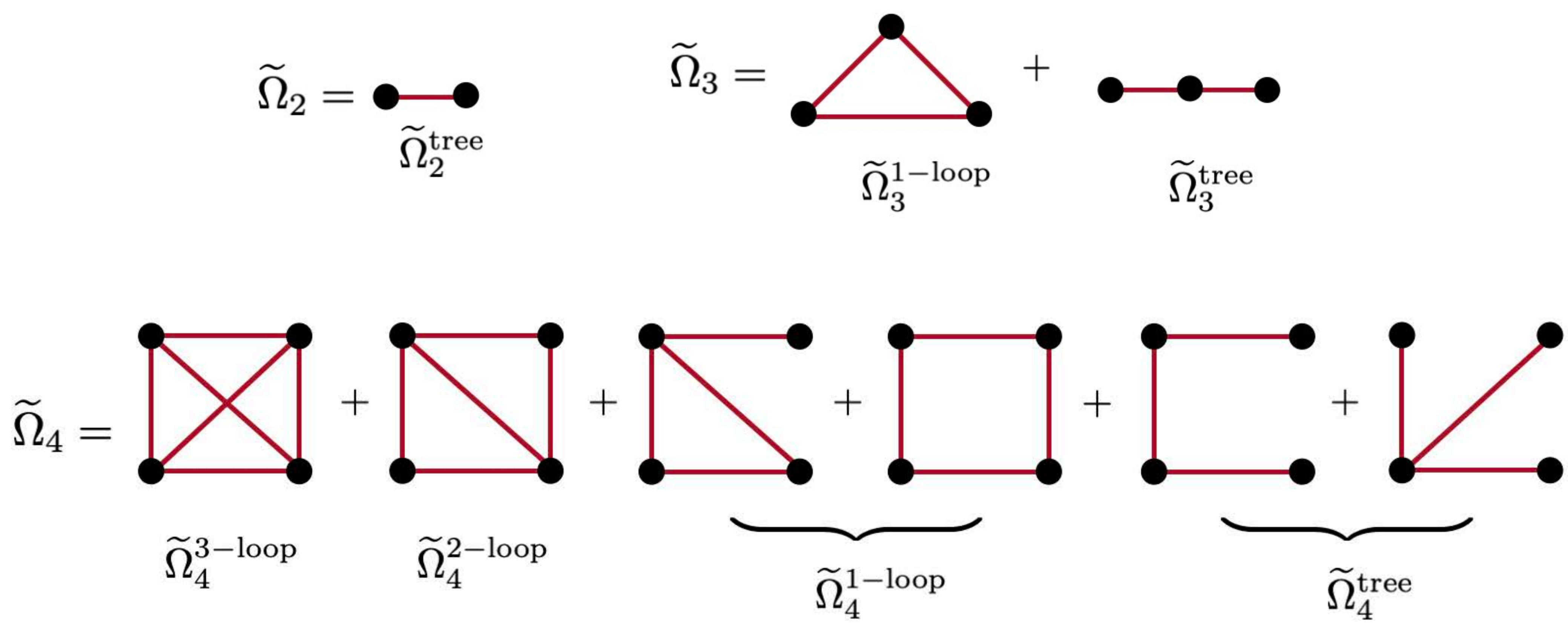
Very rough relation between geometries and diagrams: internal propagators = quadratic inequalities in the geometry

LOOPS OF LOOPS (CYCLES)

General Amplituhedron geometry

$\Omega_G^{\text{tree}} =$  $= \dfrac{d\mu \cdot \mathcal{N}_G}{\prod_i D_i \cdot \prod_{\text{links}} \langle AB_i AB_j \rangle}$

Natural hierarchy of geometries: more "loops" = more complicated



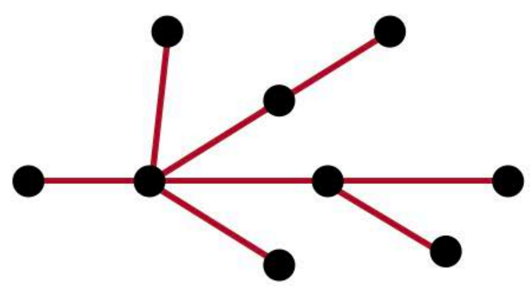
TREE-LEVEL / ZERO-CYCLE

(Arkani-Hamed, Henn, JT, 2021)

Tree-level approximation: only keep geometries with tree graphs

We found a closed form for the numerator of any tree graph!

$$\Omega_G^{\text{tree}} =$$



$$= \frac{d\mu \cdot \mathcal{N}_G}{\prod_i D_i \cdot \prod_{\text{links}} \langle AB_i AB_j \rangle}$$

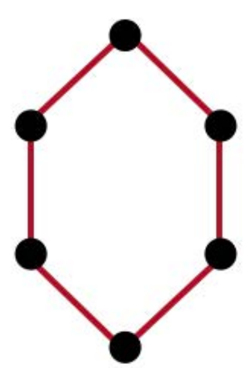
The numerator takes a factorized form

$$\mathcal{N}_G = \langle 1234 \rangle^{L+1} \times \prod_{\text{links}} N_{ij}^{(-)}$$

of the 2-loop numerators

$$N_{ij}^{(-)} = \langle AB_i 13 \rangle \langle AB_j 24 \rangle + \langle AB_i 24 \rangle \langle AB_j 13 \rangle$$

Same formula does not hold for a loop graph



$$\mathcal{N}_G = \langle 1234 \rangle^{L+1} \times \prod_{\text{links}} N_{ij}^{(-)}$$

satisfies many consistency constraints but not all (vanishing on double pole)

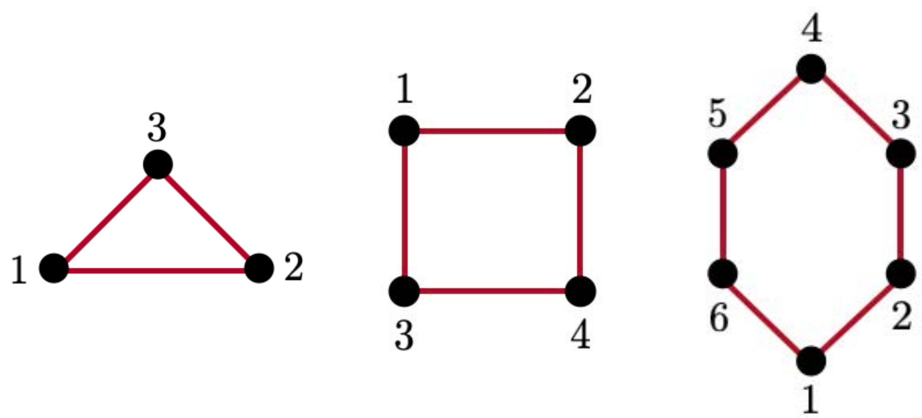
Need to find a specific correction which does not spoil any cuts we already matched

ONE-CYCLE

(Brown, Oktem, Paranjape, JT, 2023)

One-cycle: we found a numerator for a general one-cycle graph

- First step: find an integrand for a graph with a closed loop



Write the numerator as

$$\mathcal{N}_G = \prod_{\text{links}} N_{ij}^{(-)} + R_{\text{loop}}$$

extremely constrained,
write ansatz of all terms

For example: for 3-cycle we have two terms

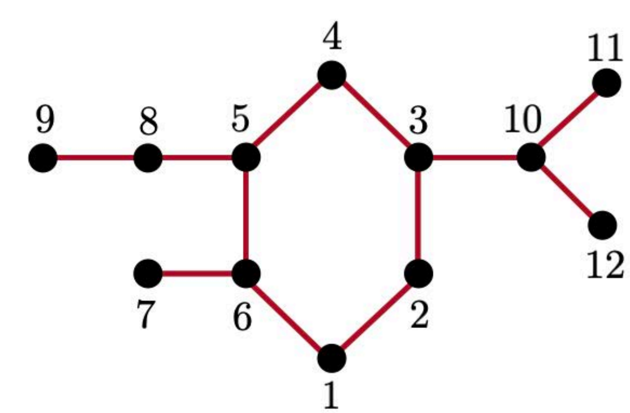
$$R^{(3)}_{\text{loop}} = c_1\{\langle AB_1 12\rangle\langle AB_1 34\rangle\langle AB_2 12\rangle\langle AB_2 34\rangle\langle AB_3 12\rangle\langle AB_3 34\rangle + \dots\}$$
$$+\, c_2\{\langle AB_1 12\rangle\langle AB_1 34\rangle(\langle AB_2 12\rangle\langle AB_3 34\rangle + \langle AB_3 12\rangle\langle AB_2 34\rangle)(\langle AB_2 13\rangle\langle AB_3 24\rangle + \langle AB_3 13\rangle\langle AB_2 24\rangle) + \dots\}$$

Solve from the double pole cancelation: $c_1 = 4,\ c_2 = -1$

Generalized to any cycle

- Second step: add tree branches

$$\mathcal{N}_G = \left(\prod_{\text{loop links}} N_{ij}^{(-)} + R_{\text{loop}}\right) \times \prod_{\text{other links}} N_{ij}^{(-)}$$



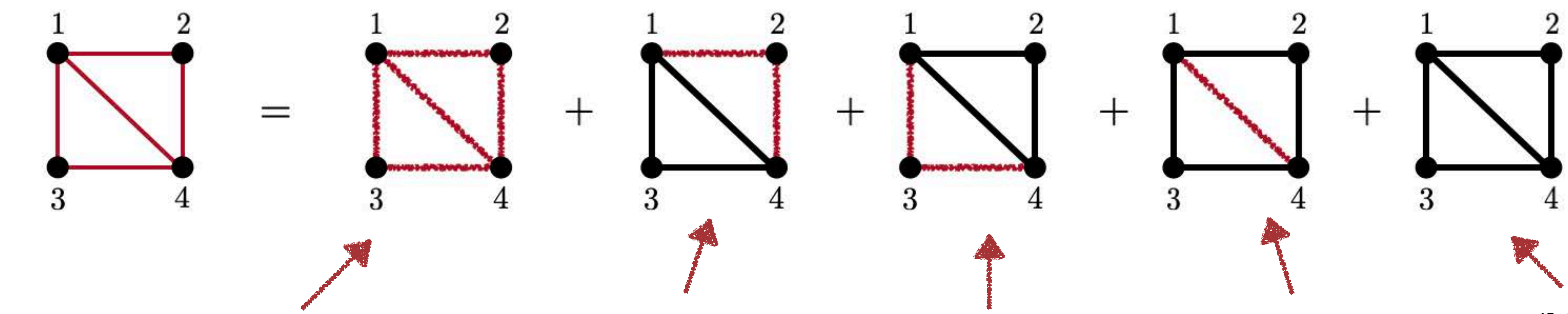
Solved for any one-cycle graph!

HIGHER CYCLES

We do not know how to find the numerator for a general higher-cycle graph
= as hard as solving the four-point problem completely

Useful strategy: **loop decomposition**

denote: $= N_{ij}^{(-)}$



$N_{12}^{(-)} N_{24}^{(-)} N_{34}^{(-)} N_{13}^{(-)} N_{14}^{(-)}$ $\quad$ $R_{\text{loop}}^{(134)} \times N_{12}^{(-)} N_{24}^{(-)}$ $\quad$ $R_{\text{loop}}^{(124)} \times N_{13}^{(-)} N_{34}^{(-)}$ $\quad$ $R_{\text{loop}}^{(1243)} \times N_{14}^{(-)}$ $\quad$ new two-loop object

tree objects

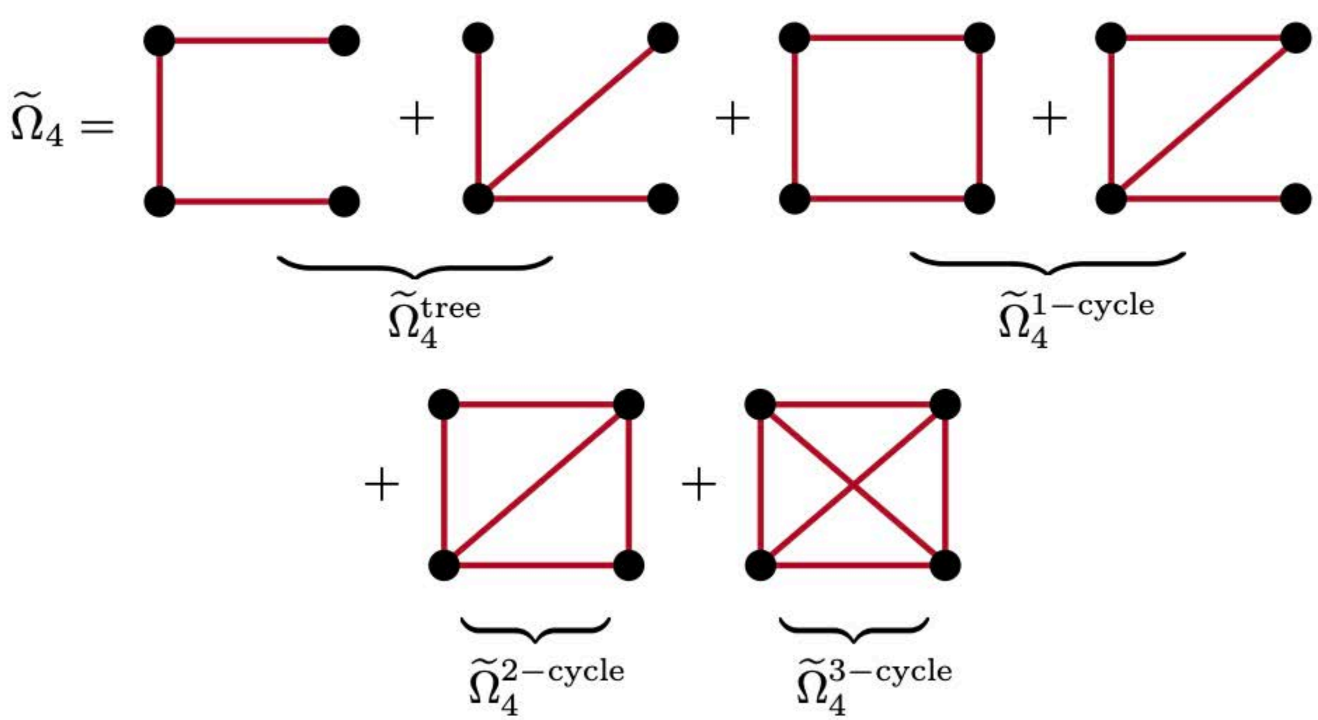
$\underbrace{R_{\text{loop}}^{(134)} \times N_{12}^{(-)} N_{24}^{(-)} \quad R_{\text{loop}}^{(124)} \times N_{13}^{(-)} N_{34}^{(-)} \quad R_{\text{loop}}^{(1243)} \times N_{14}^{(-)}}_{\text{one-loop (and tree) objects}}$

Only few cases solved at the moment, new ideas needed

FULL FOUR-LOOP RESULT

(Dixon, Oktem, Paranjape, JT, Xu, Zhang, to appear)

Using a hybrid Amplituhedron / generalized unitarity method we can construct the integrand for all L=4 negative geometries



But this is not really a power of this method (unless we solve everything): as mentioned data available up to 12-loops

IR FINITE FUNCTION
FROM NEGATIVE GEOMETRIES

INTEGRATING GEOMETRIES

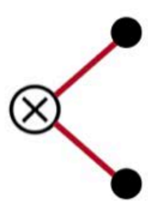
(Arkani-Hamed, Henn, JT, 2021)

Wilson loop: freeze one of the loops and integrate over others

$$\boxed{\mathcal{F}_{L-1}(z) \;\text{ from }\; \widetilde{\Omega}_L}$$

Simplest one-loop result:  $= \left[\pi^2 + \log^2 z\right]$

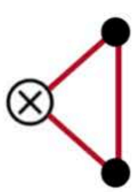
Three-loop result: three different contributions



$$= -\frac{1}{2}\left[\pi^2 + \log^2 z\right]^2$$

$$= -\frac{1}{12}\left[\pi^2 + \log^2 z\right] \times \left[5\pi^2 + \log^2 z\right]$$

Tree graphs



$$\begin{aligned} &= -\frac{1}{6}\log^4 z + \log^2 z\left[-\frac{2}{3}\mathrm{Li}_2\left(\frac{1}{z+1}\right) - \frac{2}{3}\mathrm{Li}_2\left(\frac{z}{z+1}\right) + \frac{\pi^2}{9}\right] \\ &+ \log z\left[4\mathrm{Li}_3\left(\frac{z}{z+1}\right) - 4\mathrm{Li}_3\left(\frac{1}{z+1}\right)\right] - \frac{2}{3}\left[\mathrm{Li}_2\left(\frac{1}{z+1}\right) + \mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{\pi^2}{6}\right]^2 \\ &- \frac{8}{3}\pi^2\left[\mathrm{Li}_2\left(\frac{1}{z+1}\right) + \mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{\pi^2}{6}\right] - 8\mathrm{Li}_4\left(\frac{1}{z+1}\right) - 8\mathrm{Li}_4\left(\frac{z}{z+1}\right) - \frac{\pi^4}{18} \end{aligned}$$

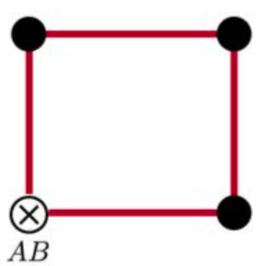
One-loop graph

Same loop order, tree graphs are simpler - consistent with simple integrand

THREE-LOOP WILSON LOOP

(Dixon, Oktem, Paranjape, JT, Xu, Zhang, to appear)

All contributions calculated:



$$
\begin{aligned}
I_4^{1-\mathrm{cyc}}(x) = 96\Big(& -8\zeta(3)H_{-2,0} + 4\zeta(3)H_{-1,0,0} + 2\zeta(3)H_{0,0,0} + \frac{1}{2}\pi^2 H_{-3,0} + \pi^2 H_{-2,-2} - \frac{5}{2}\pi^2 H_{-1,-3} \\
& + \frac{1}{8}\pi^4 H_{-1,-1} + \frac{11}{45}\pi^4 H_{-1,0} + \frac{5}{72}\pi^4 H_{0,0} + \frac{3}{2}H_{-4,0,0} - \frac{2}{3}\pi^2 H_{-2,-1,0} - \frac{5}{12}\pi^2 H_{-2,0,0} - \frac{1}{3}\pi^2 H_{-1,-2,0} \\
& + 3H_{-3,0,0,0} + 2H_{-2,-2,0,0} - 5H_{-1,-3,0,0} + \frac{1}{2}\pi^2 H_{-1,-1,0,0} + \frac{7}{6}\pi^2 H_{-1,0,0,0} + \frac{1}{6}\pi^2 H_{0,0,0,0} \\
& - 4H_{-2,-1,0,0,0} - \frac{1}{2}H_{-2,0,0,0,0} - 2H_{-1,-2,0,0,0} + 3H_{-1,-1,0,0,0,0} + 2H_{-1,0,0,0,0,0} + 3H_{0,0,0,0,0,0} \\
& - \frac{1}{3}\pi^2 H_{-1}\zeta(3) + \frac{3}{2}\pi^2 H_0\zeta(3) - 19H_{-1}\zeta(5) + \frac{31H_0\zeta(5)}{2} + \frac{3}{4}\pi^2 H_{-4} - \frac{43}{144}\pi^4 H_{-2} + \frac{505\pi^6}{9072} + 11\zeta_3^2\Big)
\end{aligned}
$$

all HPLs (harmonic polylogs) of variable z

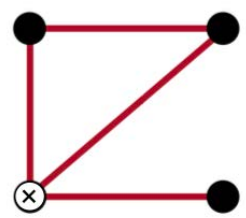
Symbol:

$$
\begin{aligned}
&\mathrm{Symbol}(I_4^{1-\mathrm{cyc}}) = \\
&96\Big(3\,\mathrm{SB}(x,x,x,x,x,x) + 2\,\mathrm{SB}(x,x,x,x,x,1{+}x) - \frac{1}{2}\,\mathrm{SB}(x,x,x,x,1{+}x,x) + 3\,\mathrm{SB}(x,x,x,x,1{+}x,1{+}x) \\
&+ 3\,\mathrm{SB}(x,x,x,1{+}x,x,x) - 2\,\mathrm{SB}(x,x,x,1{+}x,x,1{+}x) - 4\,\mathrm{SB}(x,x,x,1{+}x,1{+}x,x) \\
&+ \frac{3}{2}\,\mathrm{SB}(x,x,1{+}x,x,x,x) - 5\,\mathrm{SB}(x,x,1{+}x,x,x,1{+}x) + 2\,\mathrm{SB}(x,x,1{+}x,x,1{+}x,x)\Big)\,.
\end{aligned}
$$

THREE-LOOP WILSON LOOP

(Dixon, Oktem, Paranjape, JT, Xu, Zhang, to appear)

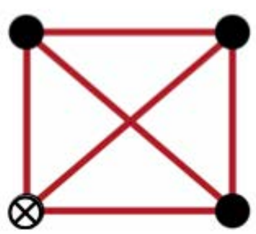
All contributions calculated:



$$\begin{aligned}\mathrm{Symbol}(I_{3,3}) = 96\, \big(&15\,\mathrm{SB}(x,x,x,x,x,x) + 10\,\mathrm{SB}(x,x,x,x,x,1+x) + 10\,\mathrm{SB}(x,x,x,x,1+x,x)\\ &- 12\,\mathrm{SB}(x,x,x,x,1+x,1+x) + 7\,\mathrm{SB}(x,x,x,1+x,x,x) - 6\,\mathrm{SB}(x,x,x,1+x,x,1+x)\\ &- 6\,\mathrm{SB}(x,x,x,1+x,1+x,x) + 3\,\mathrm{SB}(x,x,1+x,x,x,x) - 2\,\mathrm{SB}(x,x,1+x,x,x,1+x)\\ &- 2\,\mathrm{SB}(x,x,1+x,x,1+x,x) - 2\,\mathrm{SB}(x,x,1+x,1+x,x,x)\big)\,,\end{aligned}$$

More cycles: more complicated symbol

spurious symbol letters $b(4) = \dfrac{1+i\sqrt{x}}{1-i\sqrt{x}}$



$$\begin{aligned}&\mathrm{Symbol}(I^{3-\mathrm{cyc}}) =\\ &144\,\big(4\,\mathrm{SB}(x,x,x,x,b(4),b(4)) + \mathrm{SB}(x,x,x,b(4),x,b(4)) + 3\,\mathrm{SB}(x,x,x,b(4),b(4),x) + 2\,\mathrm{SB}(x,x,x,x,x,x)\\ &+ \mathrm{SB}(x,x,x,x,x,1+x) + 11\,\mathrm{SB}(x,x,x,x,1+x,x) - 10\,\mathrm{SB}(x,x,x,x,1+x,1+x)\\ &+ 9\,\mathrm{SB}(x,x,x,1+x,x,x) + 5\,\mathrm{SB}(x,x,x,1+x,x,1+x) - 7\,\mathrm{SB}(x,x,x,1+x,1+x,x)\\ &- 4\,\mathrm{SB}(x,x,x,1+x,1+x,1+x) + 5\,\mathrm{SB}(x,x,1+x,x,x,x) + 7\,\mathrm{SB}(x,x,1+x,x,x,1+x)\\ &- \mathrm{SB}(x,x,1+x,x,1+x,x) - 4\,\mathrm{SB}(x,x,1+x,x,1+x,1+x) - 12\,\mathrm{SB}(x,x,1+x,1+x,x,x)\\ &- 4\,\mathrm{SB}(x,x,1+x,1+x,x,1+x) - 4\,\mathrm{SB}(x,x,1+x,1+x,1+x,x)\\ &+ 8\,\mathrm{SB}(x,x,1+x,1+x,1+x,1+x)\big)\,.\end{aligned}$$

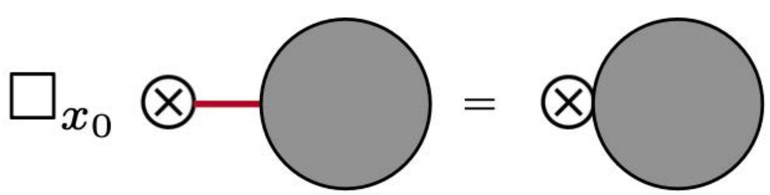
It would be great to link it to spurious cuts of the integrand

HIGHER LOOPS

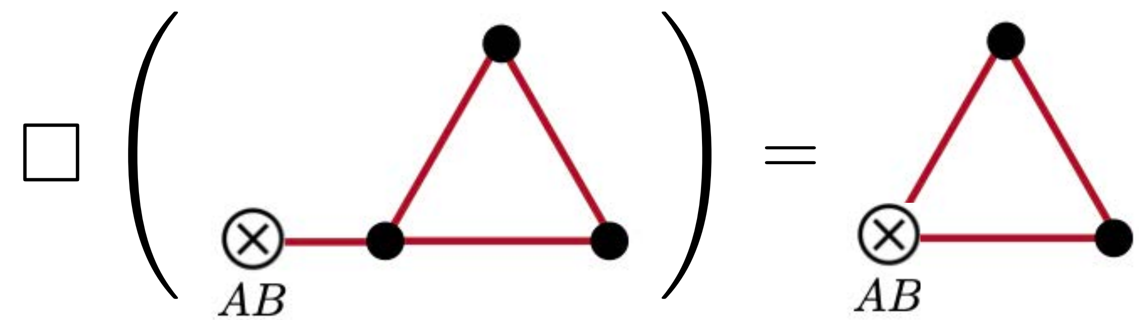
(Dixon, Oktem, Paranjape, JT, Xu, Zhang, to appear)

At higher loops, the integration technology does not exist

For certain geometries, we can use a differential equation trick:

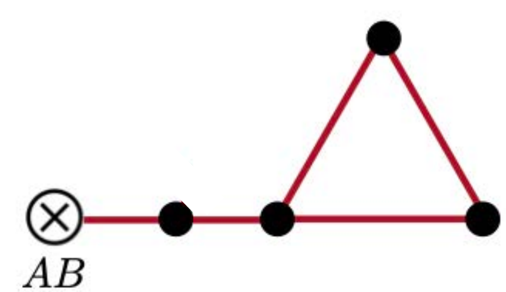


for example:



This allows us to solve classes of geometries:

$$
\begin{aligned}
I_{3,2}(z) =& 8H_{0,0,0,0,0,0,0,0,0}(z) + 8H_{0,0,0,0,-1,0,0,0}(z) - 16H_{0,0,0,0,-1,-1,0,0}(z) + 8H_{0,0,0,0,-2,0,0}(z) \\
& - 8\zeta_3(2H_{0,0,0,0,-1}(z) - H_{0,0,0,0,0}(z)) + 4\pi^2(H_{0,0,0,0,-1,0}(z) - 2H_{0,0,0,0,-1,-1}(z) \\
& + H_{0,0,0,0,-2}(z)) + \frac{13\pi^4}{1080}\log^4(z) + \frac{C_{3,1}}{6}\log^3(z) + \frac{D_{3,1}}{2}\log(z)^2 + C_{3,2}\log(z) + D_{3,2}\,,
\end{aligned}
$$



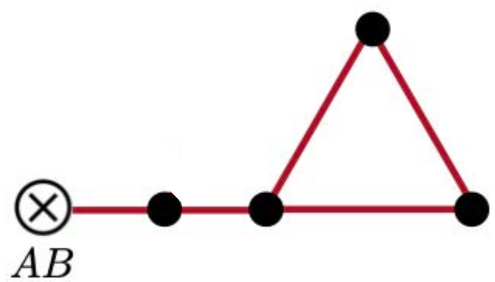
$$
C_{3,2} = -\frac{8}{3}\pi^2\zeta_{3,2} - 16\zeta_{5,2} + \frac{127\pi^4\zeta_3}{45} + \frac{8\pi^2\zeta_5}{3} - 168\zeta_7,
$$

$$
D_{3,2} = 8\pi^2\zeta_{3,3} + 16\zeta_{5,3} + 80\zeta_{6,2} - 8\zeta_3\left(\pi^2\zeta_3 + 32\zeta_5\right) + \frac{6779\pi^8}{75600}.
$$

GAMMA CUSP CONTRIBUTION

(Dixon, Oktem, Paranjape, JT, Xu, Zhang, to appear)

We can extract the contribution to gamma cusp from full function



$$= \frac{g^{10}}{10}\left(16\zeta_{5,3} + 96\zeta_{6,2} - \frac{1}{3}8\pi^2\zeta_3^2 - 352\zeta_5\zeta_3 + \frac{34\pi^8}{567}\right)$$

Multiple Zeta Values (MZV) $\quad \zeta_{a,b} = \sum_{n_1>n_2>0} \frac{1}{n_1^a n_2^b}$

Simple analysis suggest that higher cycles: $\zeta_{a_1,a_2,\ldots,a_{\ell+1}}$

They all cancel in the Wilson loop!

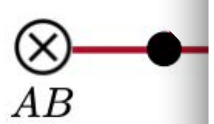
Integrability? How is this related to the integrand/geometry?

This is something totally hidden in the negative geometry expansion
(also likely in Feynman diagrams -- there are no data)

GAMMA CUSP CONTRIBUTION

ear)

We can extract the ... sp from full function



$$\ldots \frac{1}{3} 8\pi^2\zeta_3^2 - 352\zeta_5\zeta_3 + \frac{34\pi^8}{567})$$

$$\zeta_{a,b} = \sum_{n_1>n_2>0} \frac{1}{n_1^a n_2^b}$$

Simple analysis su ... $1, a_2, \ldots, a_{\ell+1}$

They ... oop!

Integrability? Ho ... d/geometry?

This is something ... ometry expansion
(also likely in Feynman diagrams -- there are no data)

See Umut's talk
in Gong show

RESUMMATION

(Dixon, Oktem, Paranjape, JT, Xu, Zhang, to appear)

RESUMMATION

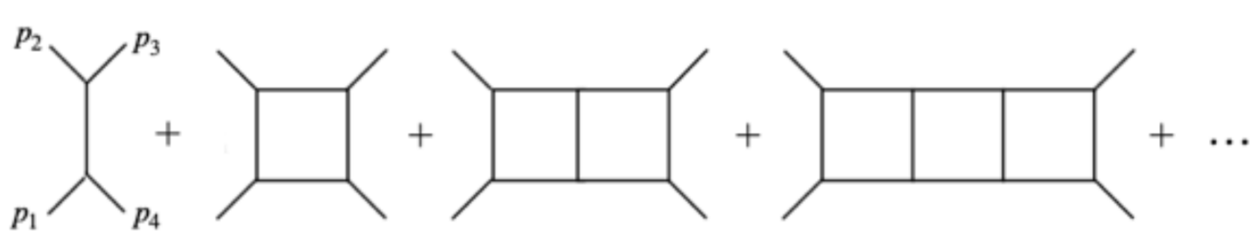
Planar N=4 SYM is special: no non-perturbative contributions

$$\mathcal{M}_4^{\text{exact}} = \sum_{L=0}^{\infty} g^{2L} \mathcal{M}_4^{(L)}$$

- approximate amplitude at each loop order and resum to all loops
- compare to strong coupling expansion (if available)

Only known (to me) example is the **ladder resummation**

(Broadhurst, Davydychev, 2010)



$\sim e^{-g} \quad \text{for } g \gg 1$

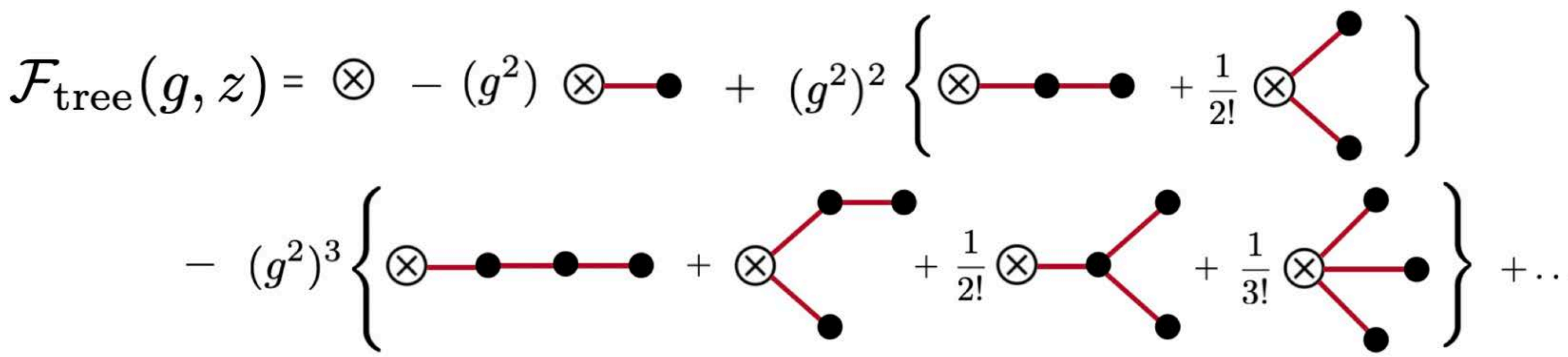
exponentially suppressed vs linear growth

We can try to resum our tree contributions (if possible to integrate all of them)

ZERO CYCLE CONTRIBUTIONS

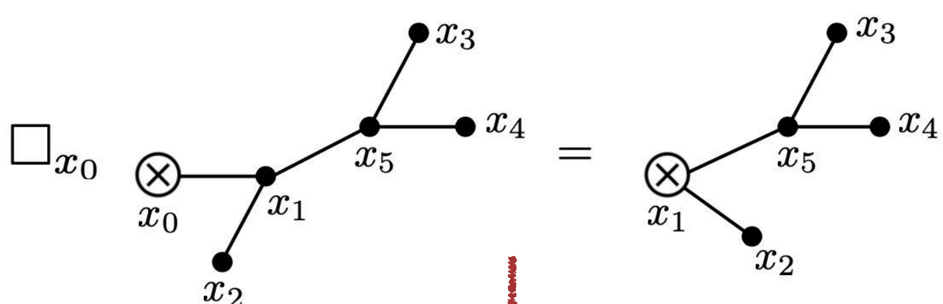
(Arkani-Hamed, Henn, JT, 2021)

Let us only consider "**tree" graphs** (no internal cycles)



Differential operator acting on the graphs (integrand)

box operator



Same operator does not work for loop graphs: search for its generalization

differential equation for the generating function

$$\mathcal{F}_{\text{tree}}(g,z) = e^{\mathcal{H}_{\text{tree}}(g,z)}$$

$$\frac{1}{2}(z\partial_z)^2\mathcal{H}_{\text{tree}}(g,z) + g^2 e^{\mathcal{H}_{\text{tree}}(g,z)} = 0$$

solve with boundary conditions

$$\boxed{\mathcal{F}_{\text{tree}}(g,z) = \frac{A^2}{g^2}\frac{z^A}{(z^A+1)^2}}$$

where $\quad \dfrac{A}{2g\cos\frac{\pi A}{2}} = 1$

STRONG COUPLING

(Arkani-Hamed, Henn, JT, 2021)

$$\boxed{\mathcal{F}_{\text{tree}}(g,z) = \frac{A^2}{g^2}\frac{z^A}{(z^A+1)^2}}$$

How good is this approximation to the exact result?

Naively, we expect it to be very bad — for infinite L vanishing part of diagrams contribute

Easy to expand at strong coupling:

$$\mathcal{F}_{\text{tree}}(g,z) = -\frac{z}{(1+z)^2} + \mathcal{O}\left(\frac{1}{g}\right)$$

misses the leading term

has 1/g expansion

For $\Gamma_{\text{cusp}}(g)$ we get even more surprising result:

$$\Gamma_{\text{tree}}(g) \to \begin{cases} 2g - \frac{3\log 2}{2\pi} + \cdots & \longrightarrow \text{exact} \\ \frac{8}{\pi}g - 1 + \ldots & \longrightarrow \text{our tree approximation} \end{cases}$$

qualitatively correct behavior at strong coupling

OPEN QUESTIONS

How do we reconstruct $\mathcal{F}(g,z) \sim g$ behavior at strong coupling?

And what about cusp anomalous dimension?

$$\Gamma_{\text{tree}}(g) \to g\left(\frac{8}{\pi} + \gamma^{(1)} + \gamma^{(2)} + \ldots\right) = 2g$$

$\simeq 2.55$

Is it just a small negative correction?
Do tree graphs dominate or are there massive cancelations between various orders?

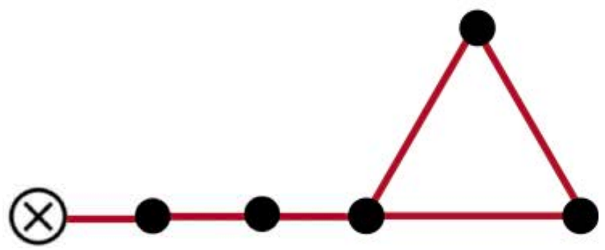
Can we **derive** BES equation from amplitudes?

Natural next step: **one-cycle resummation** --- too hard at the moment

TRIANGLE RESUMMATION

(Dixon, Oktem, Paranjape, JT, Xu, Zhang, to appear)

Consider classes of geometries starting with triangle:



they also satisfy the same differential equation with a source term

We have an explicit result for this sum (not very compact formula)

Gamma cusp calculation, strong coupling expansion

$$\gamma^{\text{cusp}}(g) \sim g^5$$

too strong but "partially summable"

We can now extend it to box being a "source" -- same feature

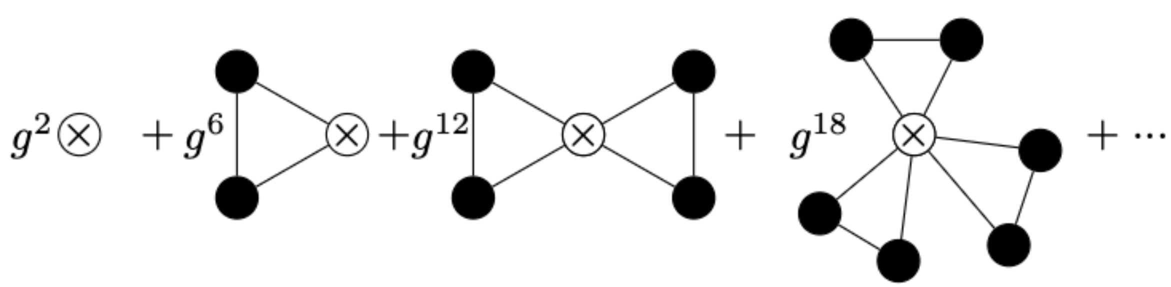
HOW TO RESUM?

Question: which negative geometries we can resum?

Gamma cusp: are there any which dominate in the computation?

The exact strong coupling behavior is something between what we have and a geometric series:

Geometric series of diagrams suppressed at strong coupling



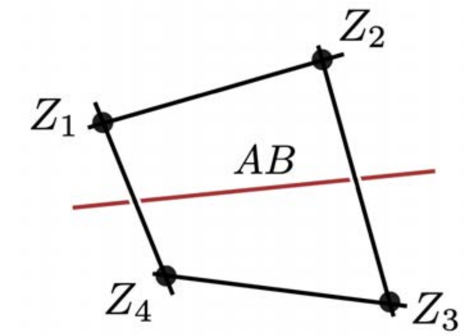
$$= \sum_{k=0}^{\infty} (g^6 \mathcal{F}_{\text{tri}}(z))^k = \frac{1}{1 - g^6\, \mathcal{F}_{\text{tri}}(z)} \sim \frac{1}{g^6}$$

In any case, this is a great playground -- nice resummation toys, something we do not have with normal diagrams

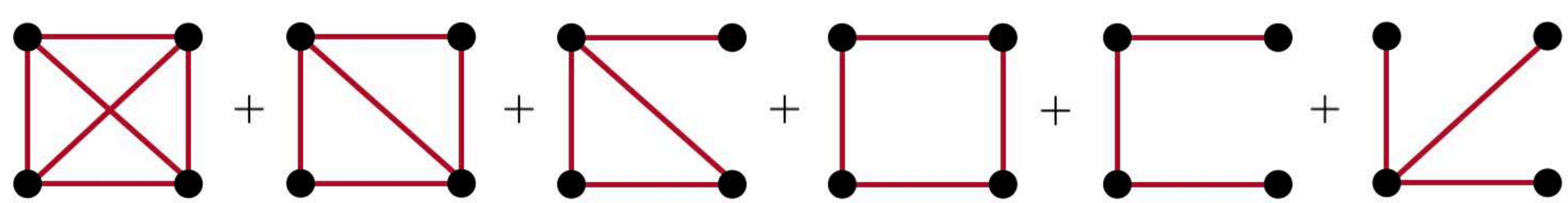
CONCLUSION

SOLVING N=4 SYM THEORY

Nice IR finite object (amplitude / Wilson loop):
new expansion using **negative geometries**



At **four-point**: function of a single cross-ratio, some data available at weak and strong coupling, our method: **negative geometries**



MZV in gamma cusp --> how do they cancel in gamma cusp?

integrability / geometry connection

Can we find a subset of negative geometries that dominate at strong coupling? Derive BES formula for gamma cusp?

THANK YOU!