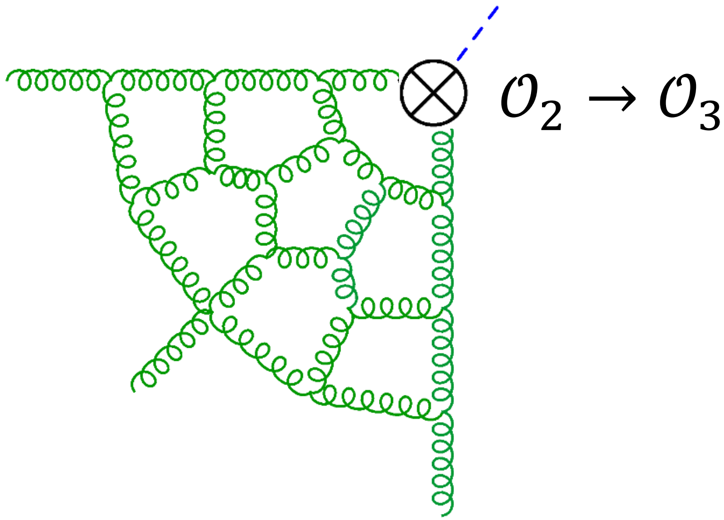
Another 8 Loop Form Factor in Planar N=4 SYM



LD, Z. Li to appear

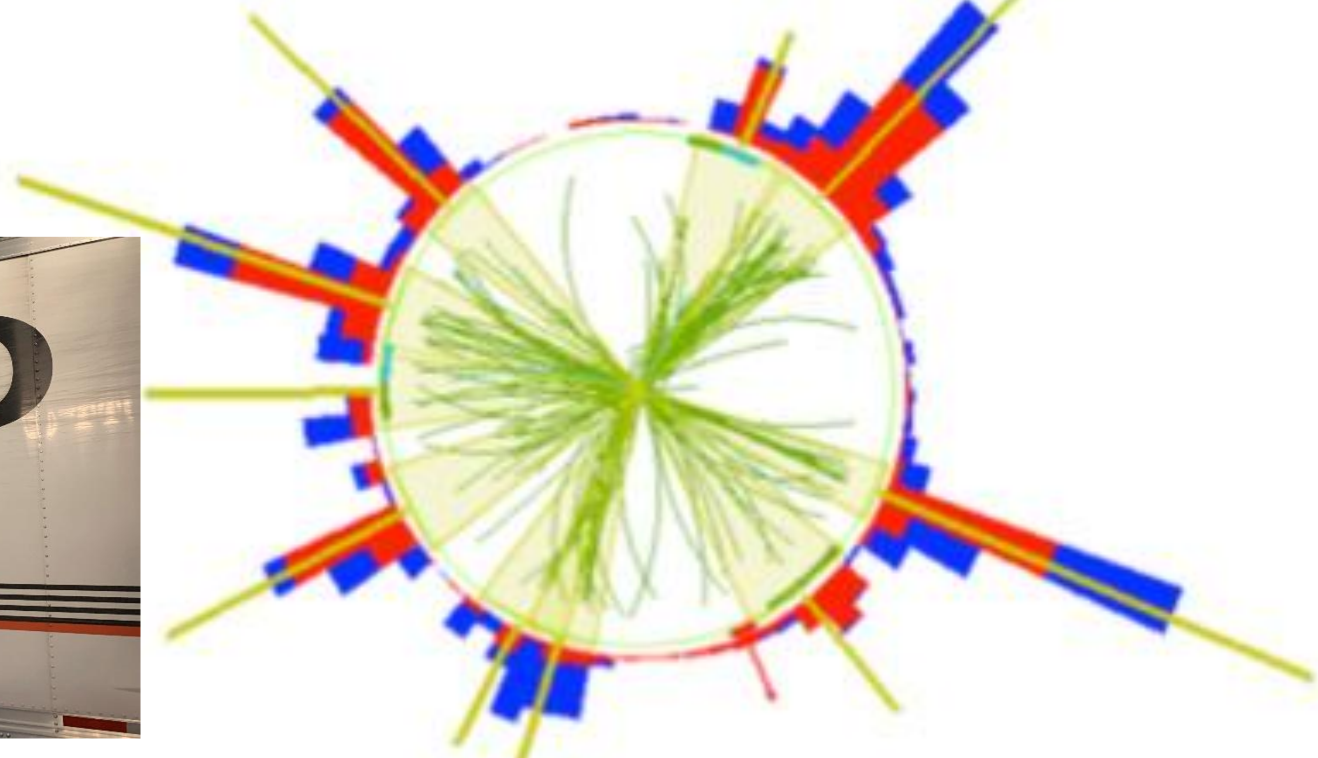
CAmps, UCLA

November 9, 2025

Lance Dixon (SLAC)

LHC is QCD Machine

Copious production of quarks and gluons, materialize as collimated jets of hadrons, predicted by Quantum Chromodynamics



- Confrontation between experiment and theory at high precision requires higher order corrections in the strong coupling α_s

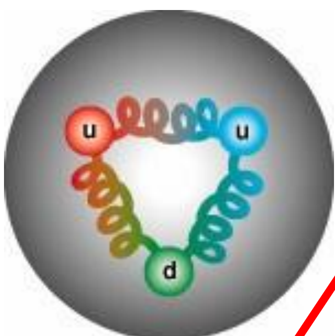
The Key of Asymptotic Freedom

- Gluons anti-screen charge: Interaction strength α_s for QCD

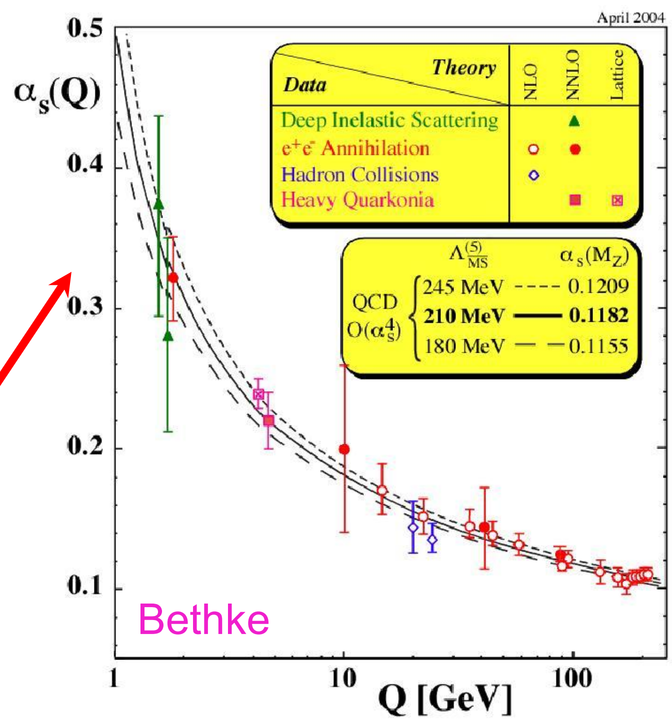
is small at short distances, so we can series expand in α_s

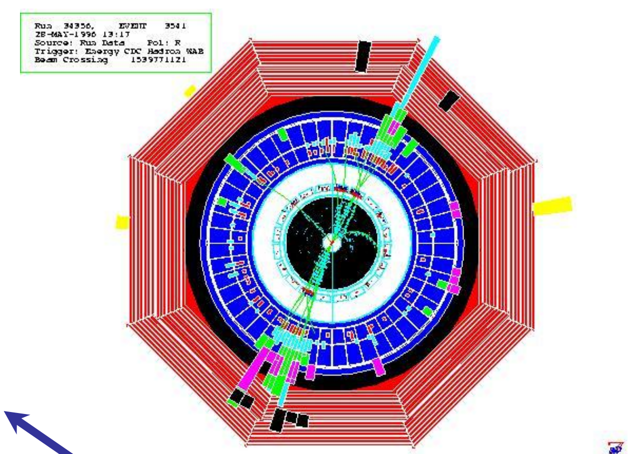
→ perturbation theory → Feynman diagrams

Gross, Wilczek, Politzer (1973)

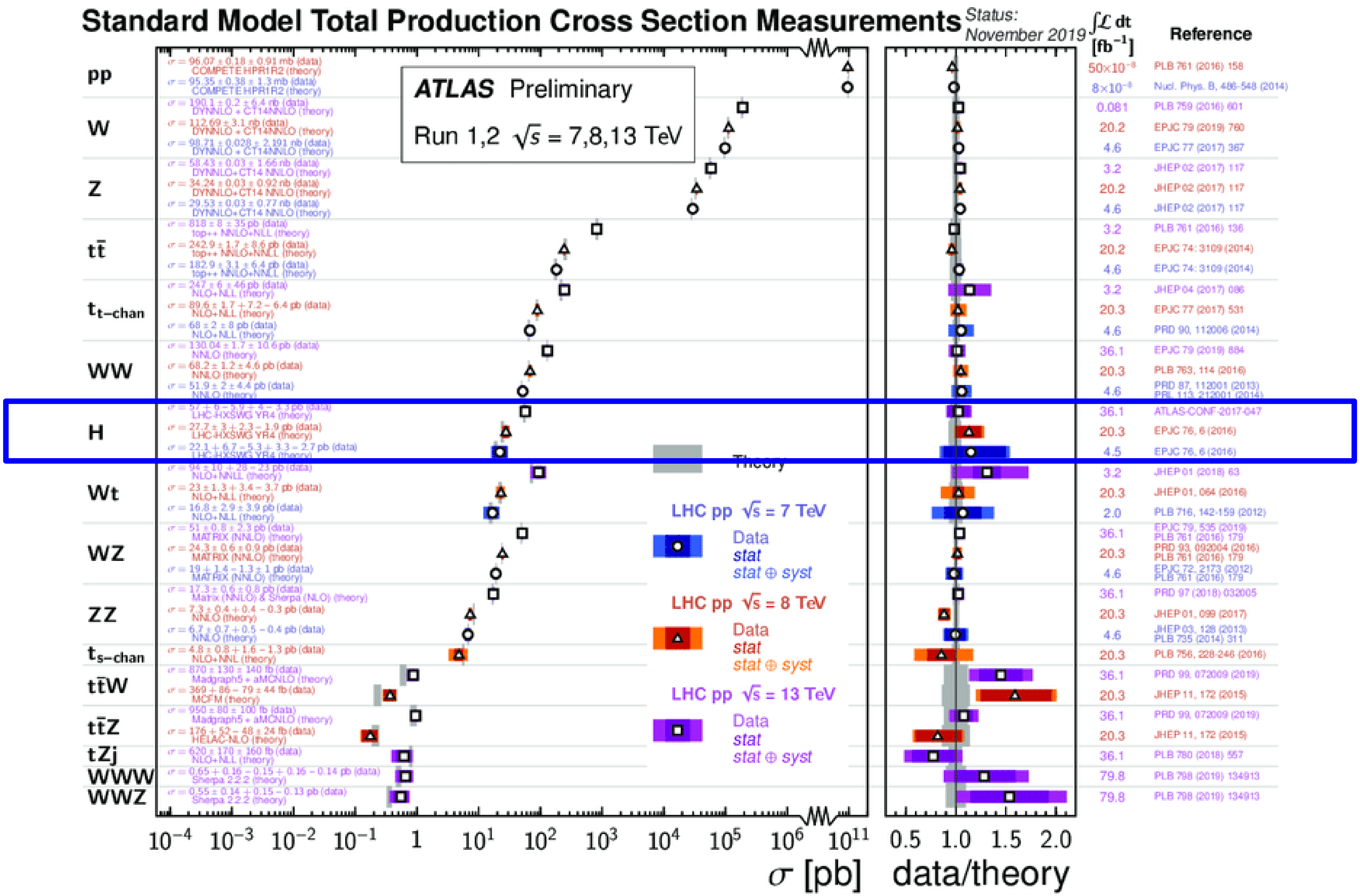


confining





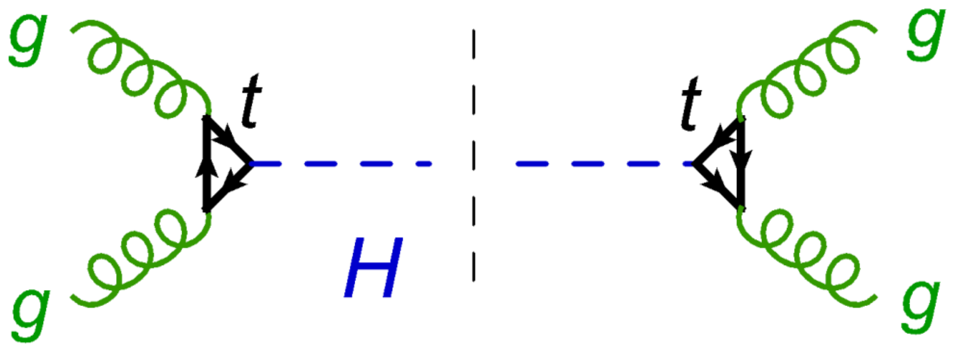
calculable


Standard Model Total Production Cross Section Measurements
Status: November 2019
ATLAS Preliminary
Run 1,2 $\sqrt{s}$ = 7,8,13 TeV
pp
W
Z
$t\bar{t}$
t_{t-chan}
WW
H
Wt
WZ
ZZ
t_{s-chan}
$t\bar{t}W$
$t\bar{t}Z$
tZj
WWW
WWZ
Theory
LHC pp $\sqrt{s}$ = 7 TeV
Data
stat
stat ⊕ syst
LHC pp $\sqrt{s}$ = 8 TeV
Data
stat
stat ⊕ syst
LHC pp $\sqrt{s}$ = 13 TeV
Data
stat
stat ⊕ syst
$\int\mathcal{L}$ dt [fb^{-1}]
Reference
σ [pb]
data/theory

Producing Higgs bosons at LHC

Higgs boson dominantly produced by **gluon fusion**, a quantum process at “one loop”, mediated by **top quark,** because ***t*** couples strongly to both gluons and Higgs

Leading Order (LO) cross section $= |\mathrm{one-loop\ amplitude}|^2$



- Since $\boldsymbol{2m_{top}} = 350\ \mathrm{GeV}$
  $\gg \boldsymbol{m_{Higgs}} = 125\ \mathrm{GeV}$,

interaction between gluons and Higgs is approximately local (mediated by an operator $HG^a_{\mu\nu}G^{\mu\nu\,a}$)

Perturbative Short-Distance Cross Section

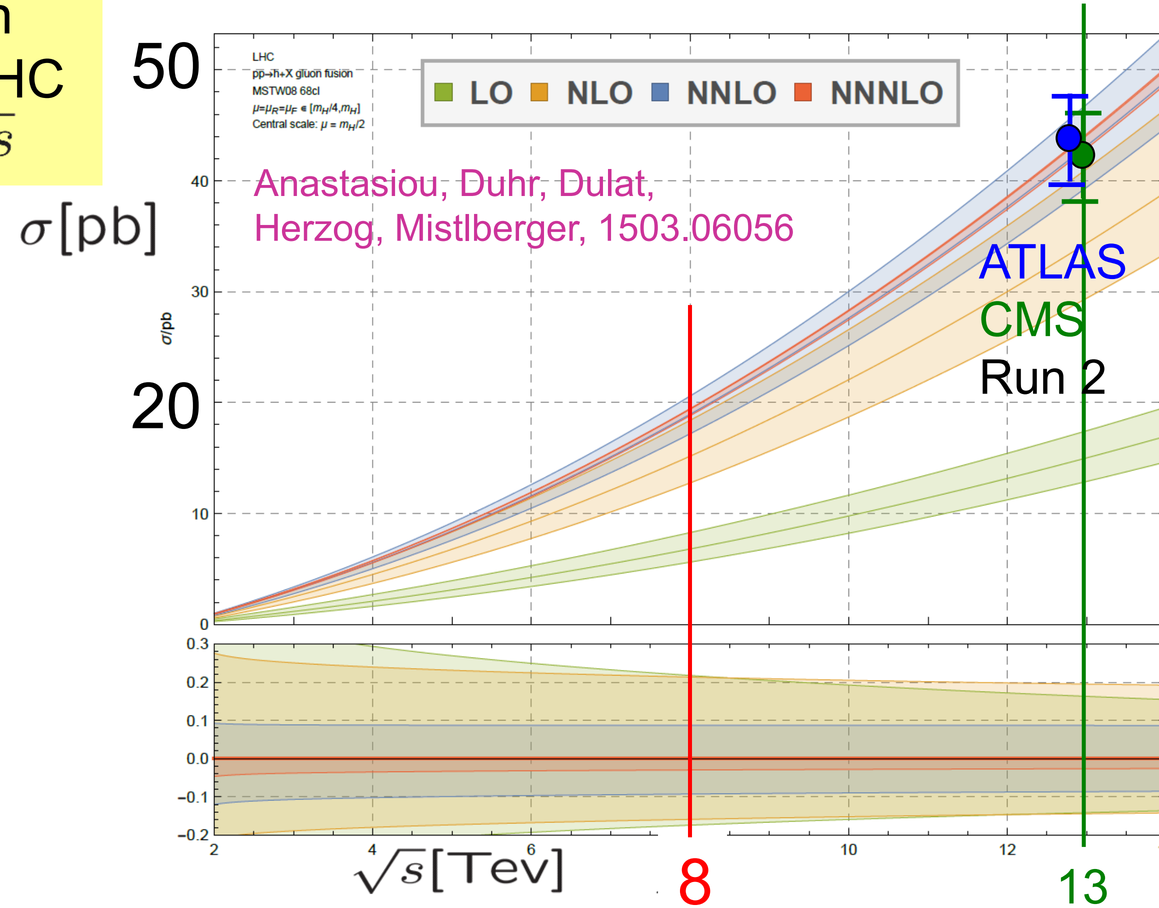
$$\hat{\sigma}(\alpha_s,\mu_F,\mu_R) \;=\; [\alpha_s(\mu_R)]^{n_\alpha}\left[\hat{\sigma}^{(0)} + \frac{\alpha_s}{2\pi}\hat{\sigma}^{(1)}(\mu_F,\mu_R) + \left(\frac{\alpha_s}{2\pi}\right)^2 \hat{\sigma}^{(2)}(\mu_F,\mu_R) + \cdots\right]$$

LO NLO NNLO

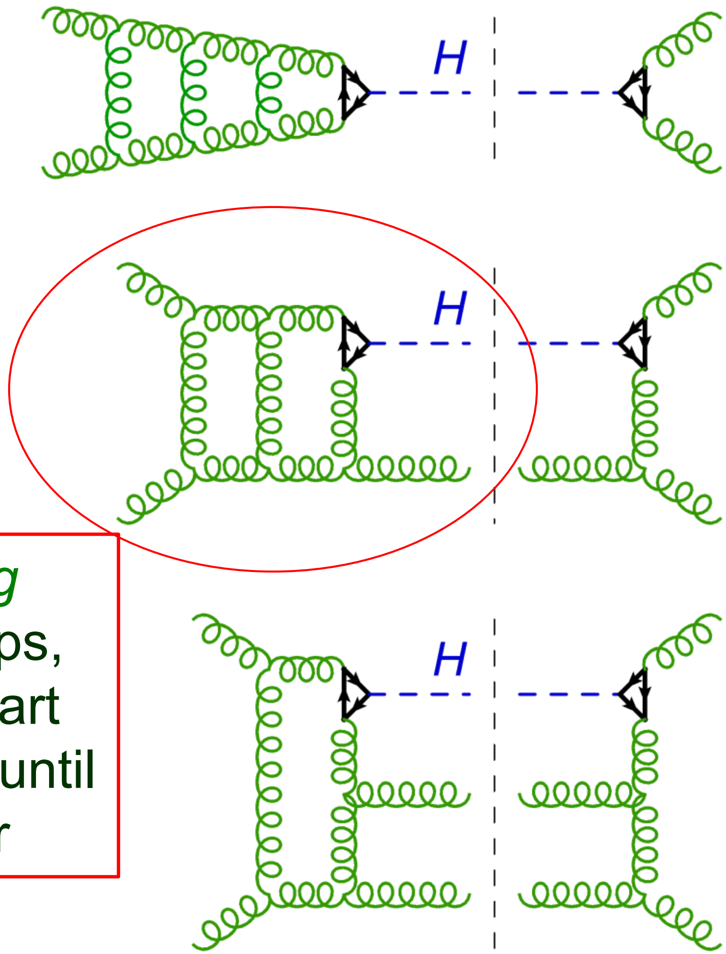
Higgs gluon fusion cross section at LHC vs. CM energy $\sqrt{s}$

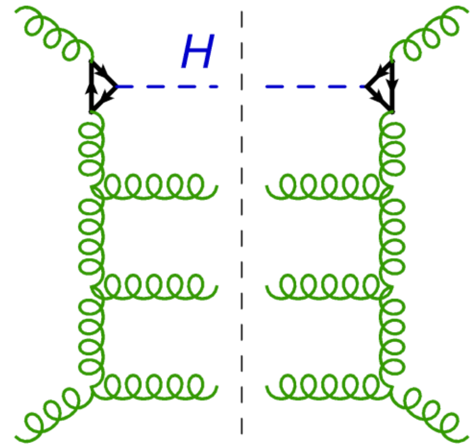
LO terrible!
LO → NNNLO
→ factor of 2 or 3 increase!

Poor convergence of expansion in $\alpha_s(\mu)$ necessitates high orders!



Very few of the NNNLO QCD diagrams





gg → *Hg* @ 2 loops, state of art in QCD until this year

+ …
+ quarks
+ operator renormalization
+ $1/m_t^2$ corrections
+ parton distributions

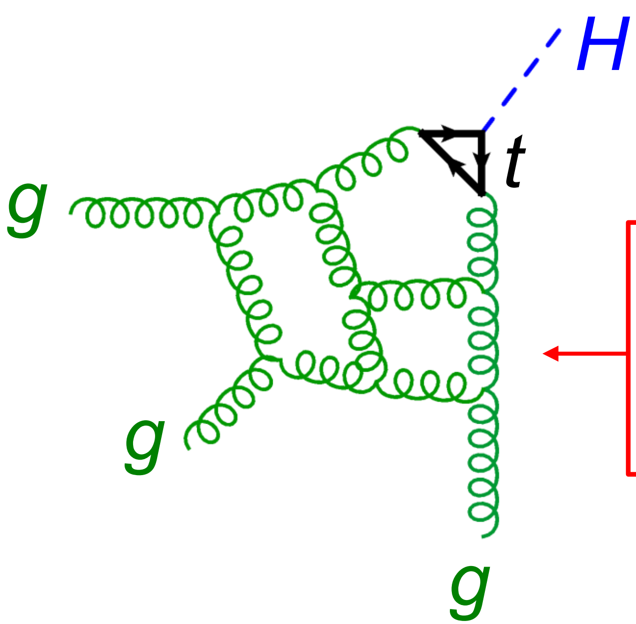
Scattering amplitudes are underlying building blocks

"Goldilocks Process" ~ $gg \rightarrow Hg$

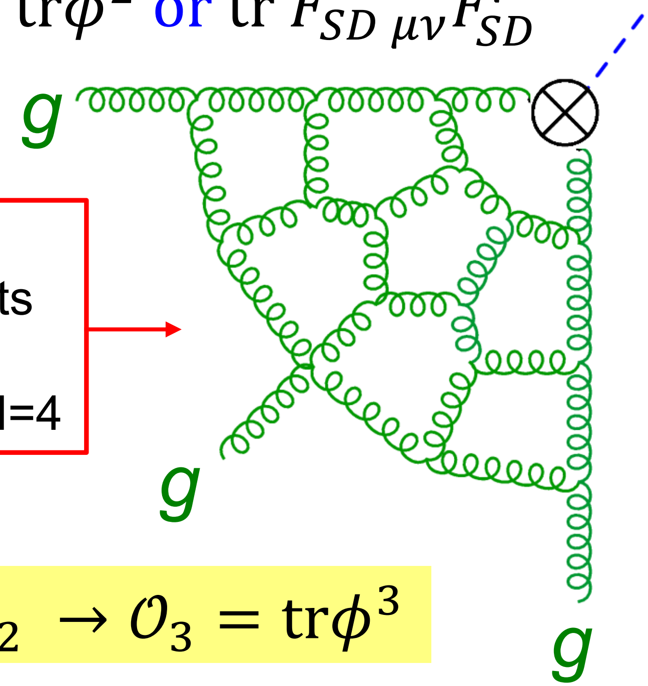
QCD state of art is now three loops @ large N_c (not counting top quark loop)
Chen, Guan, Mistlberger, 2504.06490

Can get to **eight** loops – in analog process: 3-point form factor of $\mathrm{tr}\phi^2$ operator in planar N=4 SYM
LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm
2012.12286, 2112.06243, 2204.11901

$$\mathcal{O}_2 = \mathrm{tr}\phi^2 \text{ or } \mathrm{tr}\, F_{SD\,\mu\nu}F_{SD}^{\mu\nu}$$



Leading weight $2L$ transcendental parts match between QCD ←→ planar N=4



Today: $\mathcal{O}_2 \rightarrow \mathcal{O}_3 = \mathrm{tr}\phi^3$

Planar N=4 SYM is great

- See Jara's talk for why
- Teaches us "something" about the leading transcendental part of QCD
- For simple processes, the answer is a polylogarithmic function of known weight ($2L$ at loop order L) and high loop order computations can be relatively simple.
- We know how to remove all infrared divergences, via the BDS(-like) ansatz.
- Finite remaining functions $\mathcal{E}_3^{(L)}$ for $\mathrm{tr}\phi^2$; $\mathcal{E}_{3,3}^{(L)}$ for $\mathrm{tr}\phi^3$

$\mathrm{tr}\phi^3$ form factor was known to 6 loops

Basso, LD, Tumanov, 2410.22402

- 3 loops agrees with integrand-based computation

  Henn, Lim, Torres Bobadilla, 2410.22465

- 6 loops based on amplitude bootstrap, with exactly the same function space $\mathcal{C}$ as for the $\mathrm{tr}\phi^2$ form factor, and boundary conditions provided by the FFOPE

  Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367, 2112.10569; Basso, Tumanov, 2308.08432

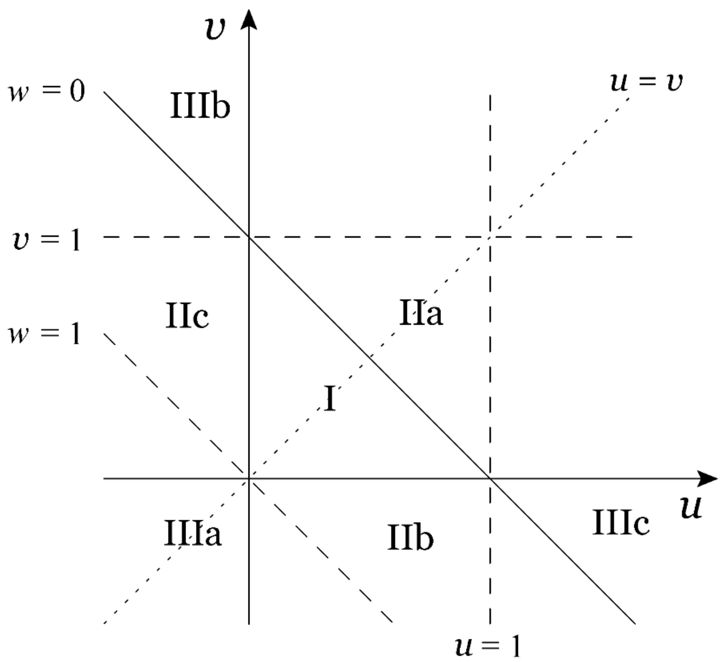
- The fact that the same function space works for $\mathrm{tr}\phi^2$ and $\mathrm{tr}\phi^3$ is not at all obvious

3-point form factor kinematics is two-dimensional

$$k_1 + k_2 + k_3 = -k_H$$

$$s_{123} = s_{12} + s_{23} + s_{31} = m_H^2$$

$$s_{ij} = (k_i + k_j)^2 \qquad k_i^2 = 0$$



$$u = \frac{s_{12}}{s_{123}} \qquad v = \frac{s_{23}}{s_{123}} \qquad w = \frac{s_{31}}{s_{123}}$$

$$u + v + w = 1$$

I = decay / Euclidean

IIa,b,c = scattering / spacelike operator

IIIa,b,c = scattering / timelike operator

N=4 amplitude is S_3 invariant

$D_3 \equiv S_3$ dihedral symmetry generated by:

a. cycle: $i \to i+1 \pmod 3$, or

$$u \to v \to w \to u$$

b. flip: $u \leftrightarrow v$

Map polylogs to **symbols**

- Iterative differentiation of a polylogarithmic function F

n times for a weight n function,
can be used to define its **symbol:**

$$["\ln" \text{ is implicit in } s_{i_k}]$$

$$\mathcal{S}[F] \equiv \sum_{s_{i_1},\ldots,s_{i_n} \in \mathcal{L}} F^{s_{i_1},\ldots,s_{i_n}} \, s_{i_1} \otimes \ldots \otimes s_{i_n}$$

where s_{i_k} are **letters** in the symbol **alphabet** $\mathcal{L}$
and $F^{s_{i_1},\ldots,s_{i_n}}$ are just rational numbers (often integers!)
Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Common alphabet for $\mathrm{tr}\phi^2$ *and* $\mathrm{tr}\phi^3$

Let $u = \frac{s_{12}}{s_{123}}$, $v = \frac{s_{23}}{s_{123}}$, $w = \frac{s_{31}}{s_{123}} = 1 - u - v$

Then $$\mathcal{L} = \{a, b, c, d, e, f\}$$

where $a = \sqrt{\frac{u}{vw}},\ b = \sqrt{\frac{v}{wu}},\ c = \sqrt{\frac{w}{uv}},\ d = \frac{1-u}{u},\ e = \frac{1-v}{v},\ f = \frac{1-w}{w}$

- Symbols of $\mathrm{tr}\phi^2$ form factor $\mathcal{E}_3^{(L)}$ simplify (remarkably) at $L = 1$ and 2 loops, to just 6 and 12 terms:

$$\mathcal{S}\left[\mathcal{E}_3^{(1)}\right] = (-2)[\, b\otimes d + c\otimes e + a\otimes f + b\otimes f + c\otimes d + a\otimes e]$$

$$\mathcal{S}\left[\mathcal{E}_3^{(2)}\right] = 8[\, b\otimes d\otimes d\otimes d + c\otimes e\otimes e\otimes e + a\otimes f\otimes f\otimes f + b\otimes f\otimes f\otimes f + c\otimes d\otimes d\otimes d + a\otimes e\otimes e\otimes e] \\ +16\,[b\otimes b\otimes b\otimes d + c\otimes c\otimes c\otimes e + a\otimes a\otimes a\otimes f + b\otimes b\otimes b\otimes f + c\otimes c\otimes c\otimes d + a\otimes a\otimes a\otimes e]$$

(really only 1 and 2 terms, plus images under dihedral symmetry)

Low loop $\mathrm{tr}\phi^3$ symbols

- Symbols of $\mathrm{tr}\phi^3$ form factor $F_{3,3}^{(L)}$ at $L=1,2$ loops are just 2 and 18 terms, plus D_3 dihedral images:

$$\mathcal{S}\left[\mathcal{E}_{3,3}^{(1)}\right] = a\otimes b - a\otimes a + \text{dihedral}$$

$$\begin{aligned}\mathcal{S}\left[\mathcal{E}_{3,3}^{(2)}\right] = {} & 6\, a\otimes a\otimes a\otimes a - 6\, a\otimes a\otimes b\otimes a - 6\, a\otimes b\otimes a\otimes a + 5\, a\otimes b\otimes b\otimes a \\ & + a\otimes b\otimes c\otimes a - 3\, a\otimes e\otimes a\otimes a + 3\, a\otimes e\otimes c\otimes a + 8\, a\otimes a\otimes b\otimes b \\ & - 6\, a\otimes a\otimes a\otimes b - 6\, a\otimes b\otimes b\otimes b + 5\, a\otimes b\otimes a\otimes b \\ & - 2\, a\otimes a\otimes c\otimes b - 2\, a\otimes c\otimes b\otimes b + 3\, a\otimes f\otimes a\otimes b - 3\, a\otimes f\otimes b\otimes b \\ & + a\otimes b\otimes c\otimes b + a\otimes c\otimes a\otimes b + a\otimes c\otimes c\otimes b \\ & + \text{dihedral}\end{aligned}$$

The bootstrap

Combines:

1. knowledge of the polylogarithmic function space (steal from $\mathrm{tr}\phi^2$)
2. empirical multi-final entry conditions (similar to $\mathrm{tr}\phi^2$)
3. leading and next-to-leading discontinuities (in bulk!)
4. the FFOPE at level of “one flux tube excitation” (fixes collinear limit of the function)

1. Function space from $\mathrm{tr}\phi^2$

- Symbol alphabet $\{a, b, c, d, e, f\}$
- Every term in the symbol starts with a, b, c; never d, e, f
- Physical reason related to causality, which dictates where branch cuts can appear: only for $(p_i + p_j)^2 \sim 0$
- 12 pairs of adjacent letters are forbidden:

$$\begin{array}{lll} \ldots a \otimes d \ldots , & \ldots b \otimes e \ldots , & \ldots c \otimes f \ldots \\ \ldots d \otimes a \ldots , & \ldots e \otimes b \ldots , & \ldots f \otimes c \ldots \\ \ldots d \otimes e \ldots , & \ldots e \otimes f \ldots , & \ldots f \otimes d \ldots \\ \ldots e \otimes d \ldots , & \ldots f \otimes e \ldots , & \ldots d \otimes f \ldots \end{array}$$

- Resemble constraints from causality: Steinmann relations
  Steinmann, Helv. Phys. Acta (1960)
- But not really, which mystified us for a while…
- However, the relations are antipodally dual to (extended) Steinmann relations for the 6-gluon amplitude!!

2. (multi-)final entries

- $\mathrm{tr}\phi^2$ symbol only ends in $d, e, \mathrm{or}\ f$ (3 final entries)
- $\mathrm{tr}\phi^2$ symbol only ends in $a, b, \mathrm{or}\ c$;
  but also $\mathcal{E}^{a}_{3,3} + \mathcal{E}^{b}_{3,3} + \mathcal{E}^{c}_{3,3} = 0$ (2 final entries)
- At each loop order, we learn another layer of multi-final entry conditions, e.g. by 3 loops we know that:

$$\mathcal{E}^{d,a}_{3,3} = \mathcal{E}^{e,b}_{3,3} = 0, \qquad \mathcal{E}^{f,b}_{3,3} = -\mathcal{E}^{f,a}_{3,3}, \qquad \mathcal{E}^{c,a}_{3,3} = -\mathcal{E}^{a,a}_{3,3} - \mathcal{E}^{b,a}_{3,3}$$

$$\mathcal{E}^{c,b}_{3,3} = -\mathcal{E}^{a,b}_{3,3} - \mathcal{E}^{b,b}_{3,3}, \qquad \mathcal{E}^{d,b}_{3,3} = \mathcal{E}^{e,a}_{3,3} - \mathcal{E}^{f,a}_{3,3} + \mathcal{E}^{a,b}_{3,3} - \mathcal{E}^{b,a}_{3,3}.$$

3. Leading discontinuities

- Compact formula from FFOPE:

$$\mathcal{W}^L_{3,3}|_{T^1 \ln^L T} = \int du \frac{\pi}{\cos(\pi i u)} S^{2iu} \frac{1}{L!} [2(\psi(\tfrac{1}{2}+iu)+\psi(\tfrac{1}{2}-iu)-2\psi(1))]^L$$

- Normally such formulas only hold in the collinear limit $v \to 0$, but this one actually holds in the bulk (generic u, v) for a suitable extrapolation.
- There is only a 2-letter symbol for this discontinuity, with letters $A \equiv u/v$ and $C \equiv (1-w)/v$ and it is possible to predict it recursively to all loop orders.

3. Leading discontinuity (cont.)

After converting the leading discontinuity from $\mathcal{W}^{L}_{3,3}$ to $\mathcal{R} \equiv \exp(R_{3,3})$ normalization, we find for the symbols of $\mathcal{R}^{(L)}$:

$$c_L(\mathtt{XCA}^k) = 2L\, c_{L-1}(\mathtt{XA}^k) - \xi_k \frac{L!}{(L-k-1)!}\, c_{L-k-1}(\mathtt{X}),$$

$$\xi_k = \sum_{m=0}^{k} \frac{(-2)^{m+1}}{(m+1)!} \frac{k!}{m!(k-m)!}$$

$$= (-1)^k\, 2\, {}_1F_1(-k,2;2)$$

$$= 0, -\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{2}{5}, \tfrac{4}{45}, \tfrac{10}{63}, -\tfrac{32}{105}, \tfrac{142}{405}, \ldots\,.$$

$$\mathcal{R}^{(0)} = 1,$$

$$\mathcal{R}^{(1)} = 0,$$

$$\mathcal{R}^{(2)} = -\,\mathtt{AA}, \qquad (1.9)$$

$$\mathcal{R}^{(3)} = -2\,\mathtt{CAA} - 6\,\mathtt{ACA} + 4\,\mathtt{A} \ldots \qquad (1.10)$$

$$\mathcal{R}^{(4)} = -16\,\mathtt{CCAA} - 16\,\mathtt{CACA} \ldots \qquad (1.11)$$

$$\begin{aligned}\mathcal{R}^{(5)} = &-160\,\mathtt{CCCAA} - 160\,\mathtt{CA} \ldots \mathtt{A}\\ &+160\,\mathtt{CAACA} + 320\,\mathtt{AC} \ldots \mathtt{A}\\ &-190\,\mathtt{AACAA} - 150\,\mathtt{AA} \ldots\end{aligned} \qquad (1.12)$$

$$\begin{aligned}\mathcal{R}^{(6)} = &-1920\,\mathtt{CCCCAA} - 1920\,\mathtt{CCACCA} - 1920\,\mathtt{CCCACA} - 1920\,\mathtt{CACCCA} - 5760\,\mathtt{ACCCCA}\\ &+1920\,\mathtt{CCCAAA} + 1920\,\mathtt{CCACAA} + 1920\,\mathtt{CCAACA} + 1920\,\mathtt{CACCAA} + 1920\,\mathtt{CACACA}\\ &+1920\,\mathtt{CAACCA} + 3840\,\mathtt{ACCCAA} + 3840\,\mathtt{ACCACA} + 3840\,\mathtt{ACACCA} + 3840\,\mathtt{AA,CCCA}\\ &-1224\,\mathtt{CCAAAA} - 1224\,\mathtt{CACAAA} - 1384\,\mathtt{CAACAA} - 1224\,\mathtt{CAAACA} - 1800\,\mathtt{ACCAAA}\\ &-2280\,\mathtt{ACACAA} - 1800\,\mathtt{ACAACA} - 2280\,\mathtt{AACCAA} - 2280\,\mathtt{AACACA} - 1800\,\mathtt{AAACCA}\\ &+608\,\mathtt{CAAAAA} + 672\,\mathtt{ACAAAA} + 912\,\mathtt{AACAAA} + 992\,\mathtt{AAACAA} + 672\,\mathtt{AAAACA}\\ &-185\,\mathtt{AAAAAA}\,.\end{aligned} \qquad (1.13)$$

Solve a large system of linear equations

- At 8 loops, 1495 x 403 = 602,485 unknowns
- Solve mod p
- See Zhenjie's: https://github.com/munuxi/SparseRREF
- One 10 digit prime was enough to reconstruct over the rational numbers.
- Symbol level unknowns, 1251 x 403 = 504,153 of them, are all integers!

Higher loop $\mathrm{tr}\phi^2$ and $\mathrm{tr}\phi^3$ symbols are lengthy!

loop order L	$\mathrm{tr}\phi^2$ symbol terms	$\mathrm{tr}\phi^3$ symbol terms
1	6	9
2	12	105
3	636	1,773
4	11,208	44,391
5	263,880	747,837
6	4,916,466	14,637,501
7	92,954,568	????
8	1,671,656,292	????

LD, Gürdoğan, McLeod, Wilhelm, 2204.11901;
Basso, LD, Tumanov, 2410.22402; LD, Z. Li, to appear

trϕ^2: number of (symbol-level) linearly independent $\{n, 1, \ldots, 1\}$ coproducts ($2L - n$ derivatives)

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$L = 1$	1	3	1														
$L = 2$	1	3	6	3	1												
$L = 3$	1	3	9	12	6	3	1										
$L = 4$	1	3	9	21	24	12	6	3	1								
$L = 5$	1	3	9	21	46	45	24	12	6	3	1						
$L = 6$	1	3	9	21	48	99	85	45	24	12	6	3	1				
$L = 7$	1	3	9	21	48	108	236	155	85	45	24	12	6	3	1		
$L = 8$	1	3	9	21	48	108	242	466	279	155	85	45	24	12	6	3	1

- Properly normalized L loop trϕ^2 form factors $F_3^{(L)}$ belong to a small space $\mathcal{C}$, dimension saturates on left
- $F_3^{(L)}$ also obeys multiple-final-entry relations, saturation on right

trϕ^3: number of (symbol-level) linearly independent $\{n, 1, \ldots, 1\}$ coproducts ($2L - n$ derivatives)

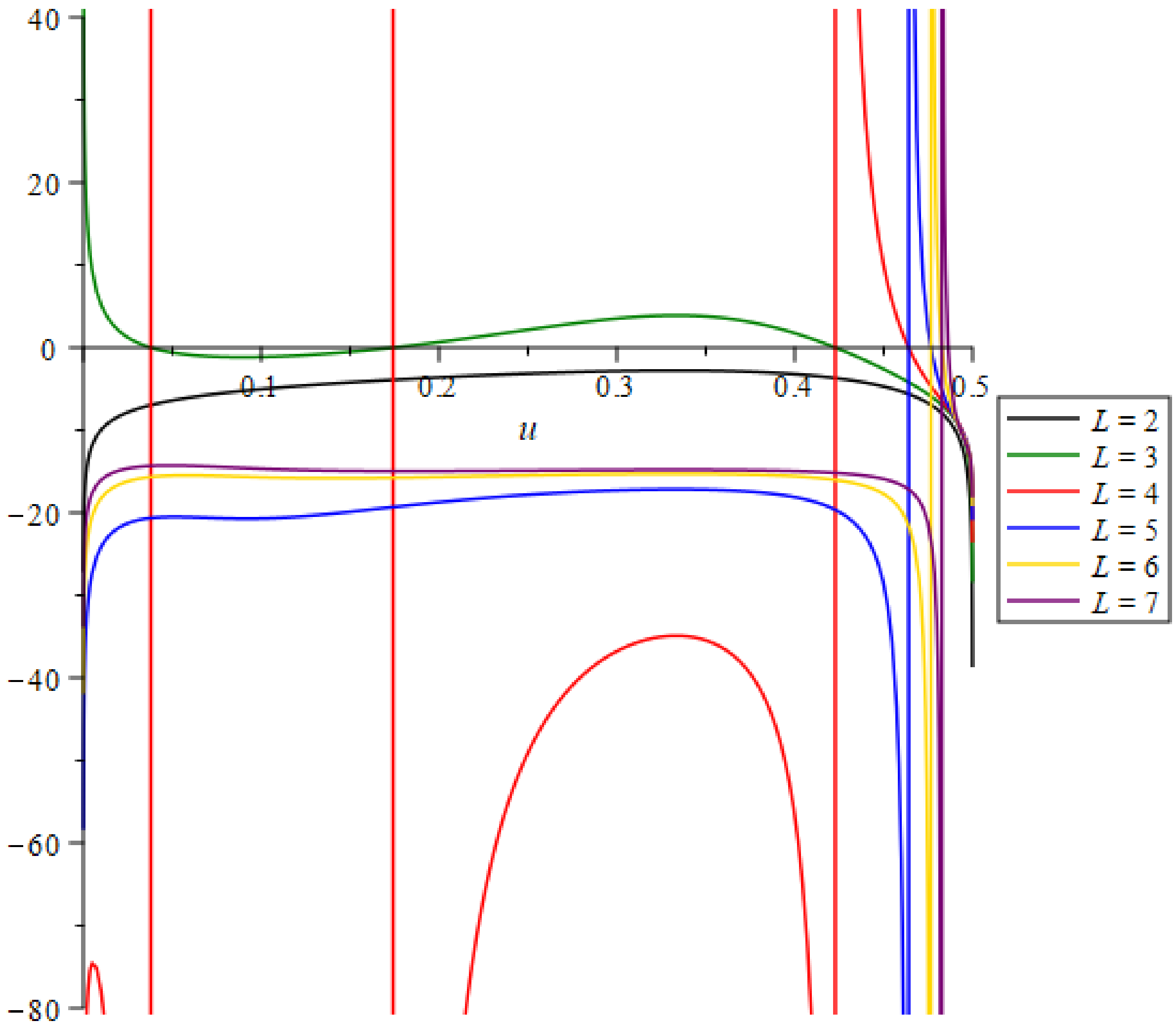
weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$L = 1$	1	2	1														
$L = 2$	1	3	3	2	1												
$L = 3$	1	3	9	13	6	2	1										
$L = 4$	1	3	9	21	29	13	6	2	1								
$L = 5$	1	3	9	21	48	57	29	13	6	2	1						
$L = 6$	1	3	9	21	48	105	112	57	29	13	6	2	1				
$L = 7$	1	3	9	21	48	108	242	206	112	57	29	13	6	2	1		
$L = 8$	1	3	9	21	48	108	242	519	375	206	112	57	29	13	6	2	1

- Properly normalized L loop trϕ^3 form factors $F_{3,3}^{(L)}$ belong to **same** small space $\mathcal{C}$, dimension
- $F_{3,3}^{(L)}$ obeys **different** multiple-final-entry relations, saturation on right

Successive loop ratios reveal finite radius of convergence

$\mathrm{tr}\phi^3$

$$\frac{\mathcal{E}_{3,3}^{(L)}(u,u,1-2u)}{\mathcal{E}_{3,3}^{(L-1)}(u,u,1-2u)}$$



High energy at fixed angles

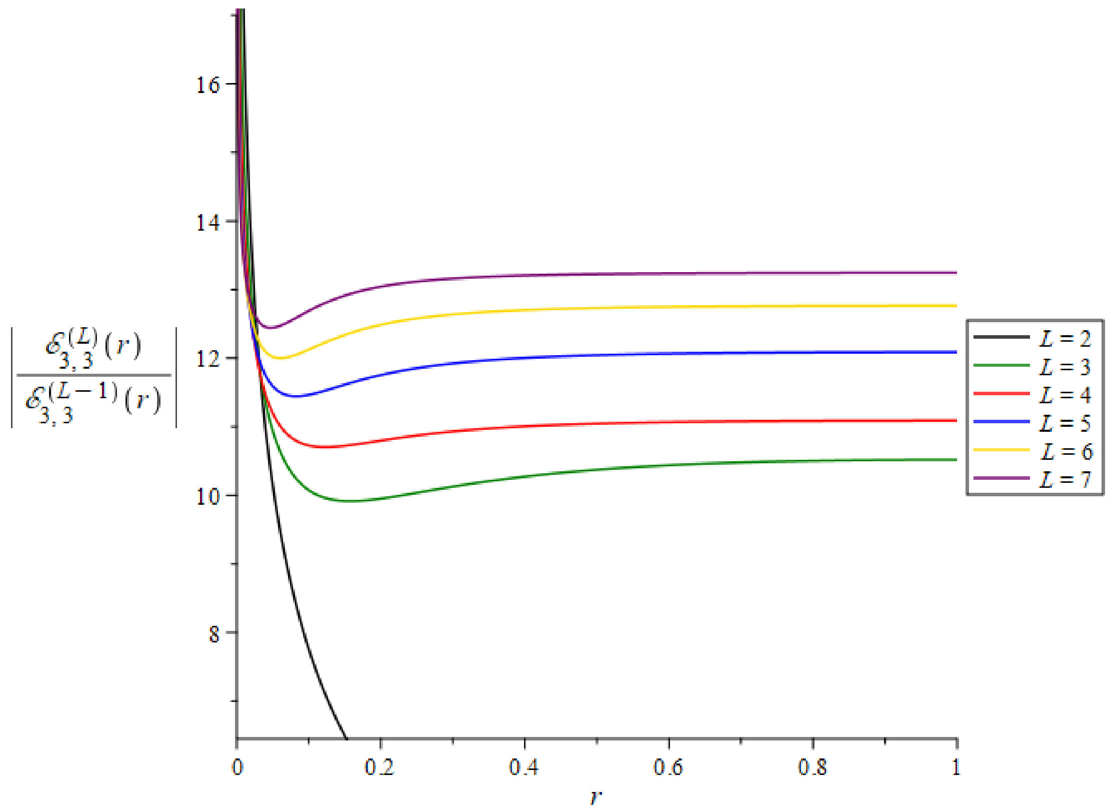
- $u, v \to \infty$, with $r = u/v$ held fixed
- trϕ^2 form factor

$\mathcal{E}_3 \to constant$

(indep. of r)

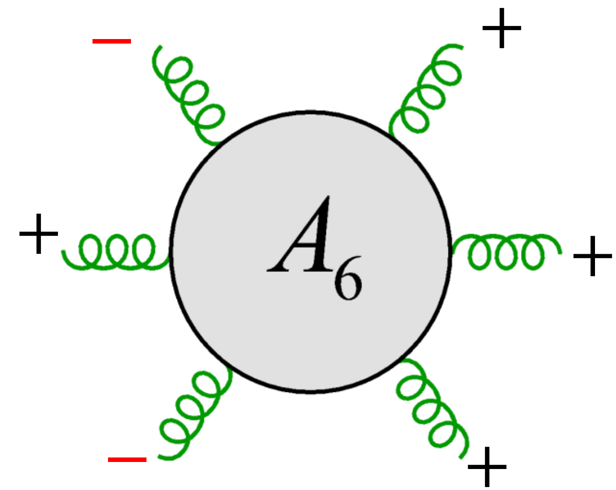
- trϕ^3 form factor

$\mathcal{E}_{3,3}$ is nontrivial →



Six-gluon MHV amplitude

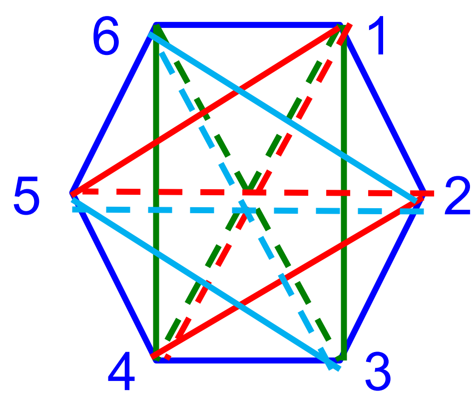
- Dual to Wilson hexagon, invariant under dual conformal transformations; it only depends on 3 dual conformal cross ratios, $\hat{u}, \hat{v}, \hat{w}$:



$$\hat{u} = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}}$$

$$\hat{v} = \frac{s_{23} s_{56}}{s_{234} s_{123}}$$

$$\hat{w} = \frac{s_{34} s_{61}}{s_{345} s_{234}}$$



D_6 dihedral symmetry: cycle $(\mathrm{mod}\ 6)$ and flip, but it acts on $\hat{u}, \hat{v}, \hat{w}$ as $D_3 = S_3$

Parity-preserving surface: (lower-dimensional kinematics)

$$\boxed{\Delta \equiv (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0}$$

Antipodal duality (AD) for trϕ^2

LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, 2112.06243

$$\boxed{F_3^{(L)}(u,v,w) = S\left(A_6^{(L)}(\hat{u},\hat{v},\hat{w}\right)}$$

- Antipode map S, is a "coinverse" for the Hopf algebra.
- At symbol level, it simply reverses order of all letters:

$$S(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = (-1)^m \, x_m \otimes \cdots \otimes x_2 \otimes x_1$$

- Kinematic map in terms of underlying variables is:

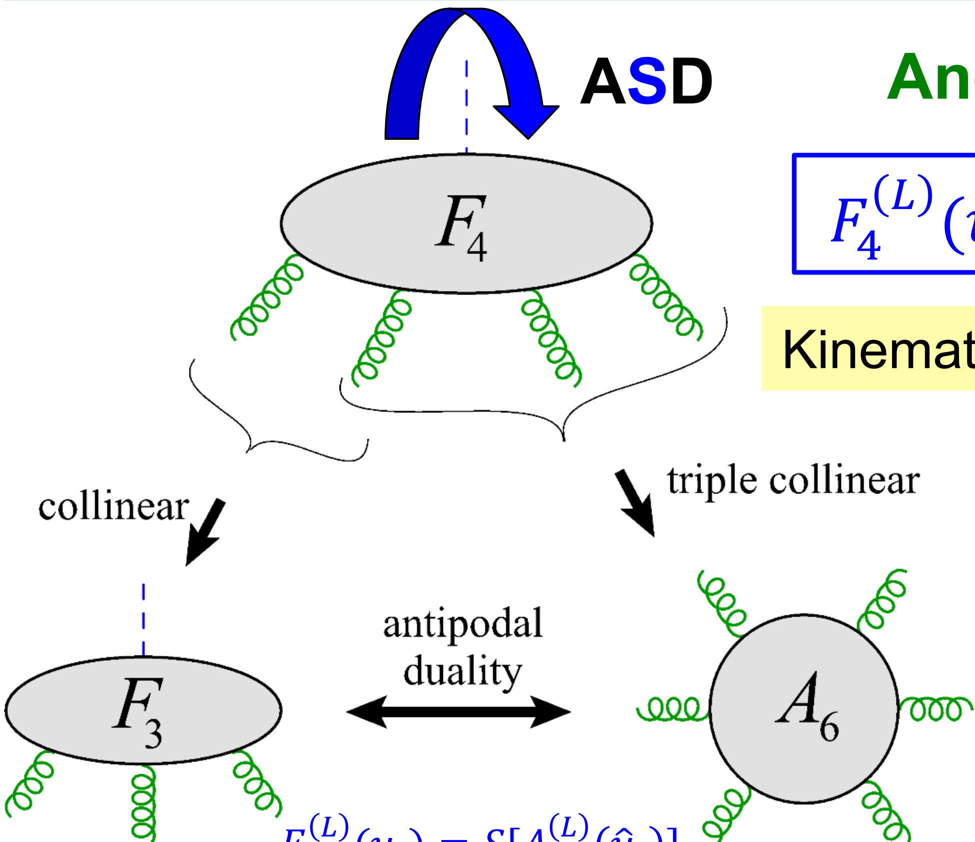
$$\hat{u} = \frac{vw}{(1-v)(1-w)}, \qquad \hat{v} = \frac{wu}{(1-w)(1-u)}, \qquad \hat{w} = \frac{uv}{(1-u)(1-v)}$$

Maps $u+v+w=1$ to $\Delta = 0$ parity-preserving surface for A_6

- Role of branch cuts and derivatives exchanged by S

Antipodal Self Duality

Given antipodal duality relating 2-collinear and 3-collinear limits of F_4, natural to search for self-duality of F_4 that holds for all parity-preserving bulk kinematics



$$F_3^{(L)}(u_i) = S[A_6^{(L)}(\hat{u}_i)]$$

And it's there!

$$\boxed{F_4^{(L)}(u_i, v_i) = S[F_4^{(L)}(\mathbf{C}(u_i), \mathbf{C}(v_i))]}$$

Kinematic map $\mathbf{C}$ simple in FFOPE variables:

$$\mathbf{C}: \quad T_2 \to \frac{T}{S}\,, \quad S_2 \to \frac{1}{TS}$$

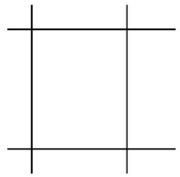
$$T \to \sqrt{\frac{T_2}{S_2}}\,, \qquad S \to \sqrt{\frac{1}{T_2 S_2}}$$

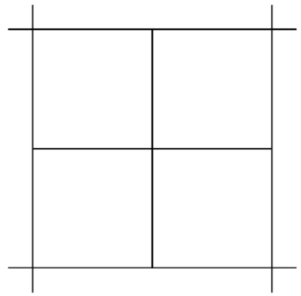
$$F_2 = 1$$

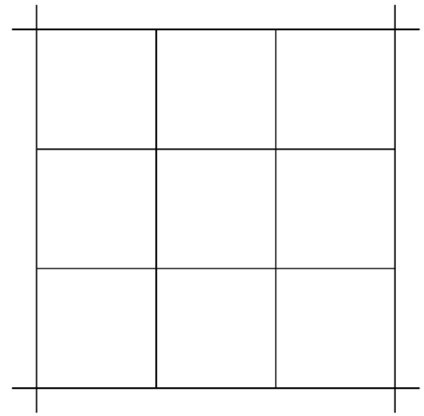
Main Open Question

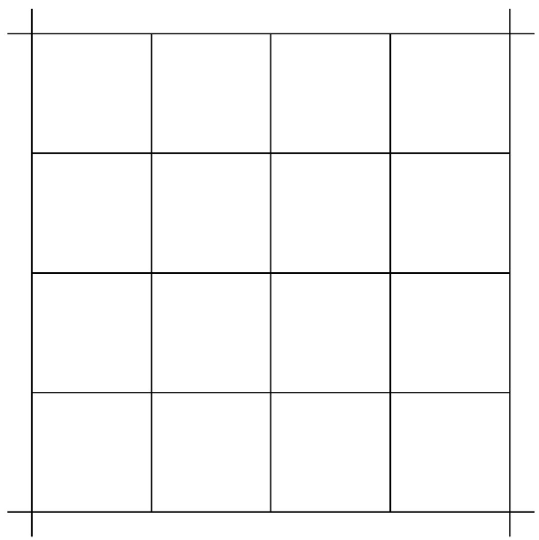
- Why does trϕ^3 3-point form factor live in same space as the trϕ^2 one?
- This space has powerful adjacency conditions whose only explanation is via antipodal duality.
- But the 2 final entries for trϕ^3, $\mathcal{E}_{3,3}^{a/b}$ and $\mathcal{E}_{3,3}^{b/c}$, are very different from the 3 final entries for trϕ^2, $\mathcal{E}_3^d$, $\mathcal{E}_3^e$, and $\mathcal{E}_3^f$, and so they map to the "wrong" first entries for the hexagon functions (A_6 amplitude). ???

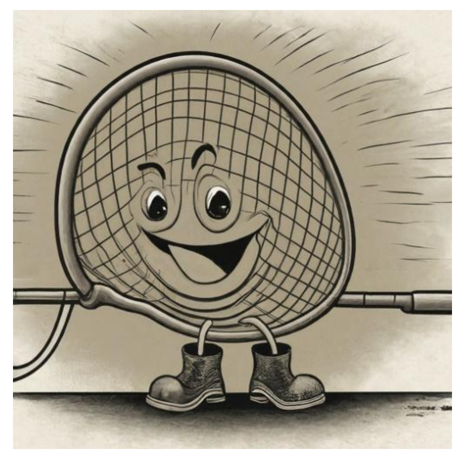
Extra Slides







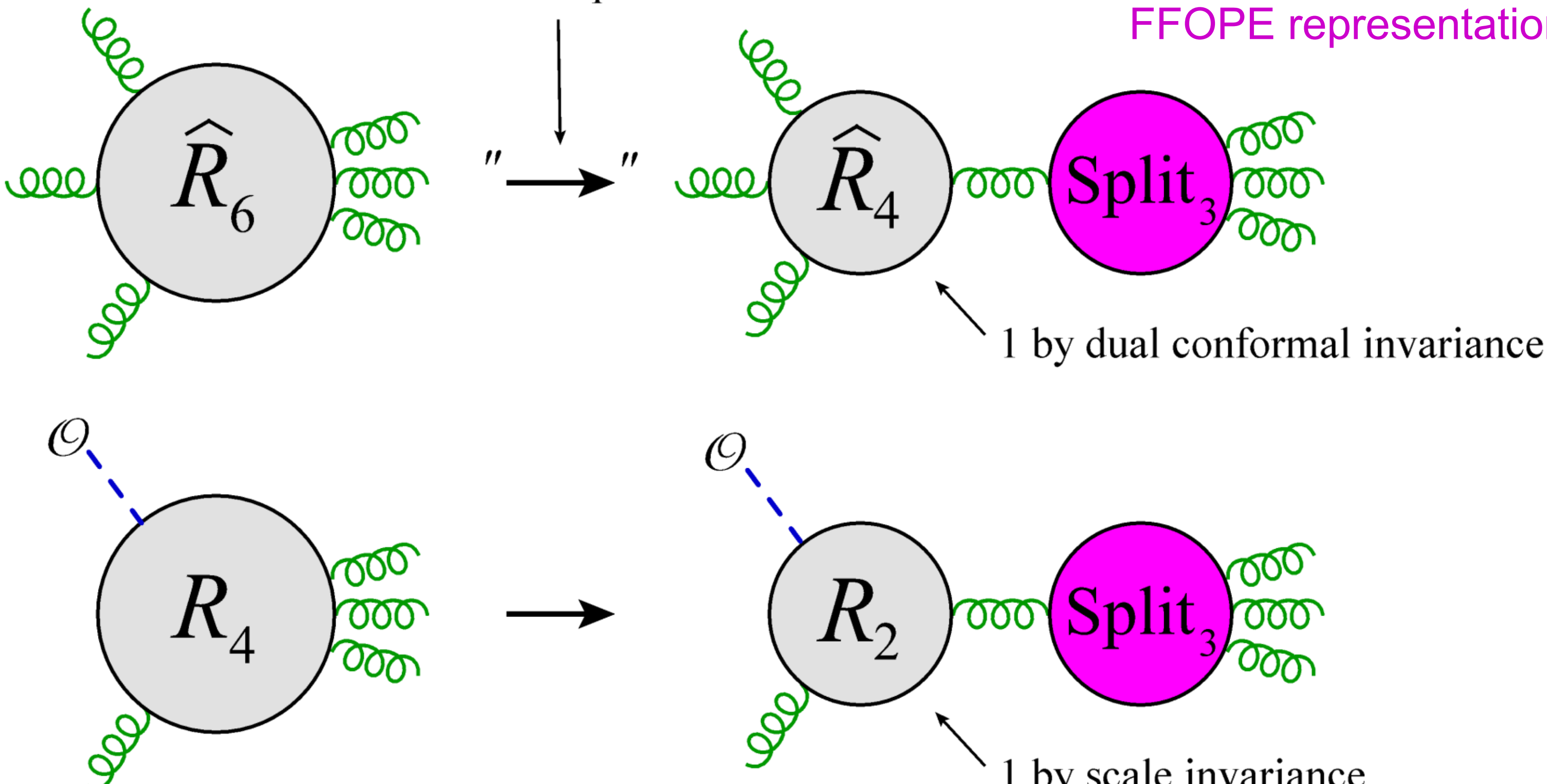




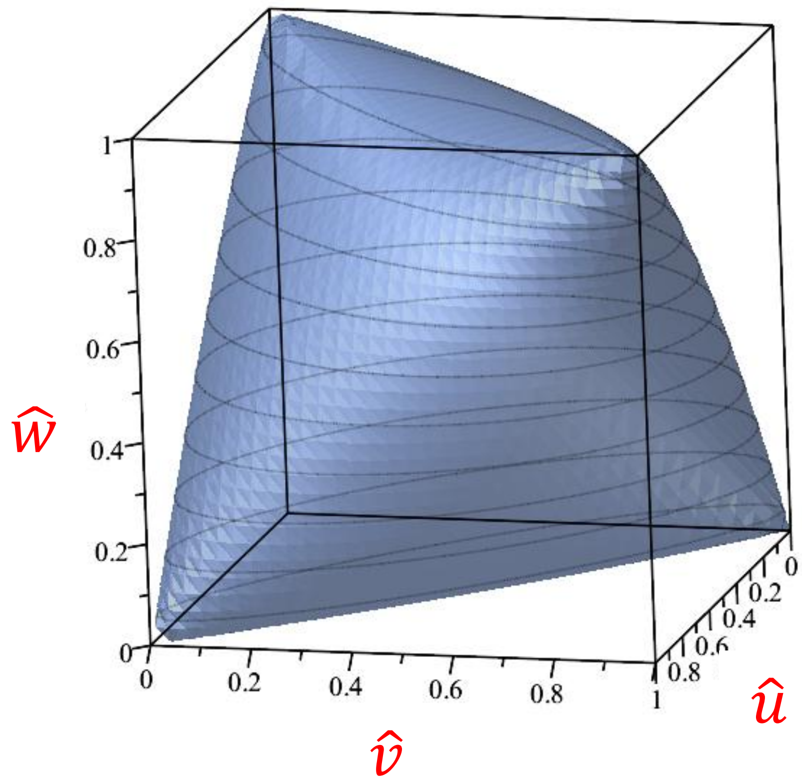
Triple Collinear Limit of 4-point form factor → 6-gluon amplitude

Bern et al., 0803.1465; (also apparent from FFOPE representation)

dual conformal transformations map all kinematics to triple collinear limit!



Parity-preserving surface



$$\boxed{\Delta \equiv (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0}$$

$\Delta = 0$ means that kinematics lies in a
3d subspace of 4d spacetime → parity invariant

FFOPE kinematical variables for F_4

$$u_i = \frac{s_{i,i+1}}{s_{1234}}, \qquad v_i = \frac{s_{i,i+1,i+2}}{s_{1234}}$$

$$-u_1 + u_3 + v_4 + v_1 = 1$$
$$-u_2 + u_4 + v_1 + v_2 = 1$$
$$-u_3 + u_1 + v_2 + v_3 = 1$$

$$u_1 = \frac{T^2 T_2^2}{(T^2+1)(S^2+T^2+T_2^2+1)}$$

$$u_2 = \{1 + T^2 + \frac{S^2[(1+F_2^2)S_2T_2 + F_2(1+S_2^2+T^2+T_2^2)]}{F_2 S_2^2}\}^{-1}$$

$$u_3 = \frac{S^2}{(T^2+1)(S^2+T^2+T_2^2+1)}$$

$$u_4 = \frac{S^2T^2}{S_2^2}\, u_2$$

$$v_1 = \frac{T_2^2+1}{S^2+T^2+T_2^2+1}$$

- OPE limit takes $T, T_2 \to 0$, interpolates between 2-collinear limit $T_2 \to 0$ and 3-collinear limit $T \to 0$

ASD beyond 2 loops

LD, Ö. Gürdoğan, Y.-T. Liu, A. McLeod, M. Wilhelm, 2212.02410v2;
LD, Z. Li, to appear

- Bootstrapped symbol of F_4 at 3 loops, using same 113 letter (2-loop) alphabet.
- **Unique result**, which obeys all the FFOPE predictions we could check.
- 2 loop symbol uses only 34 of the letters [3,784 terms]
- 3 loop symbol uses only 88 of the letters [3,621,202 terms]
- 4 loop symbol uses only 88 of the letters [???? terms]
- No OPE checks yet at 4 loops, but few constraints needed to fix it.
- **ASD holds in 3D at 4 loops!**