

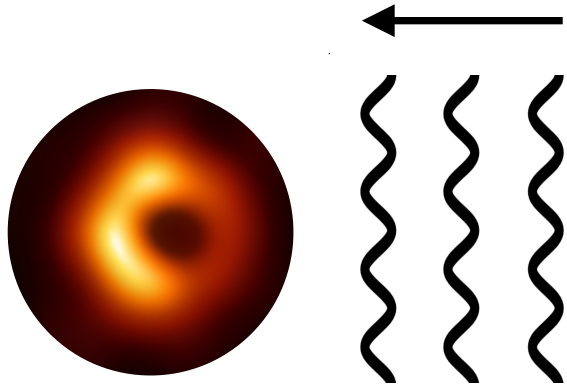
Analyticity of the Black Hole S-Matrix

Anna Wolz

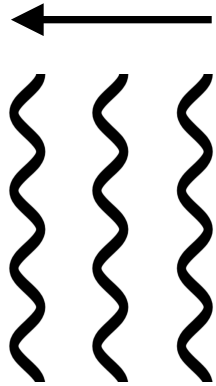
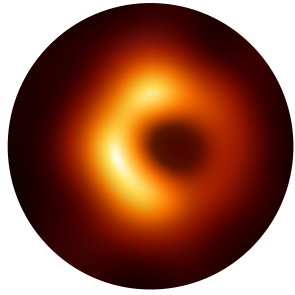
CA Amplitudes 11/8/25

with M. Correia, T. Gopalka, G. Isabella

Wave scattering against Schwarzschild black hole



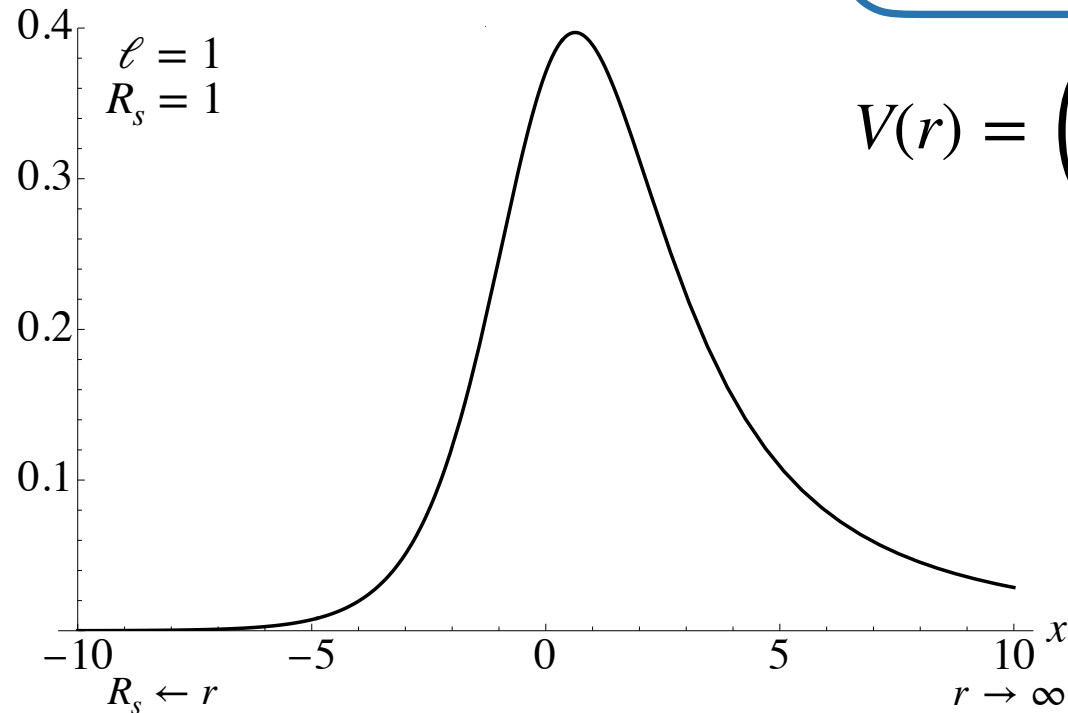
Wave scattering against Schwarzschild black hole



Regge-Wheeler equation

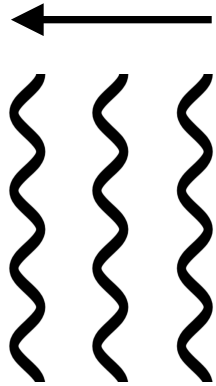
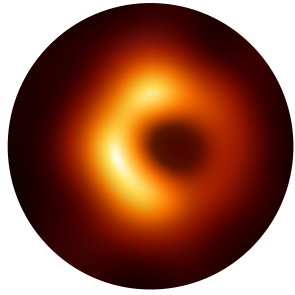
$$\left(\frac{d^2}{dx^2} + \omega^2 - V(r(x)) \right) \phi(x) = 0$$

tortoise coordinate
 $x = r + R_s \log \left(\frac{r}{R_s} - 1 \right)$
 $\Rightarrow r = r(x)$



$$V(r) = \left(1 - \frac{R_s}{r} \right) \left(\frac{\ell(\ell + 1)}{r^2} + \frac{R_s}{r^3} \right)$$

Wave scattering against Schwarzschild black hole

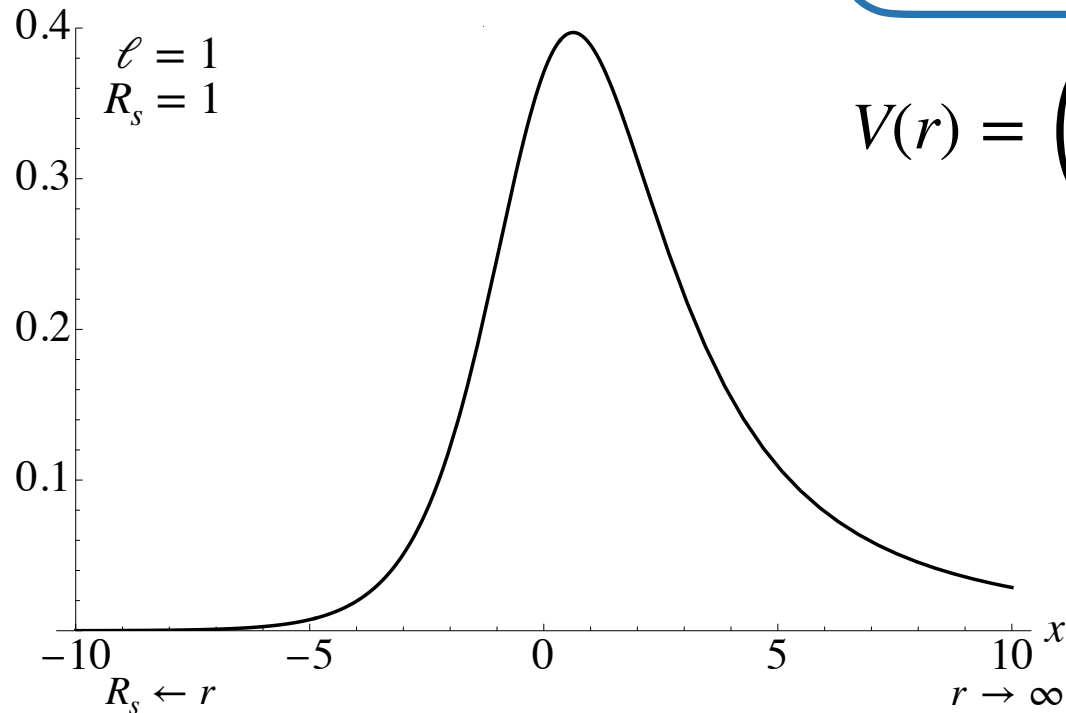


Regge-Wheeler equation

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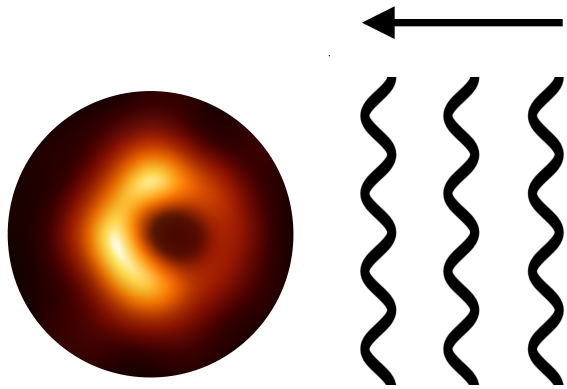
tortoise coordinate
 $x = r + R_s \log \left(\frac{r}{R_s} - 1 \right)$
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$\phi(x) \xrightarrow{x \rightarrow \infty}$ Extract partial-wave
 S-matrix $S_\ell(\omega)$

Wave scattering against Schwarzschild black hole



Regge-Wheeler equation

$$\left(\frac{d^2}{dx^2} + \omega^2 - V(r(x)) \right) \phi(x) = 0$$

- Black hole ringdown / BHPT
- Analytic continuation of $S_\ell(\omega)$ to complex ω
- Applications to gravitational S-matrix bootstrap

$\phi(x) \xrightarrow{x \rightarrow \infty}$ Extract partial-wave
S-matrix $S_\ell(\omega)$

Wave scattering against Schwarzschild black hole

$$\left(\frac{d^2}{dx^2} + \omega^2 - V(r(x)) \right) G_R(\omega, x, x') = \delta(x - x') \quad \text{retarded Green's function}$$

$$V(r) = \left(1 - \frac{R_s}{r} \right) \left(\frac{\ell(\ell + 1)}{r^2} + \frac{R_s}{r^3} \right)$$

$$G_R(\omega; x, x') \xrightarrow{x, x' \rightarrow \infty} \frac{1}{2i\omega} \left(e^{i\omega(x-x')} - S_\ell(\omega) e^{i\omega(x+x')} \right)$$

Wave scattering against Schwarzschild black hole

$$\left(\frac{d^2}{dx^2} + \omega^2 - V(r(x)) \right) G_R(\omega, x, x') = \delta(x - x') \quad \text{retarded Green's function}$$

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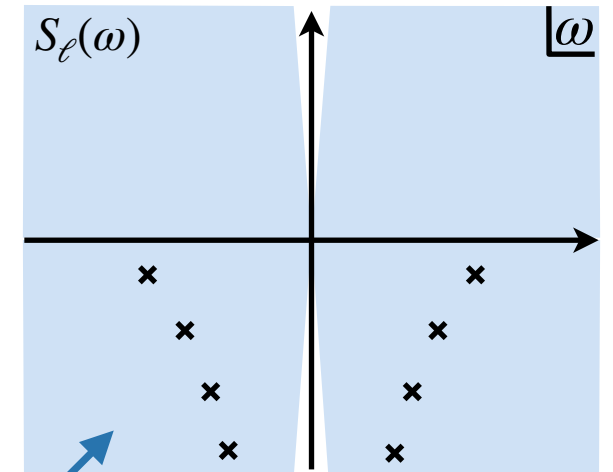
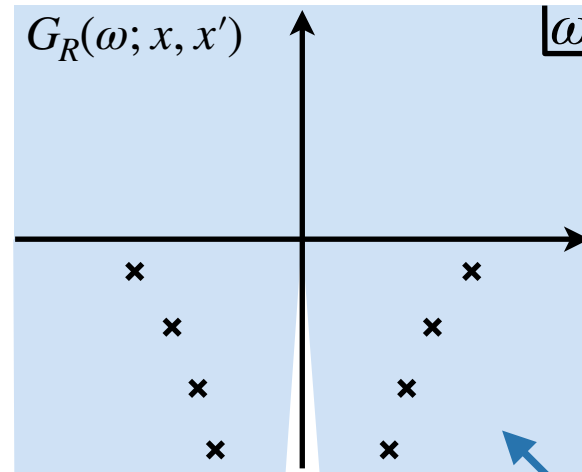
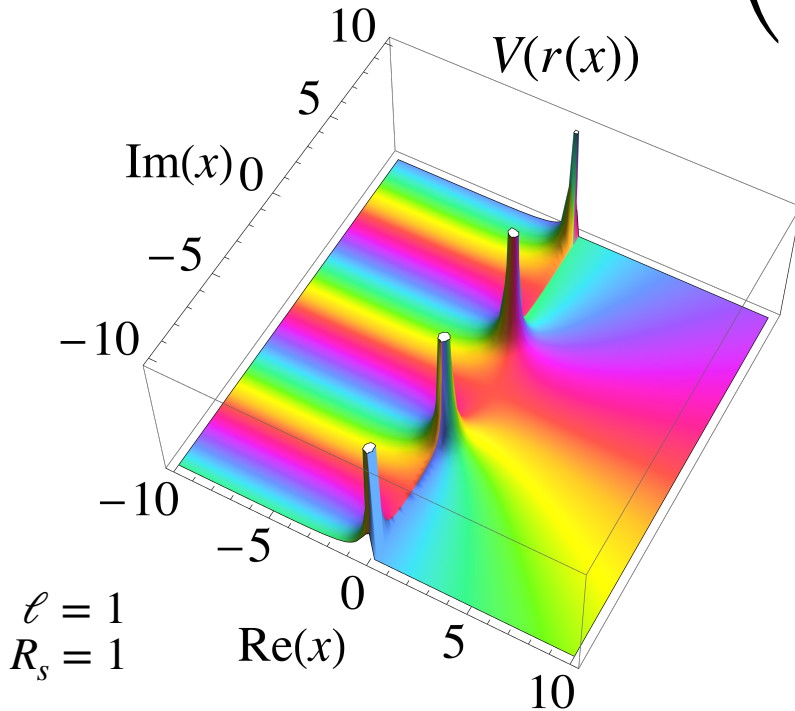
$$G_R(\omega; x, x') \xrightarrow{x, x' \rightarrow \infty} \frac{1}{2i\omega} \left(e^{i\omega(x-x')} - S_\ell(\omega) e^{i\omega(x+x')} \right)$$

- Solve perturbatively in $|R_s \omega| \ll 1$ (BHPT methods)
- Solve examples of low frequency and high frequency

Properties of black hole S-matrix

- Analyticity: $V(r) = \left(1 - \frac{R_s}{r}\right) \left(\frac{\ell(\ell+1)}{r^2} + \frac{R_s}{r^3}\right)$

(Giulia's talk)



analytic (up to poles in lower-half plane)

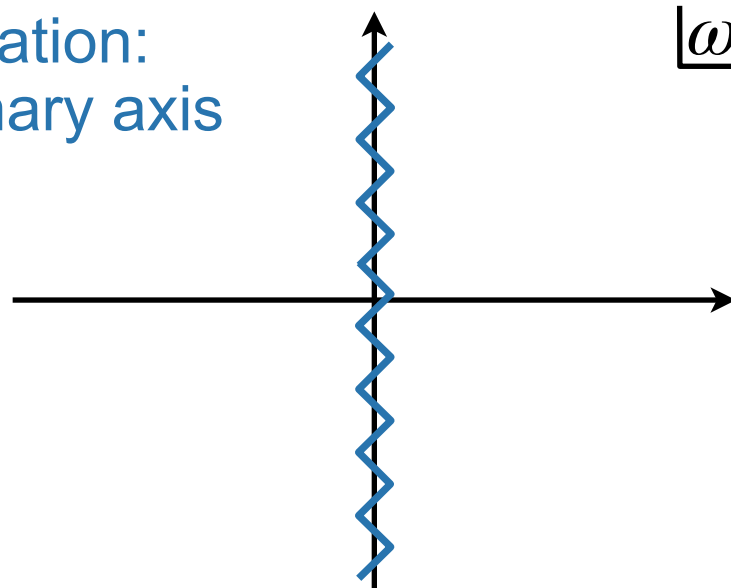
- Unitarity: $|S_\ell(\omega)| \leq 1$
- Crossing symmetry: $S_\ell^*(\omega) = S_\ell(-\omega^*)$

Low-frequency perturbative expansion

- Solve Regge-Wheeler equation perturbatively in $|R_s \omega| \ll 1$
- Known result from BHPT:

$$S_0(\omega) = 1 + 2iR_s\omega \left(\log(4R_s^2\omega^2) + 2\gamma_E - 1 \right) + \mathcal{O}\left((R_s\omega)^2\right)$$

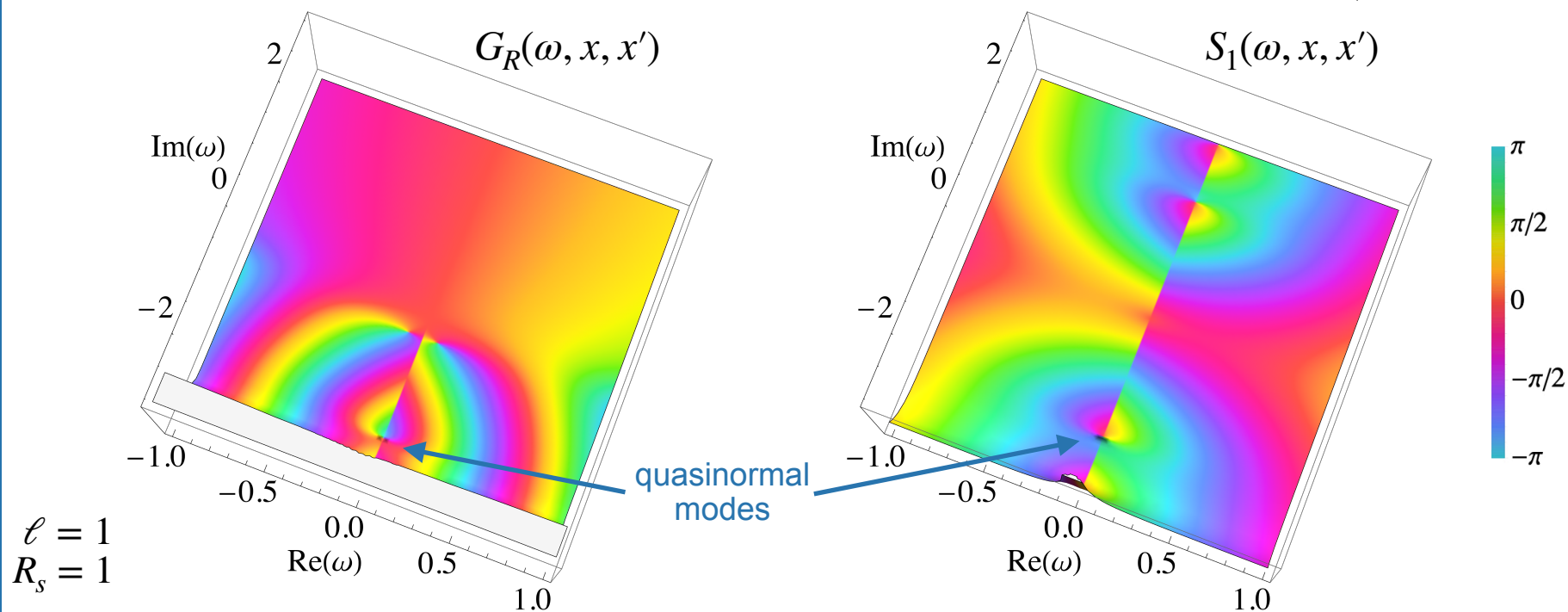
Correct analytic continuation:
branch cut along imaginary axis



consistent with
analyticity, unitarity,
crossing symmetry

Gravitational Coulomb potential

- At large r , effective Coulomb potential: $V_{eff}(r) = -\frac{2R_s\omega^2}{r} + \frac{\ell(\ell+1)}{r^2}$
- Exact solution: $S_\ell(\omega) = (-1)^\ell e^{iR_s\omega \log(4R_s^2\omega^2)} \frac{\Gamma(1+\ell-iR_s\omega)}{\Gamma(1+\ell+iR_s\omega)}$
branch cut

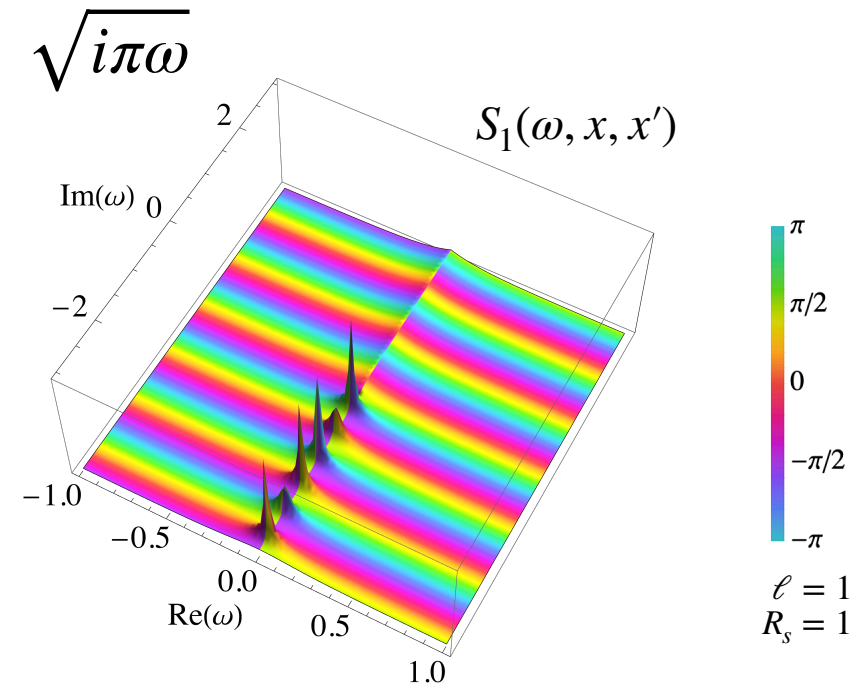
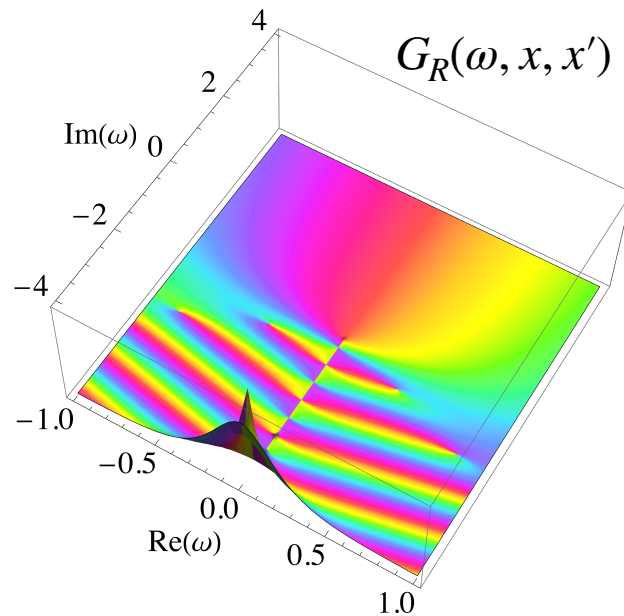


Stokes phenomenon:
 G_R analytic in UHP,
 S_ℓ has branch cut

High-frequency approximation

- $|R_s \omega| \gg 1$: wave function satisfies confluent hypergeometric equation

- Exact solution: $S_\ell(\omega) = \frac{\sqrt{-i\omega} e^{-2iR_s\omega(1-\log(2iR_s\omega))} \Gamma(1/2 - 2iR_s\omega)}{\sqrt{i\pi\omega}}$



- $S_\ell(\omega \rightarrow \infty) \sim e^{-2\pi R_s \omega}$

Summary and future directions

- Examples of black hole S-matrix consistent with analyticity, unitarity, crossing
- Future directions
 - Perturbative solution to Regge-Wheeler: **branch cuts in all diagrams (triangle)** ?
 - Can safely assume AUC, location of branch cuts → use in gravitational S-matrix bootstrap to **bound tidal Love numbers**