Bound states, spins, and the classical double copy

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w/ Ridgway (Caltech):
1711.09493
w/ J. Li and S. Prabhu (Yale)
1712.?????

Motivation: (1) Theoretical

- Does the BCJ double copy (or something like it) relate classical solutions in YM and GR? This was first raised by:

Monteiro, O'Connell, White (2014)

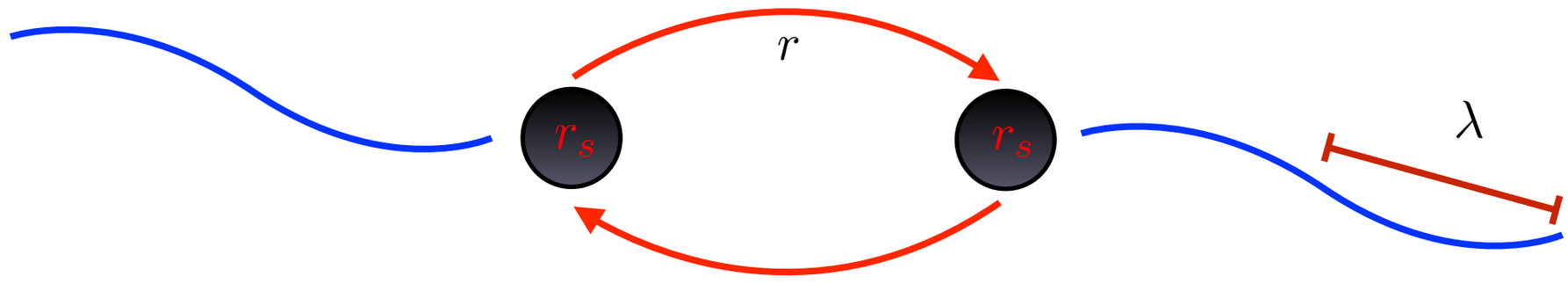
Luna, Monteiro, O'Connell, White (2015)

Luna, Monteiro, Nicholson, O'Connell, White (2016)

who provided non-perturbative examples.

Motivation: (II) Phenomenological

Gravitational dynamics of radiating classical BH (or NS) binary systems in the non-relativistic limit are experimentally relevant (LIGO/VIRGO, LISA,…)



Experiments will be sensitive to at least v^6 corrections beyond Newtonian gravity (Thorne et al 1994). Numerical GR results also motivate computing higher order corrections.

See A. Buonanno's talk

For this system, the radiation field measured by observers at infinity

$$\lim_{|\vec{x}|\to\infty} h_{ij}^{TT}(t,\vec{x})$$

encodes all the relevant physical info about this system (masses,spins, multipole moments,QNM frequencies,…). In perturbation theory, it is most conveniently computed by recasting Einstein's equations in the form (eg Weinberg 1972)

$$h_{\mu\nu}(x) = \frac{1}{2m_{Pl}} \int_k \frac{e^{-ik\cdot x}}{k^2} \left[\tilde{T}^{\mu\nu}(k) - \frac{1}{2}\eta_{\mu\nu}\tilde{T}^{\sigma}{}_{\sigma} \right]$$

(deDonder gauge)

$$\tilde{T}^{\mu\nu} = T^{\mu\nu}_{pp} + T^{\mu\nu}_{g} = \quad \text{EM pseudotensor} \qquad \partial_\mu \tilde{T}^{\mu\nu} = 0$$

$$\sim h\partial^2 h + h^2\partial^2 h + \cdots$$

$$S_{pp} = -m\int d\tau + c\int d\tau R^2_{\mu\nu\alpha\beta} + \cdots$$

$$c_{NS} \sim mR^4 \qquad c_{BH,d=4} = 0$$

(Damour et al; Poisson et al;Kol+Smolkin 2010)

The radiation field at infinity has a simple relation to the pseudo-tensor evaluated on-shell

$$h_{\pm}(t,\vec{n}) = \frac{4G_N}{r}\int \frac{d\omega}{2\pi} e^{-i\omega t}\epsilon^{*ij}_{\pm}(k)\tilde{T}_{ij}(k)$$

(Weinberg 1972)

w/

$$k^{\mu} = \omega(1, \vec{n} = \frac{\vec{x}}{r})$$

$$k^2 = 0$$

In practice computing higher order terms in perturbation theory $(v \ll 1)$ is difficult for two reasons:

Many terms in the expansion of $\tilde{T}^{\mu\nu}(x)$ at high orders in $h_{\mu\nu}$

Many physically relevant scales

Gravitational radius: $r_g = 2G_N M$

Physical radius: $r_s (= r_g \text{ for BH})$

Orbital scale: r

Radiation wavelength λ

$$r_g \sim r_s \gg r \gg \lambda$$

all correlated to the perturbative expansion parameter

$$r \sim r_g / v^2 \qquad \lambda \sim r/v \sim r_g / v^3$$

Some of these challenges can be ameliorated by employing tools from QFT:

Many terms in the expansion of $\tilde{T}^{\mu\nu}(x)$ at high orders in $h_{\mu\nu}$

Organize the expansion in terms of Feynman diagrams

Many physically relevant scales

Treat each scale separately, by constructing
a tower of gravity Effective Field Theories

$$h_{\mu\nu} = h_{\mu\nu}^{potential} + h_{\mu\nu}^{rad}$$

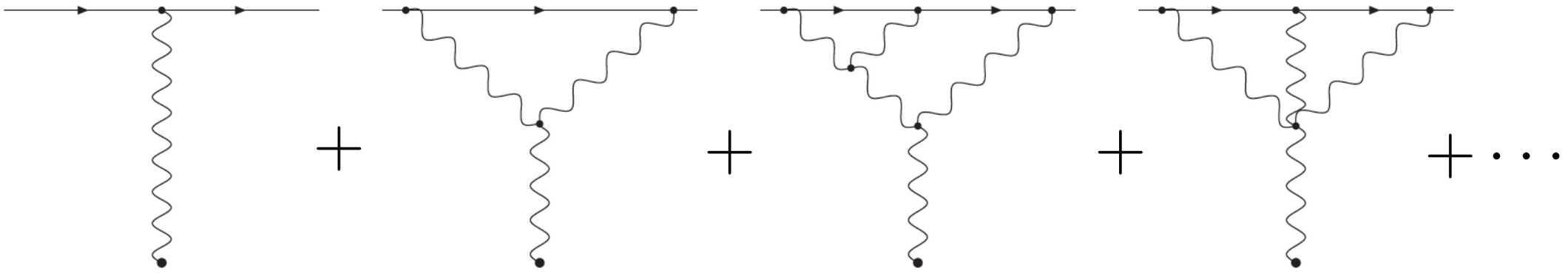
See I. Rothstein's talk

The focus of this talk is the Feynman diagram expansion.

The types of Feynman diagrams that are relevant are of the same type as in Duff's (1973) perturbative construction of the Schwarzschild solution:

(NOT A PROPAGATOR!)

$h_{\mu\nu} =$

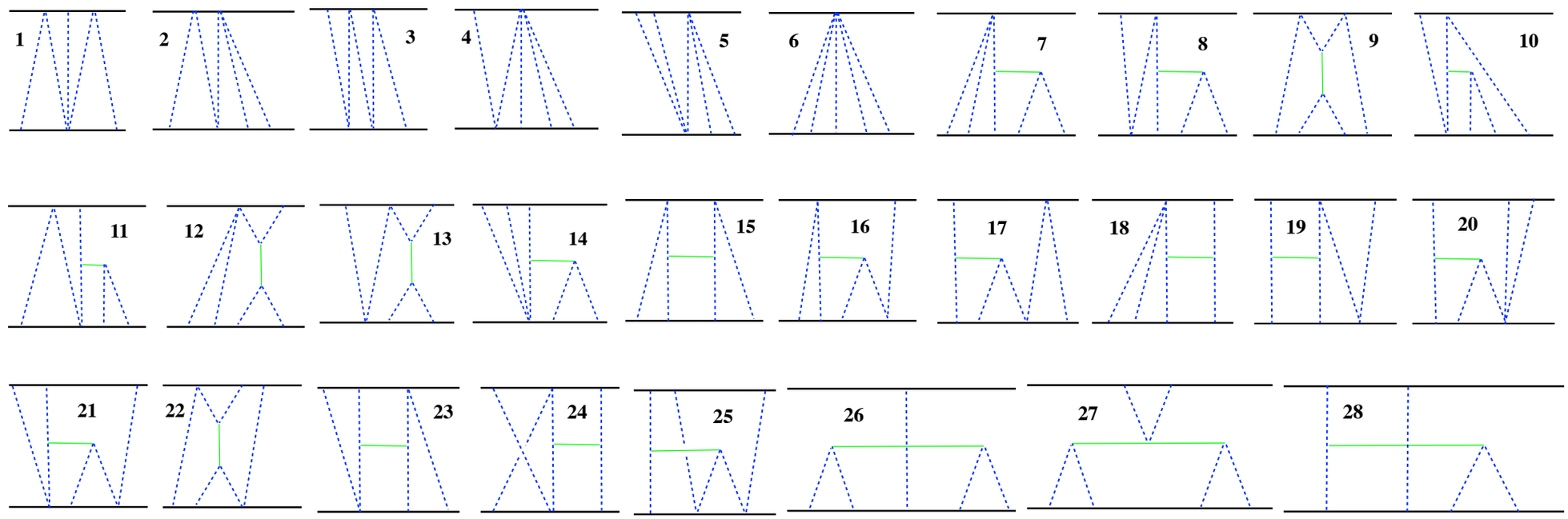


$$g_{00} = 1 - \frac{2G_N m}{r} + 2\left(\frac{G_N m}{r}\right)^2 + 2\left(\frac{G_N m}{r}\right)^3 + \cdots$$

$$g_{ij} = -\delta_{ij}\left[1 + \frac{2G_N m}{r} + 5\left(\frac{G_N m}{r}\right)^2 - \frac{2}{3}\left(\frac{G_N m}{r}\right)^3 + \cdots\right]$$

$$+ \frac{x_i x_j}{r^2}\left[7\left(\frac{G_N m}{r}\right)^2 - \frac{4}{3}\left(\frac{G_N m}{r}\right)^3 + \cdots\right]$$

Proliferation of Feynman diagrams at order 4PN =



See Mastrolia's talk

Q: Is there a more efficient way, based on color kinematics (BCJ?) to organize the perturbative expansion?

Last year: Presented some evidence (w/ A. Ridgway) of relations between perturbative radiating solutions in YM and (dilaton) gravity, in the context of classical bremsstrahlung.

This year: Generalization to bound states, spinning objects

Gauge Theory Solutions

Solve the classical Yang-Mills equations coupled to classical point color charges.

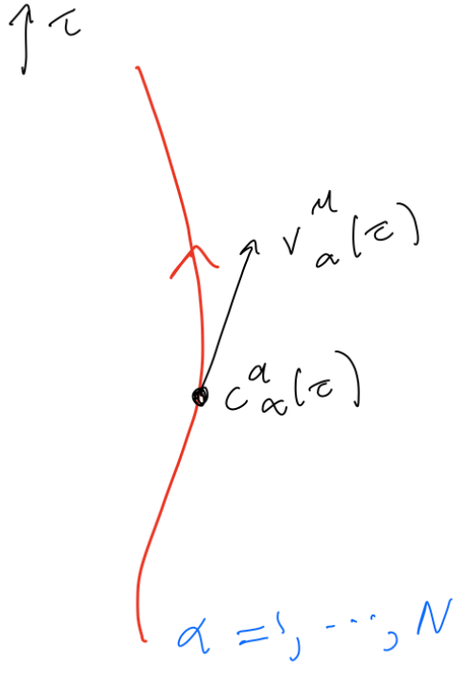
$$D_\nu F_a^{\nu\mu}(x) = g J_a^\mu(x)$$

By classical color charge we mean an object whose degrees of freedom are

$x^\mu(\tau)$ =wordline coordinate $\quad$ $c^a(\tau)$ =color d.o.f in adjoint at $x(\tau)$

(Sikivie and Weiss, 1978)

Current for a collection of charges



$$J_a^\mu(x) = \sum_\alpha \int d\tau c_\alpha^a(\tau) v_\alpha^\mu(\tau) \delta^d(x - x_\alpha(\tau)) + \cdots$$

(finite size terms and other color moments)

The particle equations of motion follow from conservation laws. Color conservation:

$$D_\mu J^\mu_a = 0 \Rightarrow v \cdot D c^a = 0$$

or in terms of adjoint rep. Wilson line $$c^a_\alpha(\tau) = W_\alpha{}^a{}_b(\tau) c^b_\alpha(-\infty)$$

$$W_\alpha{}^a{}_b(\tau) = \left[P \exp\left\{ -ig \int_{-\infty}^{\tau} dx^\mu_\alpha A_\mu \cdot T_{\rm adj} \right\} \right]^a{}_b .$$

The orbital motion is fixed by energy-momentum and gives the non-Abelian Lorentz force law:

$$\partial_\mu (T^{\mu\nu}_{YM} + T^{\mu\nu}_{pp}) = 0 \Rightarrow \frac{dp^\mu}{d\tau} = g c^a F^{\mu\nu}_a v_\nu$$

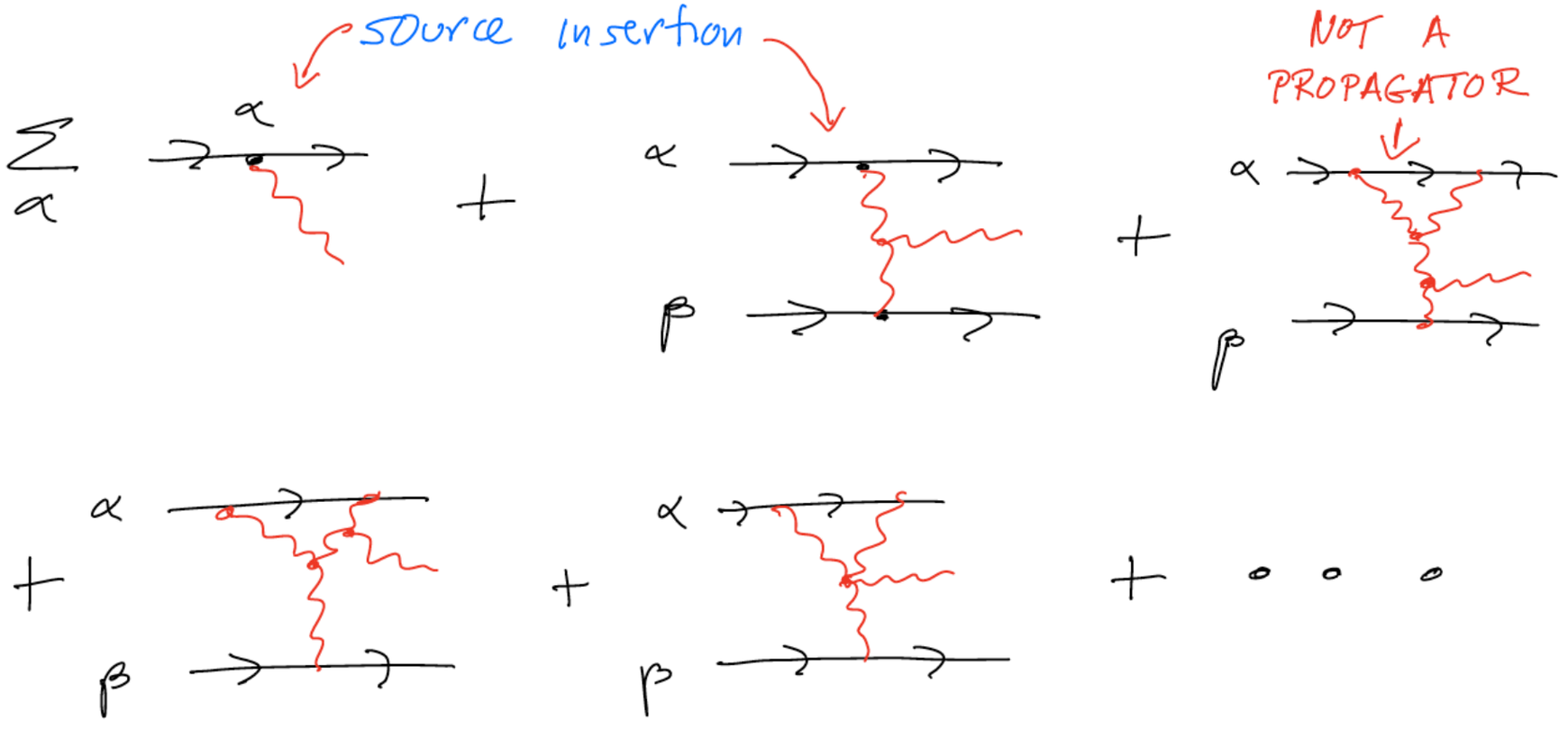
We find it convenient to solve the coupled equations in Lorentz gauge, in the form

$$\Box A_a^\mu = g\tilde{J}_a^\mu \qquad \tilde{J}_a^\mu = J_a^\mu + f^{abc}A_\nu^b(\partial^\nu A_c^\mu - F_c^{\mu\nu})$$

$$\partial_\mu \tilde{J}_a^\mu = 0$$

The classical solution is an off shell one-point function in the presence of (self-consistent) sources:

$$\langle A\rangle_J =$$



Our focus in this talk is on the radiation field measured by observers at $r \to \infty$ and fixed retarded time t (i.e. "asymptotic null future infinity"). This is related to the on-shell momentum space current

$$\tilde{J}_a^\mu(k) = \int d^d x e^{ik\cdot x} \tilde{J}_a^\mu(x)$$

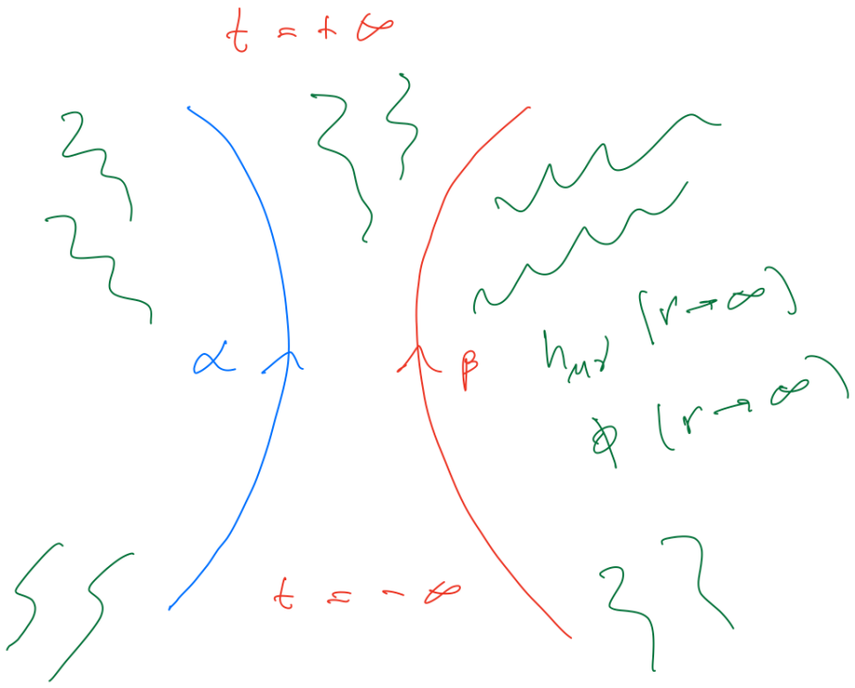
w/ $k^2 \to 0$. E.g., in four dimensions, the asymptotic gauge field is

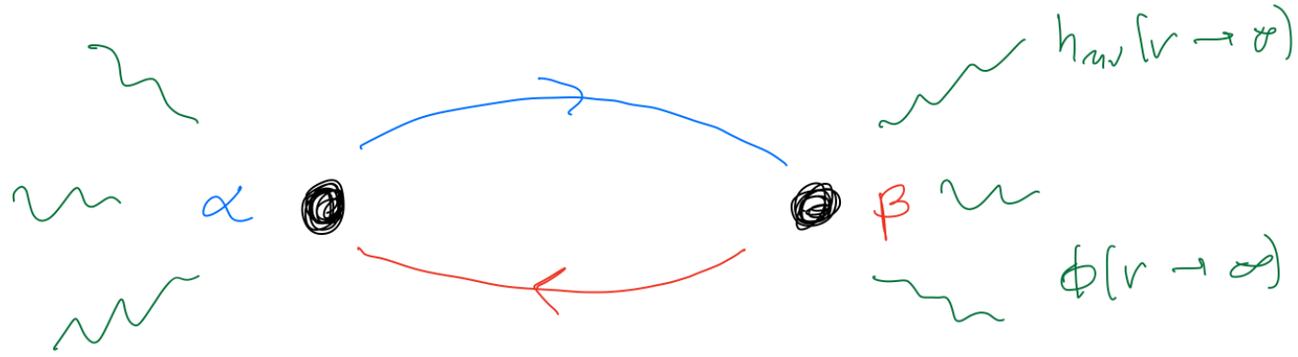
$$\lim_{r\to\infty} \langle A_\mu^a \rangle(x) = \frac{g}{4\pi r} \int \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{J}_a^\mu(k)$$

$$k^\mu = (\omega, \vec{k}) = \omega(1, \vec{x}/r)$$

This determines all observables measured by detectors at $r \to \infty$

Consider a collection of particles in generic time-dependent orbits:





$$\omega b \sim \mathcal{O}(1)$$

$$E/m \sim \mathcal{O}(1)$$

Formally, if the particles remain well separated, the perturbative solution for the radiation field can be constructed as an expansion in powers of the gauge coupling g .

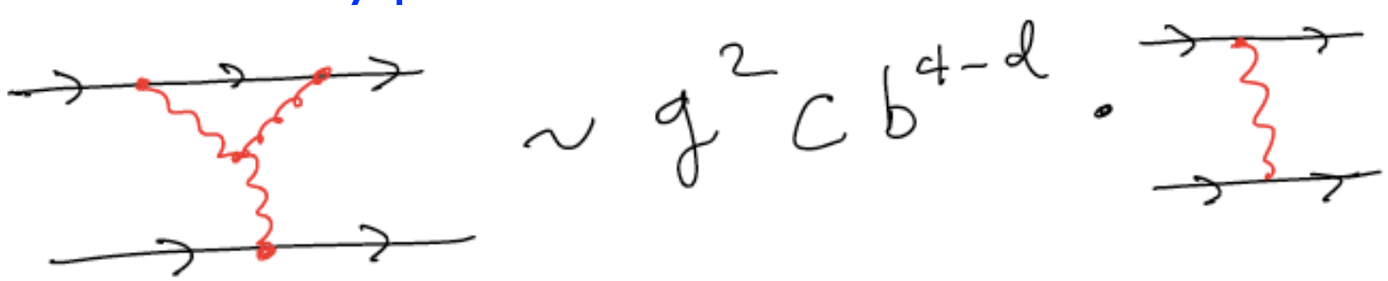
Less formally, for generic E/m and ωb, in the classical limit,

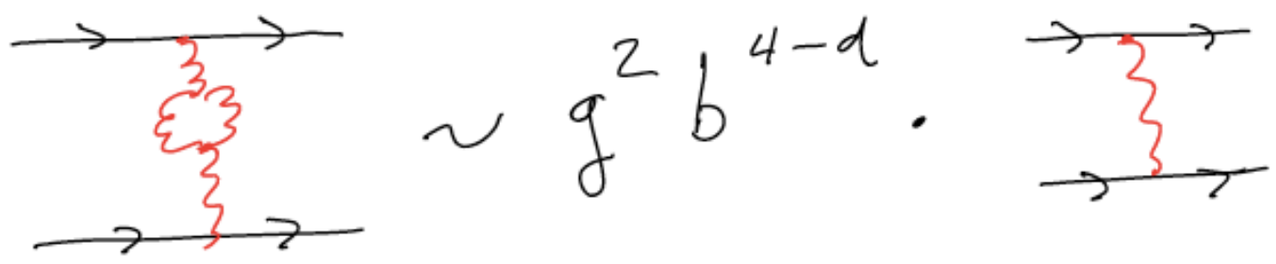
$$L \sim Eb \gg \hbar$$

perturbation theory is controlled by two independent expansion parameters. Non-linear corrections due to YM interactions are small as long as

$$g^2 c b^{4-d} \ll 1$$

E.g., corrections to the two-body potential:





2nd diagram is a quantum effect (one-loop beta fn.) which is suppressed as long as the color charges are sufficiently large $c^a \gg 1$. As we'll see, double copy works in the regime

$$c^a \sim Eb \gg 1$$

The other small expansion is in the corrections to the particle orbital and color deflections. The classical equations of motion

$$\frac{dp^\mu}{d\tau} = gc^a F_a^\mu{}_\nu v^\nu \qquad \frac{dc^a}{d\tau} = gf^{abc} v^\mu A_\mu^b(x(\tau))c^c(\tau)$$

are perturbative in the kinematic regime

$$g^2c^2/Eb^{d-3} \ll 1$$

where deflections away from straight line motion are small.

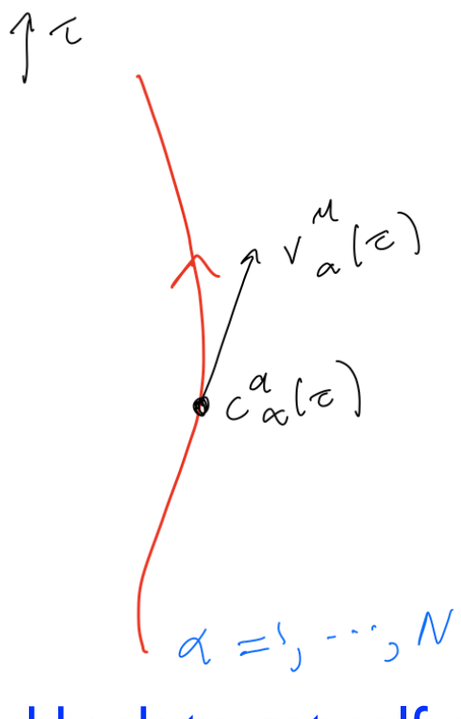
But note, for $c^a \sim Eb$

$$g^2c^2/Eb^{d-3} \sim g^2cb^{4-d}$$

the two expansion coincide, and perturbation theory is equivalent to an expansion in powers of the YM gauge coupling.

Perturbative solution: Leading order equations of motion:

A fixed configuration of worldlines $x^\mu(\tau), c^a(\tau)$ sources a gauge field



$$A^{a\mu}(x) = g\int \frac{d^4\ell}{(2\pi)^4}\frac{e^{-i\ell\cdot x}}{\ell^2}J^{a\mu}(\ell) = g\sum_\alpha \int d\tau \frac{d^4\ell}{(2\pi)^4}\frac{e^{-i\ell\cdot(x-x_\alpha)}}{\ell^2}c^a_\alpha(\tau)v^\mu_\alpha(\tau).$$

Feed back to get self-consistent eqns. of motion:

$$m_\alpha \dot{v}_\alpha = \quad ig^2\sum_\beta \int d\tau_\beta \frac{d^4\ell}{(2\pi)^4}\frac{e^{-i\ell\cdot x_{\alpha\beta}}}{\ell^2}(c_\alpha\cdot c_\beta)\left[(v_\alpha\cdot v_\beta)\ell^\mu - (\ell\cdot v_\alpha)v^\mu_\beta\right],$$

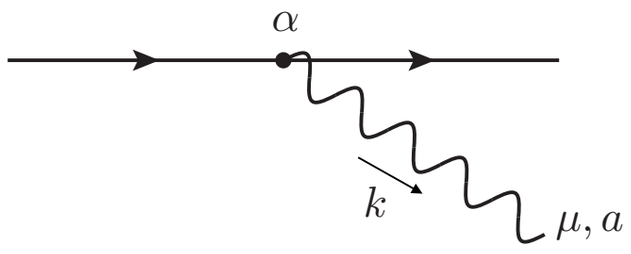
$$\dot{c}_\alpha = g^2\sum_\beta \int d\tau_\beta \frac{d^4\ell}{(2\pi)^4}\frac{e^{-i\ell\cdot x_{\alpha\beta}}}{\ell^2}f^{abc}c^b_\alpha c^c_\beta (v_\alpha\cdot v_\beta)$$

w/ $\quad x^\mu_\alpha = x^\mu_\alpha(\tau_\alpha) \qquad v^\mu_\alpha = v^\mu_\alpha(\tau_\alpha) \qquad c^a_\alpha = c^a_\alpha(\tau_\alpha)$

Radiation:

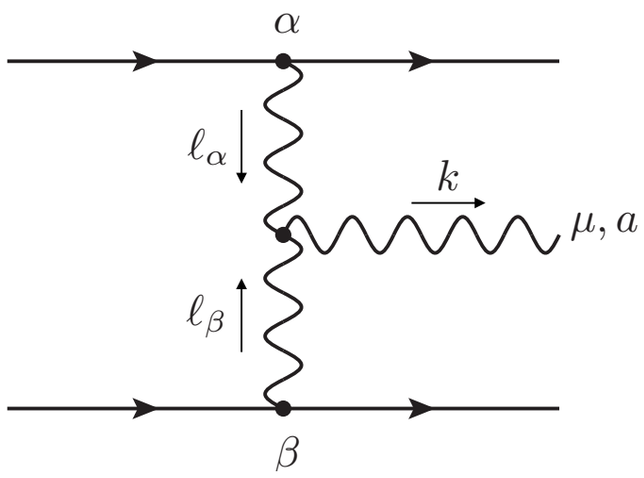
Starts at $\mathcal{O}(g^2)$. Two types of effects:

1. Direct emission by time-dependent particle current:



$$\tilde{J}^{a\mu}(k)\Big|_{\text{worldline}} = \sum_\alpha \int dx^\mu(\tau)_\alpha c^a_\alpha(\tau) e^{ik\cdot x_\alpha(\tau)}$$

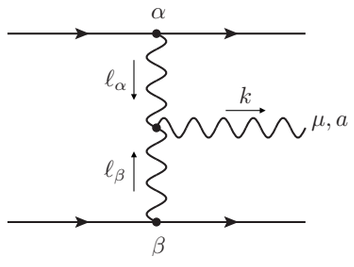
2. Gluon emission sourced by self-interaction



$$\propto f^{abc}$$

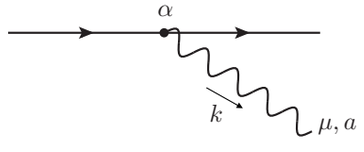
The total current has two types of color structures:

$$\tilde{J}^{a\mu}(k) = g^2 \sum_{\alpha,\beta} \int_{\alpha,\beta} d\mu_{\alpha\beta}(k) \left[i f^{abc} c_\alpha^b c_\beta^c \mathcal{A}^\mu_{adj} + (c_\alpha \cdot c_\beta) \frac{c_\alpha^a}{m_\alpha} \mathcal{A}^\mu_s \right],$$



color eoms

orbital eom's



Integration measure:

$$d\mu_{\alpha\beta}(k) = d\tau_\alpha d\tau_\beta \left[\frac{d^4\ell_\alpha}{(2\pi)^4} \frac{e^{i\ell_\alpha \cdot x_\alpha}}{\ell_\alpha^2} \right] \left[\frac{d^4\ell_\beta}{(2\pi)^4} \frac{e^{i\ell_\beta \cdot x_\beta}}{\ell_\beta^2} \right] \times (2\pi)^4 \delta(\ell_\alpha + \ell_\beta - k)$$

Partial amplitudes:

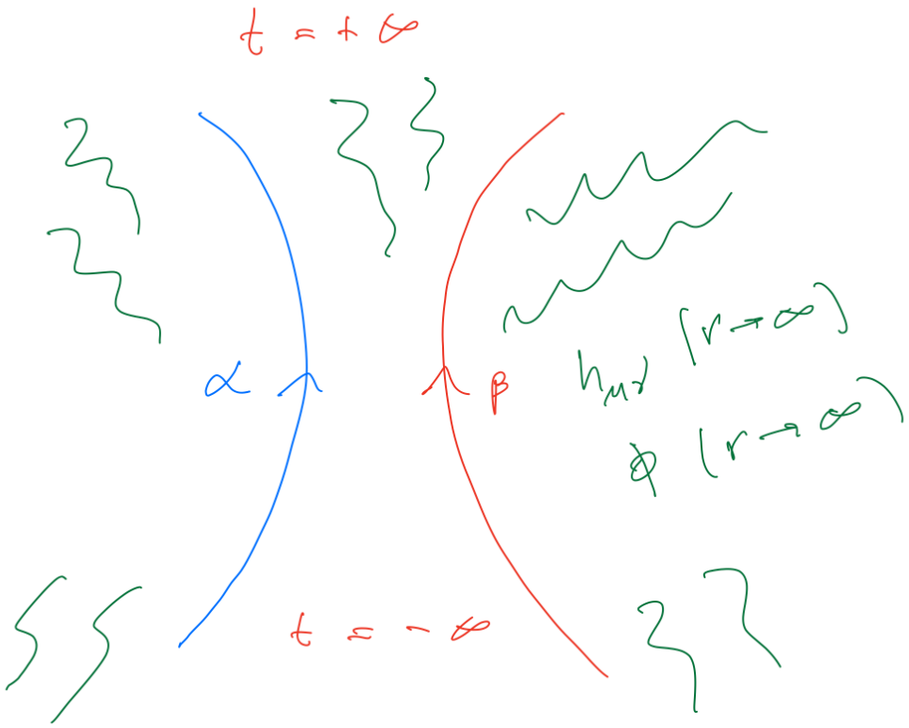
$$\mathcal{A}^\mu_{adj} = (v_\alpha \cdot v_\beta) \left[\frac{1}{2}(\ell_\beta - \ell_\alpha)^\mu + \frac{\ell_\alpha^2}{k \cdot v_\alpha} v_\alpha^\mu \right] + (k \cdot v_\beta) v_\alpha^\mu - (k \cdot v_\alpha) v_\beta^\mu,$$

$$\mathcal{A}^\mu_s = -\frac{\ell_\alpha^2}{k \cdot v_\alpha} \left[(v_\alpha \cdot v_\beta) \left(\ell_\beta^\mu - \frac{k \cdot \ell_\beta}{k \cdot v_\alpha} v_\alpha^\mu \right) - (k \cdot v_\alpha) v_\beta^\mu + (k \cdot v_\beta) v_\alpha^\mu \right].$$

As a check, the current obeys the Ward identity $k_\mu \tilde{J}^\mu{}_a(k) = 0$, but only after self-consistent particle eqns. of motion are plugged in to the diagrams.

Consistency checks I: Classical scattering and bremsstrahlung

1. Classical scattering and gluon bremsstrhalung: Initially well separated particles with constant $b^\mu_\alpha, v^\mu_\alpha, c^a_\alpha$ at early times



$$x^\mu_\alpha(\tau \to -\infty) = b^\mu_\alpha + v^\mu_\alpha \tau$$

$$c^a_\alpha(\tau \to -\infty) = c^a_\alpha$$

agrees with earlier results in $d = 4$ (Gyulassi+McLerran) and in general d (WG+Ridgway, 2016)

Consistency checks II: Non-relativistic limit

$$v^\mu_\alpha(\tau) \approx (1, \vec{v}_\alpha = d\vec{x}_\alpha/dt) \qquad |\vec{v}_\alpha| \ll 1$$

Eqns of motion reduce to (in $d = 4$)

$$\dot{c}_\alpha(t) = -\alpha_s \sum_\beta \frac{f^{abc} c^b_\alpha c^c_\beta}{|\vec{x}_{\alpha\beta}|}, \qquad m_\alpha \dot{\vec{v}}_\alpha(t) = \alpha_s \sum_\beta \frac{(c_\alpha \cdot c_\beta)\vec{x}_{\alpha\beta}}{|\vec{x}_{\alpha\beta}|^3},$$

Using the e.o.m's, the current simplifies to:

$$J^0_a(x) = \delta^3(\vec{x}) Q^a - \vec{p}_a(t) \cdot \nabla \delta^3(\vec{x}) + \mathcal{O}(v^2)$$

$$\vec{J}_a(x) = \delta^3(\vec{x}) \dot{\vec{p}}_a(t) + \mathcal{O}(v^2)$$

w/ color monopole moment $Q^a = \sum_\alpha c^a_\alpha(t) \qquad (\dot{Q}^a = 0)$ and color electric dipole

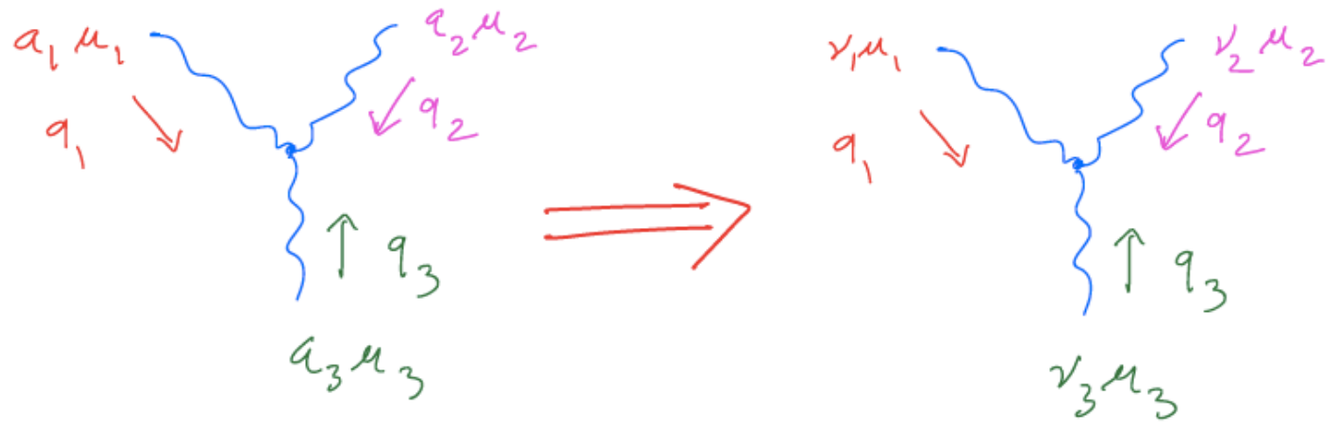
$$\vec{p}^a(t) = \sum_\alpha c^a_\alpha(t) \vec{x}^a_\alpha(t)$$

See Ridgway's talk for details…

Double copy of gauge theory

Inspired by BCJ, we make the following formal replacements to the gauge theory result

$$c^a_\alpha(\tau) \to i p^\mu_\alpha(\tau)$$



$$if^{a_1a_2a_3} \to \Gamma^{\nu_1\nu_2\nu_3}(q_1,q_2,q_3) = -\frac{1}{2}\left[\eta^{\nu_1\nu_3}(q_1-q_3)^{\nu_2} + \eta^{\nu_1\nu_2}(q_2-q_1)^{\nu_3} + \eta^{\nu_2\nu_3}(q_3-q_2)^{\nu_1}\right],$$

$$p^\mu_\alpha(\tau) \to p^\mu_\alpha(\tau)$$

$$g \to \frac{1}{2m_{Pl}^{d/2-1}}$$

Under this map:

$$\tilde{J}^{\mu}_{a}(k) \to i\tilde{T}^{\mu\nu}(k)$$

w/

$$\tilde{T}^{\mu\nu}(k) = \frac{1}{4m_{Pl}^{d-2}} \sum_{\alpha,\beta} m_\alpha m_\beta \int d\mu_{\alpha\beta}(k) \left[\left(\frac{1}{2}(v_\alpha \cdot v_\beta)(\ell_\beta - \ell_\alpha)^\nu + (v_\beta \cdot k)v_\alpha^\nu - (v_\alpha \cdot k)v_\beta^\nu \right) \mathcal{A}^{\mu}_{adj} \right.$$
$$\left. -(v_\alpha \cdot v_\beta)v_\alpha^\nu \hat{\mathcal{A}}^{\mu}_{s} \right]$$

For $k^2 = 0$ this object satisfies

$$\tilde{T}^{\mu\nu}(k) = \tilde{T}^{\nu\mu}(k) \qquad\qquad k_\mu \tilde{T}^{\mu\nu}(k) = 0$$

so it defines a consistent pseudo-tensor in some theory of gravity, with (in $d = 4$) radiation fields

$$h_\pm(t, \vec{n}) = \frac{4G_N}{r} \int \frac{d\omega}{2\pi} e^{-i\omega t} \epsilon_\pm^{*\mu\nu}(k) \tilde{T}_{\mu\nu}(k),$$

$$\phi(t, \vec{n}) = \frac{G_N}{r} \int \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{T}^{\mu}{}_{\mu}(k)$$

at future null infinity (retarded time t and $r \to \infty$).

Dilaton gravity:

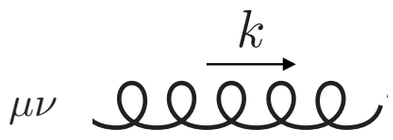
In fact, the gravity theory defined by $\tilde{T}^{\mu\nu}(k)$ is a local theory coupled to classical point sources, with Lagrangian

$$S = S_g + S_{pp}$$

$$S_g = -2m_{Pl}^{d-2}\int d^dx\sqrt{g}\left[R-(d-2)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]$$

$$S_{pp} = -m\int d\tau e^{\phi}+\cdots\,, \qquad \text{“}e^{\phi}=1+\phi+\frac{1}{2!}\phi^2+\cdots\text{”}$$

(up to higher derivatives, more powers of the dilaton…). The role of the dilaton is to cancel the explicit d's from graviton exchange Bern+Grant(1999); Scherk+Schwarz (1974):



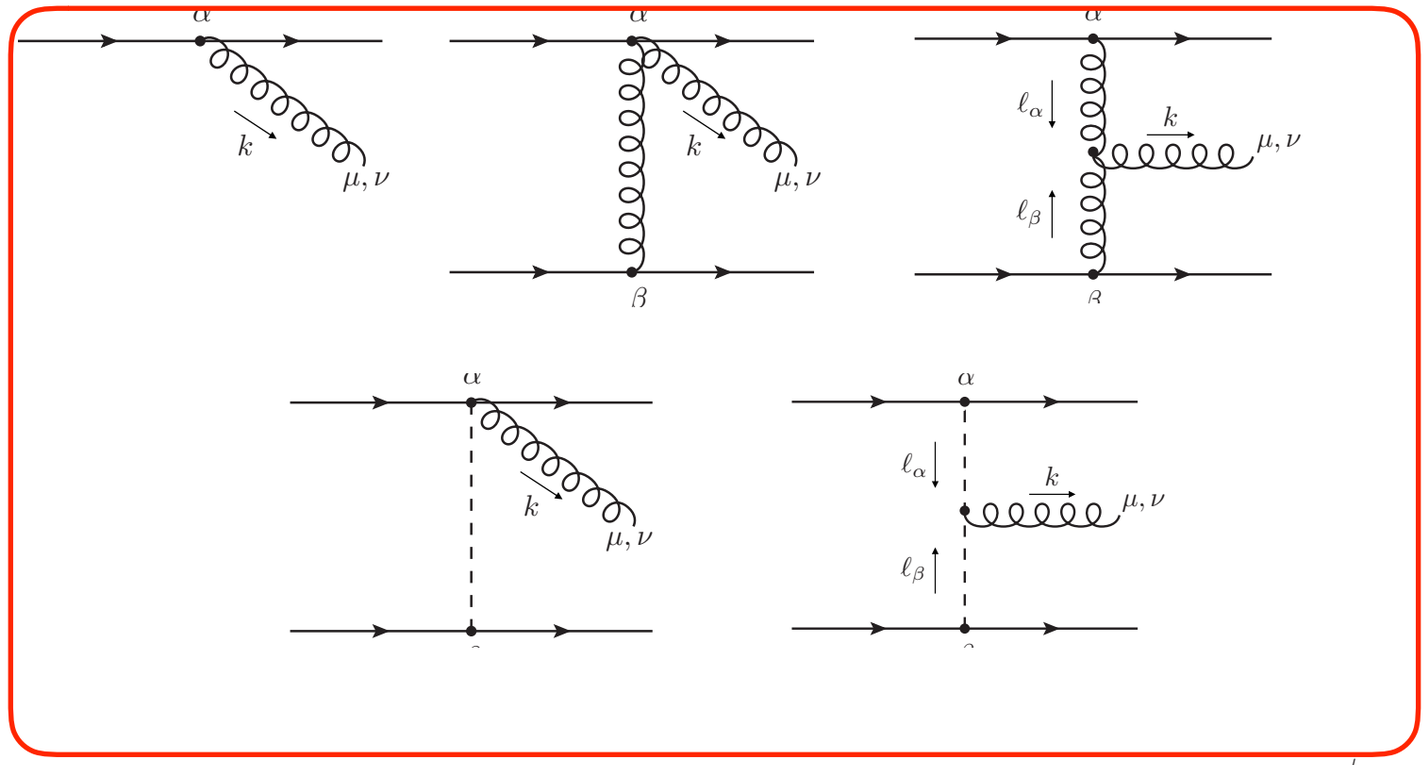
$$= \frac{i}{2m_{Pl}^{d-2}}\frac{1}{k^2}\left[\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}-\frac{2}{d-2}\eta_{\mu\nu}\eta_{\rho\sigma}\right]$$

It is also motivated by BCJ for pure gauge theory, and by results of KLT:

$$A_\mu\otimes A_\nu = \phi\oplus B_{\mu\nu}\oplus h_{\mu\nu}$$

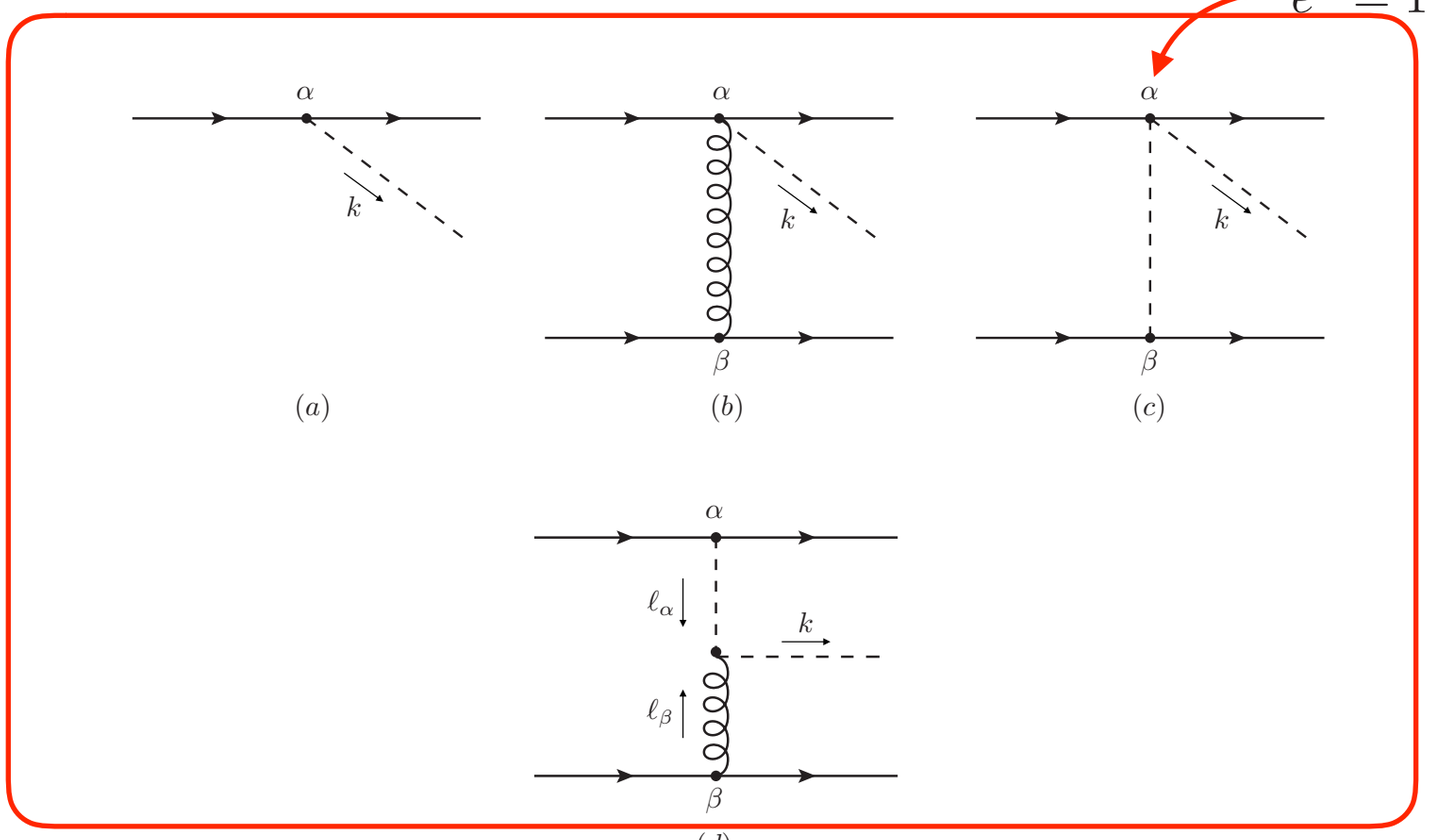
In the case of bremsstrhalung, we verified this explicitly by direct calculation (WG+Ridgway 2016) in the graviton and dilaton channels:

graviton emission channel:



$$e^{\phi} = 1 + \phi + \frac{1}{2!}\phi^2 + \cdots$$

scalar channel:



We also find that $\tilde{T}^{\mu\nu}(k)$ has the correct non-relativistic limit in $d = 4$:

$$h_{ij}(t,\vec{n}) = \frac{2G_N}{r}[\ddot{Q}_{ij}(t)]^{TT} \qquad Q^{ij} = \sum_\alpha m_\alpha \left(x^i_\alpha x^j_\alpha - \frac{1}{3}\delta^{ij}\vec{x}^2_\alpha \right)$$

$$\phi(t,\vec{n}) = \frac{G_N}{r}\sum_\alpha m_\alpha \left(\vec{v}^2_\alpha - \vec{x}_\alpha \cdot \ddot{\vec{x}}_\alpha - \frac{1}{2}\frac{d^2}{dt^2}(\vec{x}_\alpha \cdot \vec{n})^2 \right),$$

corresponding to the radiation fields sourced by a collection of interacting particles:

$$m_\alpha \dot{\vec{v}}_\alpha = -2G_N \sum_\beta \frac{m_\alpha m_\beta \vec{x}_{\alpha\beta}}{|\vec{x}_{\alpha\beta}|^3},$$

This is consistent with the post-Newtonian limit of more general scalar-tensor theories in 4D.
(Will+Zaglauer 1989; Damour+Esposito-Farrese, 1992)

(see Ridgway's talk for more details…)

Axion radiation and particle spin: (WG, Li, Prabhu, 1712.?????)

For spinless particles, we found no radiation in the anti-symmetric channel

$$a_{\mu\nu}(k)\tilde{T}^{\mu\nu}(k) = 0$$

$$a_{\mu\nu}(k) = -a_{\mu\nu}(k)$$

$$k^{\mu}a_{\mu\nu}(k) = 0$$

This is expected on the basis of symmetry: There are no linear pt. particle couplings of the $B_{\mu\nu}$ field that are consistent w/.

Two-form gauge invariance: $B \to B + d\lambda$

Diff. invariance

unless the particles carry a spin degree of freedom $S^{\mu\nu}(\tau) = -S^{\nu\mu}(\tau)$:

$$S_{pp} \supset \int dx^{\mu}\tilde{\kappa}(\phi)S^{\nu\sigma}H_{\mu\nu\sigma}$$

Does the spin interaction w/ $B_{\mu\nu}$ arise as double copy of classical YM? Add a chromomagnetic interaction to the classical color charge:

$$S_{pp} = -\int dx^\mu c_a(s) A^a_\mu + \frac{g\kappa}{2}\int ds e c_a(s) S^{\mu\nu} F^a_{\mu\nu} + \cdots ,$$

Color current is modified due to spin:

$$J^\mu_a(x) = -\frac{1}{g_s}\frac{\delta}{\delta A^a_\mu(x)} S_{int} = \sum_\alpha \int dx^\mu_\alpha c^a_\alpha(s)\delta(x - x(s)) - \kappa_\alpha \int ds e S^{\mu\sigma} D_\sigma \left[c^a_\alpha \delta(x - x(s))\right].$$

As is the energy-momentum tensor

$$T^{\mu\nu}_{pp}(x) = \int dx^{(\mu} p^{\nu)} \delta(x - x(s)) + \int dx^{(\mu} S^{\nu)\sigma} \partial_\sigma \delta(x - x(s) - \kappa g_s \int ds e \frac{\delta(x - x(s))}{\sqrt{g}} c_a F^a{}_\sigma{}^{(\mu} S^{\nu)\sigma},$$

with spin $S^{\mu\nu}(s) = -S^{\nu\mu}(s)$ defined to obey the constraint $p_\mu S^{\mu\nu} = 0$. For $F^{\mu\nu}_a = 0$:

$$P^\mu = \int d^3\mathbf{x} T^{0\mu}(\mathbf{x}, x^0)$$ = "CM" particle momentum

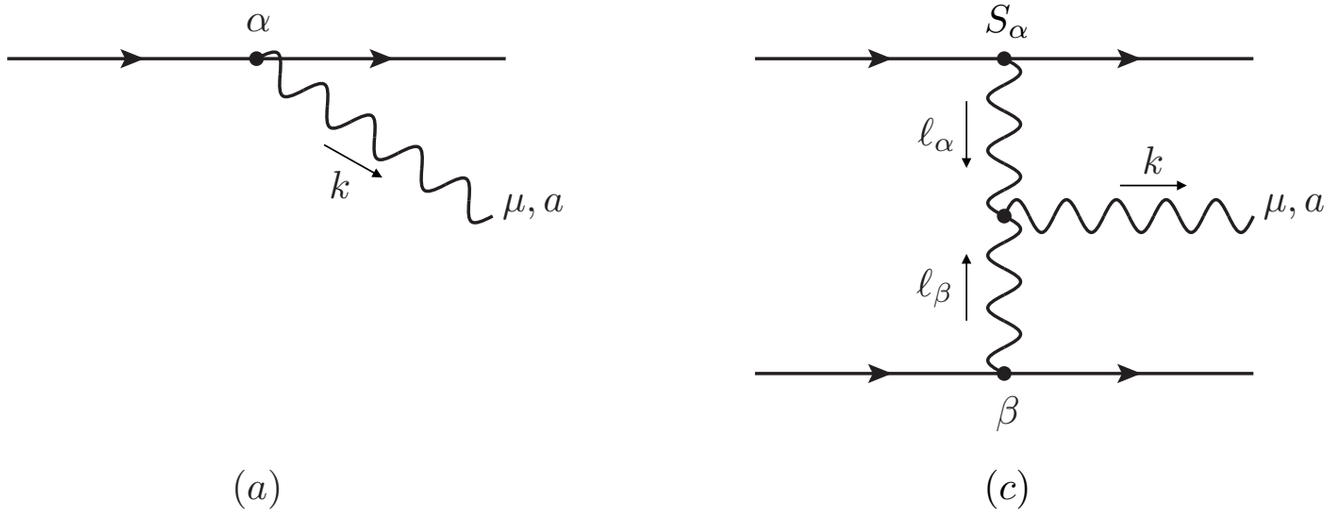
$$J^{\mu\nu} = \int d^3\mathbf{x} x^{[\mu} T^{0\nu]}(\mathbf{x}, x^0) = x^\mu p^\nu - x^\nu p^\mu + S^{\mu\nu}$$ = "CM" particle angular momemtum

Eqns of motion from conservation laws:

$$D_\mu J^\mu_a = 0 \quad \Longrightarrow \quad (v\cdot D)c^a = \frac{i}{2} g_s \kappa [S^{\mu\nu} F_{\mu\nu}, c]^a$$

$$\partial_\mu T^{\mu\nu} = 0 \quad \Longrightarrow \quad \begin{aligned} \frac{d}{ds} p^\mu &= g_s c^a F_a^{\mu\nu} v_\nu - \frac{1}{2}\kappa g_s c_a S^{\alpha\beta} D^\mu F^a_{\alpha\beta} \\ \frac{d}{ds} S^{\mu\nu} &= p^\mu v^\nu - p^\nu v^\mu - 2\kappa g_s c^a F_a^{\alpha[\mu} S_\alpha{}^{\nu]} \end{aligned}$$

Then the total current at $\mathcal{O}(g^2)$, working to linear order in spin:



Then the total current at $\mathcal{O}(g^2)$, working to linear order in spin:

$$\tilde{J}_a^\mu(k)\Big|_{\mathcal{O}(S^1)} = ig_s^2 \sum_{\alpha,\beta} \int \mu_{\alpha,\beta}(k) \left((c_\alpha \cdot c_\beta) c_\alpha^a \mathcal{A}_s^\mu + [c_\alpha, c_\beta]^a \mathcal{A}_a^\mu \right)$$

where now the partial amplitudes are

$$\mathcal{A}_a^\mu = \kappa_\alpha \left[(\ell_\alpha \wedge p_\beta)_\alpha (\ell_\beta - \ell_\alpha)^\mu - \frac{\ell_\alpha^2}{k \cdot p_\alpha} (\ell_\beta \wedge p_\beta)_\alpha p_\alpha^\mu - \frac{\ell_\beta^2}{k \cdot p_\beta} (\ell_\alpha \wedge p_\beta)_\alpha p_\beta^\mu + \ell_\alpha^2 (S_\alpha \wedge p_\beta)^\mu \right]$$
$$-2\kappa_\alpha \left[(\ell_\alpha \wedge \ell_\beta)_\alpha p_\beta^\mu + (k \cdot p_\beta)(S_\alpha \wedge \ell_\alpha)^\mu \right] - \kappa_\alpha \frac{\ell_\alpha^2}{k \cdot p_\alpha} (p_\alpha \cdot p_\beta)(S_\alpha \wedge k)^\mu .$$

$$\mathcal{A}_s^\mu = \frac{(1+\kappa_\alpha)^2}{m_\alpha^2} \ell_\alpha^2 \left[(k \cdot p_\alpha) \left\{ (S_\alpha \wedge p_\beta)^\mu - \frac{(k \wedge p_\beta)_\alpha}{k \cdot p_\alpha} p_\alpha^\mu \right\} + (p_\alpha \cdot p_\beta) \left\{ (S_\alpha \wedge \ell_\beta)^\mu - \frac{(\ell_\alpha \wedge \ell_\beta)_\alpha}{k \cdot p_\alpha} p_\alpha^\mu \right\} \right]$$
$$-\kappa_\alpha^2 \frac{\ell_\alpha^2}{k \cdot p_\alpha} \left[\left\{ (\ell_\alpha \wedge \ell_\beta)_\alpha p_\beta^\mu - (k \cdot p_\beta)(S_\alpha \wedge \ell_\beta)^\mu \right\} + \left\{ (k \cdot \ell_\beta)(S_\alpha \wedge p_\beta)^\mu - (k \wedge p_\beta)_\alpha \ell_\beta^\mu \right\} \right]$$
$$+\kappa_\alpha \frac{\ell_\alpha^2}{k \cdot p_\alpha} \left[(\ell_\beta \wedge p_\beta)_\alpha \left\{ \ell_\beta^\mu - \frac{k \cdot \ell_\beta}{k \cdot p_\alpha} p_\alpha^\mu \right\} + (k \cdot p_\beta)(S_\alpha \wedge k)^\mu \right] + \kappa_\beta \frac{\ell_\alpha^2}{k \cdot p_\alpha} (\ell_\beta \wedge p_\alpha)_\beta \left[\ell_\beta^\mu - \frac{k \cdot \ell_\beta}{k \cdot p_\alpha} p_\alpha^\mu \right]$$
$$-\kappa_\alpha \frac{\ell_\alpha^2}{(k \cdot p_\alpha)^2} (p_\alpha \cdot p_\beta)(k \cdot \ell_\beta)(S_\alpha \wedge k)^\mu$$

$$(S_\alpha \wedge a)^\mu = S_\alpha^{\mu\nu} a_\nu \qquad (a \wedge b)_\alpha = a \cdot (S_\alpha \wedge b) \qquad \left(k_\mu \tilde{J}_a^\mu(k) = 0\right)$$

Double copy

Same formal replacement rules as before, w/. $S^{\mu\nu} \to S^{\mu\nu}$ yields a gravitational field

$$\tilde{J}^{\mu}_{a}(k) \to i\tilde{T}^{\mu\nu}(k)$$

which is again of the form

$$\tilde{T}^{\mu\nu}(k) = \frac{1}{4m_{Pl}^{d-2}} \sum_{\alpha,\beta} \int d\mu_{\alpha\beta}(k) \left[\left(\frac{1}{2}(p_\alpha \cdot p_\beta)(\ell_\beta - \ell_\alpha)^\nu + (p_\beta \cdot k)p_\alpha^\nu - (p_\alpha \cdot k)p_\beta^\nu \right) \mathcal{A}^{\mu}_{adj} \right.$$
$$\left. -(p_\alpha \cdot p_\beta)p_\alpha^\nu \mathcal{A}^{\mu}_{s} \right]$$

This is now non-zero in the axion channel:

$$\tilde{T}^{\mu\nu}(k) - \tilde{T}^{\nu\mu}(k) \neq 0$$

However, demanding that this is a consistent radiation field that satisfies the Ward identity

$$k_\mu \tilde{T}^{\mu\nu}(k) = k_\nu \tilde{T}^{\mu\nu}(k) = 0 \qquad (k^2 = 0)$$

requires that the chromomagnetic interaction have the specific value

$$\kappa = -1$$

indicating that the classical double copy does not work for generic sources on the gauge theory side

(In the NR limit, for $d = 4$ the dipole interaction is

$$S_{int} = -\frac{g_s \kappa}{m} \int dt c_a \vec{S} \cdot \vec{B}^a$$

$\kappa = -1$ ⟹ g-factor $g = g_{Dirac} = 2$

So this is a classical spin $S \gg \hbar$ with the magnetic properties of a Dirac particle.)

For the choice of magnetic coupling where the double copy works, the source $\tilde{T}^{\mu\nu}(k)$ describes radiation in a local theory of massless particles $\phi, B_{\mu\nu}, h_{\mu\nu}$

At the two derivative level, the form of this theory is fixed by diffeomorphism and two-form gauge invariance to be :

$$S_g = -2m_{Pl}^{d-2}\int d^dx\sqrt{g}\left[R-(d-2)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+\frac{1}{12}f(\phi)H_{\mu\nu\sigma}H^{\mu\nu\sigma}\right]$$

$$f(\phi)=1+c\phi+\cdots$$

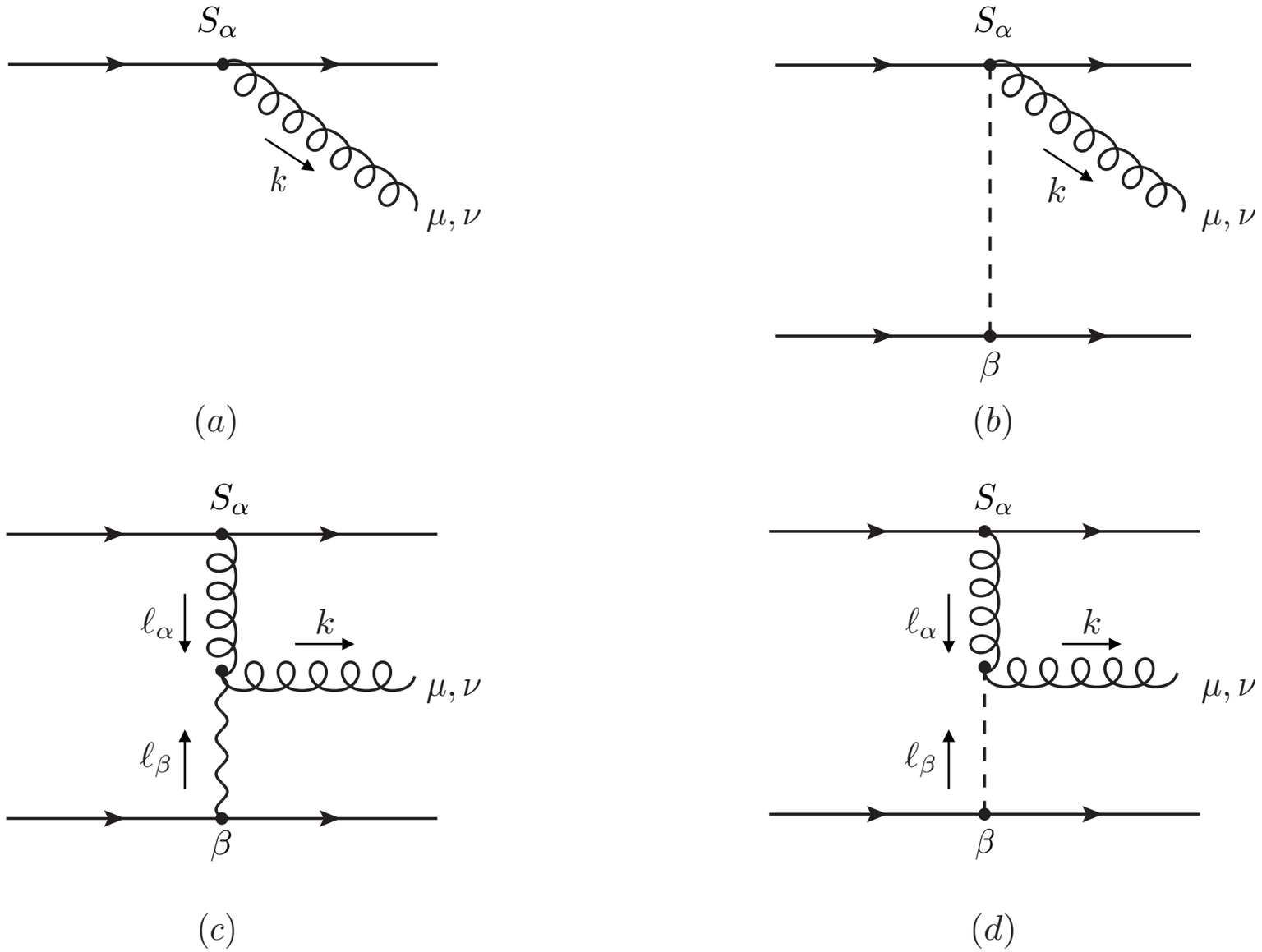
with couplings to point sources of the form

$$S_{pp}=\int dx^\mu\tilde{\kappa}(\phi)H_{\mu\nu\sigma}S^{\nu\sigma}$$

$$\tilde{\kappa}(\phi)=\tilde{\kappa}+\tilde{\kappa}'\phi+\cdots$$

We'll fix the unknown constants by comparing to the double copy to the gravitational emission amplitude.

In addition to spin corrections to particle eqns. of motion, must calculate the diagrams



(a) (b) (c) (d)

In order to cancel the explicit d dependence of the diagrams, need to fix dilaton couplings

$$c = -4, \tilde{\kappa}' = 0$$

to the axion and point particles respectively.

Furthermore, we need specific choice of axion-spin coupling $\tilde{\kappa} = 2$

$$S_{pp} = 2 \int dx^\mu H_{\mu\nu\sigma} S^{\nu\sigma}$$

in which case we find agreement between axion radiation in gravity and its double copy. At linear order in dilaton couplings, the gravity theory is consistent with the form

$$S_g = -2m_{Pl}^{d-2} \int d^d x \sqrt{g} \left[R - (d-2) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{12} e^{-4\phi} H_{\mu\nu\sigma} H^{\mu\nu\sigma} \right]$$

which precisely matches the the action for the $(\phi, g_{\mu\nu}, B_{\mu\nu})$ sector of string theory. This can also written as (Scherk, Schwarz 1974)

$$\tilde{g}_{\mu\nu} = e^{2\phi} g_{\mu\nu}$$

$$S_g = -2m_{Pl}^{d-2} \int d^d x \sqrt{\tilde{g}} R(\tilde{g}, \hat{\Gamma})$$

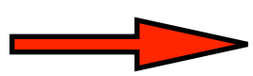
$\hat{\Gamma}^\mu_{\alpha\beta} =$ non-Riemannian connection w torsion

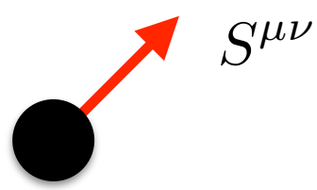
Spin correction to dilaton +graviton channels are also consistent with the double copy for this choice of parameters (Li, Prabhu, 1712.????)

What are these things?

The double copy is only consistent if on the gauge theory side, the sources have chromomagnetic coupling $\kappa = -1$.

Are there fundamental classical objects with such couplings? Look at the open (bosonic) string with Chan-Paton color charges. Taking point particle limit of the string





$$c_a = c_a(0) + c_a(\ell)$$

$$S_{int} = \sum_{\sigma=0,\ell} \int dX^\mu(\tau,\sigma) c^a(\sigma,\tau) A^a_\mu(X)$$

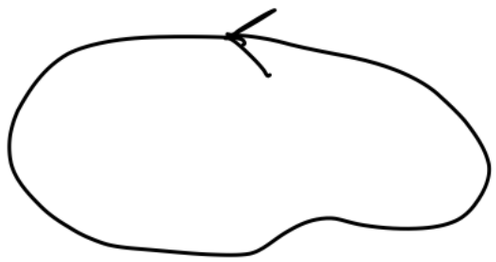
$$-g_s \int dx^\mu c_a A^a_\mu - \frac{1}{2} g_s \int d\tau c_a S^{\mu\nu} F^a_{\mu\nu}$$

$$S^{\mu\nu} = \sum_{n>0} \frac{i}{n} \left(\alpha^\mu_n \alpha^\nu_{-n} - \alpha^\mu_{-n} \alpha^\nu_n \right)$$

So the open string is an example of a fundamental massive object with $g_D = 2$

(See also d=4 Kerr-Newman BH, which has $g_D = 2$ (Horowitz, 1992))

We can also do this multipole expansion for the oriented (bosonic) closed string



$$S_{int} = \frac{1}{2\pi\alpha'} \int d^2\sigma \epsilon^{ab} B_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu$$

$$S_{int} = 2 \int dx^\mu H_{\mu\nu\sigma} \left(S_L^{\nu\sigma} - S_R^{\nu\sigma} \right)$$

$$S_R^{\mu\nu} = \sum_{n>0} \frac{i}{n} \left(\alpha_n^\mu \alpha_{-n}^\nu - \alpha_{-n}^\mu \alpha_n^\nu \right)$$

$$S_L^{\mu\nu} = \sum_{n>0} \frac{i}{n} \left(\tilde{\alpha}_n^\mu \tilde{\alpha}_{-n}^\nu - \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu \right)$$

Magnitude of coupling is correct, but relative sign between left-, right- moving spins. This is expected from worldsheet parity $P : L \leftrightarrow R$.

Axion does not couple to string's CM spin $S^{\mu\nu} = S_L^{\mu\nu} + S_R^{\mu\nu}$ due to lack of rotational invariance of the closed string configuration.

Rather, to get something whose coupling is shape is uniquely characterized by M, J, Q, couplings

$S_{pp} = 2 \int dx^\mu H_{\mu\nu\sigma} S^{\nu\sigma}$ we need an object whose ie some sort of spinning BH with dilaton+axion

Conclusions:

Feynman integral + EFT methods are a powerful tool in gravitational wave calculations. Can they be streamlined using the double copy?

Color-kinematics relations between bi-adjoint scalar, YM and gravity radiation solutions:

$$\mathcal{A}^{a\tilde{a}} \to \mathcal{A}^{a\mu} \to \mathcal{A}^{\mu\nu}$$

Under color-kinematics:

Bi-adjoint charge → YM color charge →

$$S_{pp} = -m \int d\tau e^{\phi} + \cdots ,$$

(at least to quadratic order in the scalar)

$$\int d\tau c_a S^{\mu\nu} F^a_{\mu\nu} \quad \Rightarrow \quad \int dx^{\mu} S^{\nu\sigma} H_{\mu\nu\sigma}$$

But it only works for specific numerical values of the coupling constants…

Open questions:

Higher orders in PT? Relation to BCJ? Relation to string theory?

Assuming it persists at higher orders in PT, is the classical double copy useful for gravity wave calculations? Need to remove the unwanted dilaton+axion. Recent progress: Luna, Nicholson, O'Connell, White, 1711.03901, adapting methods of Johansson+Ochirov, 1407.4772

Classical double copy as formulated here only works for specific (spinning) objects in QCD and gravity. These are effective point particle limits of more fundamental classical objects. What are they?