UV/IR Polytope: carving out the space for consistent gauge and gravity theories

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with Nima Arkani-Hamed, Tzu-Chen huang

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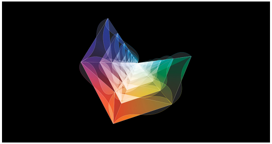
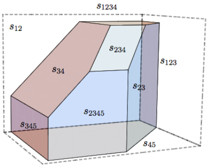
Through the lens of physical observables, a new picture has arose where physical principles (unitarity, locality, symmetries) are geometrized: see Nima, Song's and Yun-Tao's talk

Planar N=4 SYM



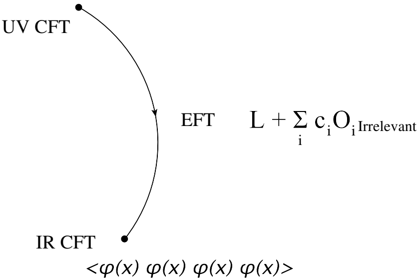
tree level ϕ^3



Is there any sense in which this picture can be universal ?

More precisely, can we geometrize the constraint of unitarity + symmetries for general QFT ?

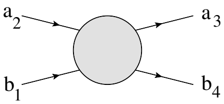
- What observables through which this geometry is realised?
- What are the content to be constrained?



- EFT: S-matrix, coefficient of higher-dimensional (irrelevant) operators
- CFT: Four point correlation function, conformal dimension and three-point coupling of primary operators.

EFT

In the IR the UV degrees of freedom are encoded in the higher dimensional operators. These information are encoded in the four-point function as



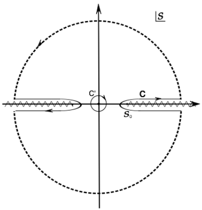
$$M(s,t) = \sum_{i,j} g_{i,j} s^i t^j$$

where $s = (p_1 + p_2)^2$ $t = (p_1 + p_4)^2$. For example:

$$\mathcal{L} = \frac{1}{2}\phi\Box\phi + a(\partial\phi\cdot\partial\phi)^2 \rightarrow M(s,t) = a(s^2 + st + t^2)$$

Naively, without an explicit UV completion, we have nothing to say. This is not true in the forward limit $t = 0$ A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis and R. Rattazzi:

$$g_{i,0} = \oint \frac{ds}{s^{i+1}} M(s,0) =$$

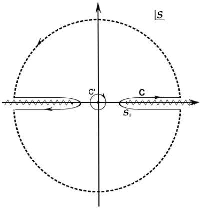


$$= \int_{s_0}^{\infty} Im[M(s,0)] \quad i \in even$$

But from the optical theorem we know that $Im[M(s,0)] = s\sigma \geq 0 \rightarrow g_{i,0} \geq 0$

For $t \neq 0$, causality bound $M(s,t)|_{s\to\infty} < s^2$. This implies that after subtraction, $M' = M - a_2 s^2 - a_1 s$, we again have a bounded function:

$$M'(s,t) = \oint_C \frac{dv}{v-s} M'(v,t) =$$

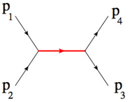


$$- \oint_{C'} \frac{dv}{v-s} M'(v,t)$$

This implies an alternative representation:

$$M'(s,t) = \sum_{i=1}^{\infty} \left(\frac{1}{s - m_i^2} + \frac{1}{u - m_i^2} \right) n(m_i, t)$$

- Lorentz invariance + Unitarity dictates



$$\rightarrow \quad A_3(\phi_1, \phi_2, h^\ell) \sim i c_\ell (p_1 - p_2)^{\mu_1} (p_1 - p_2)^{\mu_2} \cdots (p_1 - p_2)^{\mu_\ell} \epsilon_{\mu_1 \mu_2 \cdots \mu_\ell}$$

The residue must take the form ($X \equiv p_1 - p_2, \;\; Y \equiv p_3 - p_4$):

$$X^{\mu_1} X^{\mu_2} \ldots X^{\mu_\ell} \mathcal{P}_{\mu_1 \cdots \mu_\ell \nu_1 \cdots \nu_\ell} Y^{\nu_1} Y^{\nu_2} \ldots Y^{\nu_\ell}$$

where $\mathcal{P}_{\mu_1 \cdots \mu_\ell \nu_1 \cdots \nu_\ell}$ is symmetric traceless. This implies

$$\Box_X f(X, Y) = \delta^{D-1}(X - Y) \rightarrow \frac{1}{|1 - \cos\theta t + t^2|^{D-3/2}} = \sum_\ell t^\ell G_\ell^D(\cos\theta)$$

$$n(m_i, t) = c_i^2 G_{\ell_i}^D \left(1 + \frac{2t}{m_i^2}\right)$$

What constraint can we derive for the low energy EFT from the existence of a UV completion of the form:

$$M'(s,t) = -\sum_i \left(\frac{1}{s-m_i^2} + \frac{1}{u-m_i^2} \right) c_{i,\ell_i}^2 G_{\ell_i}^D \left(1 + \frac{2t}{m_i^2} \right) ??$$

Note that this is applicable for massive loop corrections

- First since the is s, u-symmetric it would be convenient to switch to

$$s = -\frac{t}{2} + z, \quad u = -\frac{t}{2} - z$$

- Second, recall that low energy EFT has a four point function of the form:

$$M^{IR}(z,t) = \sum_{i,j} g_{i,j} z^{2i} t^j$$

- For a fixed mass dimension $2L$, the space of possible higher dimensional operators has dimension that correspond to the number of (i,j)s that satisfies $2i + j = L$. Any particular theory corresponds to a particular point in this subspace:

Exp: for $L = 4$ we have

$$\tilde{g}_{0,4} t^4 + \tilde{g}_{2,2} z^2 t^4 + \tilde{g}_{4,0} z^4$$

A given EFT is represented as a specific point $(\tilde{g}_{0,4}, \tilde{g}_{2,2}, \tilde{g}_{4,0})$ in this three-dimensional space

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A given EFT is represented as a specific point $(\tilde{g}_{0,4}, \tilde{g}_{2,2}, \tilde{g}_{4,0})$ in this three-dimensional space

$$\frac{\Gamma[-\alpha' s]\Gamma[-\alpha' t]\Gamma[-\alpha' u]}{\Gamma[1+\alpha' s]\Gamma[1+\alpha' t]\Gamma[1+\alpha' u]} = \cdots + \zeta_7 (\frac{9}{8} t^4 + 3z^2 t^2 + 2z^4)\alpha'^4 + \cdots$$

type-II string theory correspond to $(\frac{9}{8}, 3, 2)$

Within this space, what is the subspace that has a UV completion of the form?

$$A(z,t) = -\sum_i \left(\frac{1}{-\frac{t}{2} + z - m_i^2} + \frac{1}{-\frac{t}{2} - z - m_i^2} \right) c_{i,\ell_i}^2 G_{\ell_i}^D \left(1 + \frac{2t}{m_i^2} \right)$$

We can also expand in low energy. For fixed L the coefficient for $z^{2a} t^{L-2a}$

$$z^{2a} t^{L-2a} : \quad \sum_i c_{i,\ell_i}^2 \left\{ \sum_{q=0}^{L-2a} \left((-)^{q+1} 2^{L-2i-2q+1} \frac{[2a+1]_q}{q!(L-2a-q)!} G_{\ell,q}^D \right) \right\} = \sum_i c_{i,\ell_i}^2 \hat{G}_{\ell,a}^D$$

where $G_{\ell,q}^D = \frac{\partial^q}{\partial x^q} G_\ell^D (1+x) \Big|_{x=0}$. For $L=4$ we have infinite set (infinte spins) of three-dimensional vectors

$$\mathcal{V}_\ell = \begin{pmatrix} \hat{G}_{\ell,4}^D \\ \hat{G}_{\ell,2}^D \\ \hat{G}_{\ell,0}^D \end{pmatrix}$$

The allowed space is spanned by the convex hull of $\mathcal{V}_\ell$s.

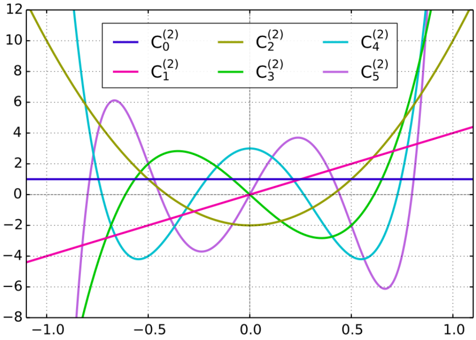
We have an infinite number of vectors spanning a finite dimensional space $\rightarrow$ there probably is no constraint?

Remarkably, it does not span the full space !

Note that if we look at the last entry of the vertices

$$\mathcal{V}_\ell = \begin{pmatrix} \hat{G}^D_{\ell,4} \\ \hat{G}^D_{\ell,2} \\ \hat{G}^D_{\ell,0} \end{pmatrix}$$

$\hat{G}^D_{\ell,0} \sim G^D_{\ell_i}(1) > 0$

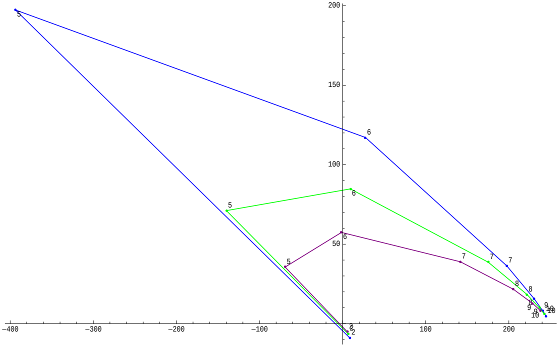


This is nothing but the old positivity bound $g_{i,0} > 0$ A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis and R. Rattazzi

i.e. it's just a tip of the iceberg.

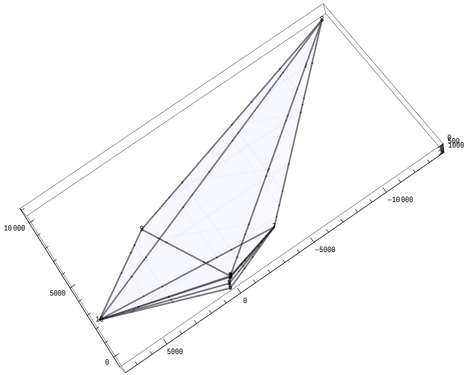
Let's consider the space projectively. Taking

$$v_\ell = \left(\frac{\mathcal{V}_\ell^1 - \mathcal{V}_\ell^2}{\mathcal{V}_\ell^3}, \frac{\mathcal{V}_\ell^2 - \mathcal{V}_\ell^3}{\mathcal{V}_\ell^3} \right)$$



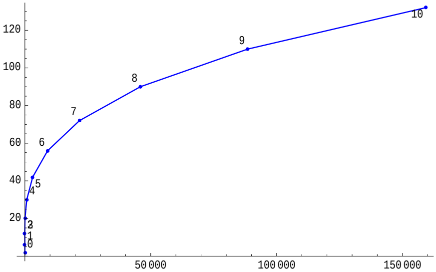
$L = 6$ is spanned by a four-dimensional space: the projective three-dimensional polytope:

$$v_\ell = \left(\frac{\mathcal{V}_\ell^1 - \mathcal{V}_\ell^2}{\mathcal{V}_\ell^4}, \frac{\mathcal{V}_\ell^2 - \mathcal{V}_\ell^3}{\mathcal{V}_\ell^4}, \frac{\mathcal{V}_\ell^3 - \mathcal{V}_\ell^4}{\mathcal{V}_\ell^4} \right)$$



There such a "UV-IR" polytope in each dimension $\to$ infinite number of constraint

What is special about these polytopes ? What is the geometric feature of Lorentz-invariance and Unitarity ?



The vertices appear to lie on a moment curve:

$$\mathcal{V}_\ell \sim (1, t, t^2, \cdots, t^d)$$

The convex hull of points on a moment curve $\rightarrow$ cyclic polytope!

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Cyclic polytope

In mathematics, a **cyclic polytope**, denoted *C*(*n*,*d*), is a convex polytope formed as a convex hull of *n* distinct points on a rational normal curve in $\mathbf{R}^d$, where *n* is greater than *d*. These polytopes were studied by Constantin Carathéodory, David Gale, Theodore Motzkin, Victor Klee, and others. They play an important role in polyhedral combinatorics: according to the upper bound theorem, proved by Peter McMullen and Richard Stanley, the boundary *Δ*(*n*,*d*) of the cyclic polytope *C*(*n*,*d*) maximizes the number f_i of *i*-dimensional faces among all simplicial spheres of dimension *d* − 1 with *n* vertices.

Consider the ordered (here by spin) matrices constructed by vertex vectors. If all ordered minor is positive, then the vertices construct a cyclic polytope.

Proof:

$$det\left[\mathcal{V}_{\ell_1}\mathcal{V}_{\ell_2}\cdots\mathcal{V}_{\ell_L}\right] = \frac{\prod_{i<j}(\ell_i-\ell_j)(D-3+\ell_i+\ell_j)\prod_i\Gamma[D-3+\ell_i]}{\prod_i\Gamma[D-3-2+2i]\Gamma[1+\ell_i]}$$

Positive if $\ell_1 < \ell_2 < \cdots < \ell_L$

- The gegenbauer polytope is a cyclic polytope
- All vectors are part of the vertices of the cyclic polytope

Important property of cyclic polytope:

- All boundaries are known (CP^{d-1})

$$d{=}3: \ (\mathcal{V}_{\ell_i}\mathcal{V}_{\ell_{i+1}}), \quad d=4\ (\mathcal{V}_{\ell_1}\mathcal{V}_{\ell_i}\mathcal{V}_{\ell_{i+1}}),\ (\mathcal{V}_{\ell_i}\mathcal{V}_{\ell_{i+1}}\mathcal{V}_{\ell_N}), d=5\ (\mathcal{V}_{\ell_i}\mathcal{V}_{\ell_{i+1}}\mathcal{V}_{\ell_j}\mathcal{V}_{\ell_{j+1}})$$

- Projecting through a vertex, the higher-dimensional cyclic polytope lands on a lower-dimensional one

$$\langle \mathcal{V}_{\ell_1}, \mathcal{V}_{\ell_2}, \mathcal{V}_{\ell_3}, \mathcal{V}_{\ell_4} \rangle = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} > 0 \to \langle \bar{\mathcal{V}}_{\ell_2}, \bar{\mathcal{V}}_{\ell_3}, \bar{\mathcal{V}}_{\ell_4} \rangle > 0$$

- Through projection, what's inside the higher-dimensional polytope is also inside the lower dimensional one:

$$\langle \mathcal{V}_{\ell_1}, \mathcal{V}_{\ell_i}, \mathcal{V}_{\ell_{i+1}}, X \rangle > 0 \to \langle \mathcal{V}_{\ell_i}, \mathcal{V}_{\ell_{i+1}}, X \rangle > 0$$

The fact that it's a cyclic polytope resolves an apparent tension with RG

For simplicity let's consider only s-channel contributions:

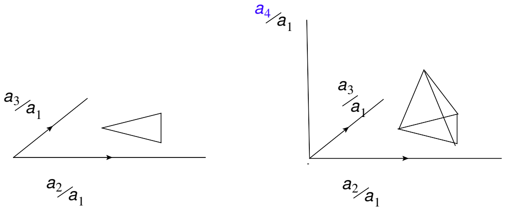
$$A(z,t) = -\sum_i \left(\frac{1}{s-m_i^2}\right) c_{i,\ell_i}^2 G_{\ell_i}^D \left(1+\frac{2t}{m_i^2}\right) \to \sum_{i,j} g_{i,j} s^i t^j$$

Let's say we have for mass-dimension four: $a_1 s^2 + a_2 st + a_3 t^2$. For mass-dimension six, we then have $a_1 s^3 + a_2 s^2 t + a_3 st^2 + a_4 t^6$

- We will have a three-dimensional constraint for (a_1, a_2, a_3), but this is not enough since
- We will also have a four-dimensional constraint for (a_1, a_2, a_3, a_4) and so fourth
- This would appear that the coefficients of lower mass-dimension operators are very sensitive to that of higher mass-dimension operators inconsistent with RG

However since the coefficients are bounded by a cyclic polytope

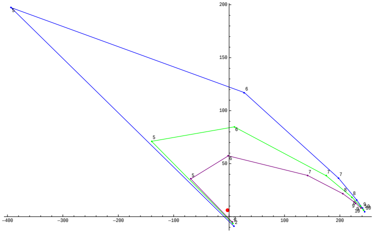
- The projection of a higher-dimensional polytope lands on a lower dimensional cyclic polytope, implies that any point in the lower dimensional polytope is guaranteed to have an image upstairs

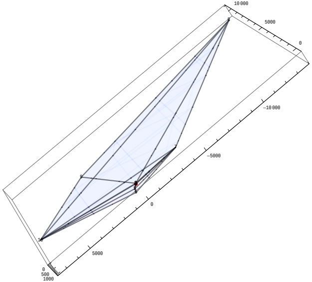


$$(a_1, a_2, a_3) \rightarrow (a_1, a_2, a_3, a_4)$$

- The the higher-dimensional polytope only constrains the new coefficient!

We can ask where does super string theory lies:





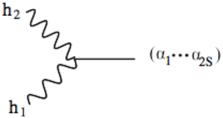
What about gauge and gravity?

In four-dimensions, 2 massless 1 massive N. Arkani-Hamed, T-z Huang, Y-t The three-point amplitude is also unique

$$\epsilon^{\mu_1\mu_2\cdots\mu_\ell}X_{\mu_1}\cdots X_{\mu_i}q_{\mu_{i+1}}\cdots q_{\mu_\ell} \quad q^{\alpha\dot{\alpha}}=\lambda_1\tilde{\lambda}_2$$

The degree of q depends on the helicity (h_1, h_2).

It is convenient to use manifest SL(2,C) on-shell representations



$M^{h_1h_2}{}_{\{\alpha_1\alpha_2\cdots\alpha_{2S}\}}$

$$M^{h_1h_2}{}_{\{\alpha_1\alpha_2\cdots\alpha_{2S}\}}=\frac{g}{m^{2S+h_1+h_2-1}}\left(\lambda_1^{S+h_2-h_1}\lambda_2^{S+h_1-h_2}\right)_{\{\alpha_1\alpha_2\cdots\alpha_{2S}\}}[12]^{S+h_1+h_2}$$

$$M^{h_1 h_2}{}_{\{\alpha_1\alpha_2\cdots\alpha_{2S}\}} = \frac{g}{m^{2S+h_1+h_2-1}} \left(\lambda_1^{S+h_2-h_1}\lambda_2^{S+h_1-h_2}\right)_{\{\alpha_1\alpha_2\cdots\alpha_{2S}\}} [12]^{S+h_1+h_2}$$

We can orient

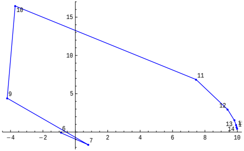
$$\lambda_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \lambda_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; \lambda_3 = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix}, \; \lambda_4 = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

The numerator is simply the Wigner d-matrix $d^j_{m',m}(\theta) = \langle j, m'|e^{-i\theta\mathcal{J}_y}|j, m\rangle$, or equivalently

$$d^j_{m',m}(\theta) = \mathcal{J}(\ell + 4h, 0, -4h, \cos\theta)$$

We also have a universal polynomial, $G_\ell^D(\cos\theta) \to \mathcal{J}(\ell + 4h, 0, -4h, \cos\theta)$ We again have

$$M(s,t) = \frac{n(h_i)}{s t^2} + \sum_i \left(\frac{1}{s - m_i^2} + \frac{1}{u - m_i^2} \right) \mathcal{J}(\ell + 4h, 0, -4h, \cos\theta)$$



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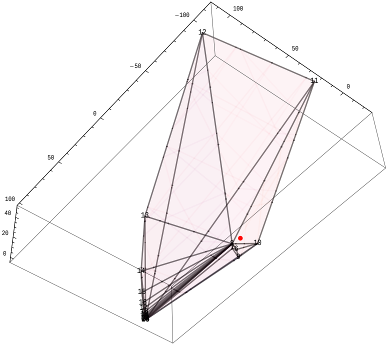


- The polytopes are also cyclic !
- These are real world predictions, for any theory involving weakly coupled matter.

Remarkably, the same polytope is present for CFT four-point function! w Nima, Shu-Heng Shao

Consider the a 1D four-point function:

$$\langle\phi(1)\phi(2)\phi(3)\phi(4)\rangle \equiv F(z)$$

$$F(z) = \sum_{\Delta} p_{\Delta}^2 C_{\Delta}(z), \quad C_{\Delta}(z) = z^{\Delta}\, {}_2F_1(\Delta, \Delta, 2\Delta, z)$$

We can again expand the four-point function, say around $z = \frac{1}{2}$

$$F\left(\frac{1}{2} + y\right) = \sum_{q=0}^{\infty} F_q y^q$$

The 1-D blocks also yield an infinite set of vectors

$$C_{\Delta}\left(\frac{1}{2} + y\right) = \sum_{q=0}^{\infty} c_{\Delta,q} y^q$$

Unitarity then requires that

$$\mathbf{F} = \begin{pmatrix} F_0 \\ F_1 \\ \vdots \\ F_{L-1} \end{pmatrix} \subset \sum_{\Delta} p_{\Delta}^2 \begin{pmatrix} c_{\Delta,0} \\ c_{\Delta,1} \\ \vdots \\ c_{\Delta,L-1} \end{pmatrix}$$

Now crossing is just

$$z^{-2\Delta_\phi} F(z) = (1-z)^{-2\Delta_\phi} F(1-z) \to F(z) = \left(\frac{z}{1-z}\right)^{2\Delta_\phi} F(1-z)$$

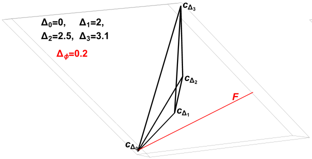
Again expanded around $z = \frac{1}{2}$ we find

$$\sum_q F_q y^q = \left(\frac{1+2y}{1-2y}\right)^{2\Delta_\phi} \sum_q (-)^q F_q y^q$$

This tells us that **F** must lie within the crossing plane X

We have the polytope $P(\Delta_i) = \sum_i p^2_{\Delta_i} c_{\Delta_i}$ and a crossing plane $X(\Delta_\phi)$, and they must intersect. $P(\Delta_i)$ is a cylic polytope! See Nima's talk

For example:


Δ_0=0, Δ_1=2,
Δ_2=2.5, Δ_3=3.1
Δ_ϕ=0.2
c_{Δ_3}
c_{Δ_2}
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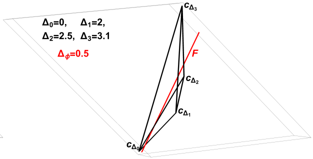
c_{Δ_0}
F


Δ_0=0, Δ_1=2,
Δ_2=2.5, Δ_3=3.1
Δ_ϕ=0.5
c_{Δ_3}
F
c_{Δ_2}
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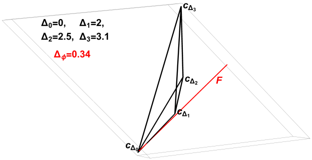
c_{Δ_0}


Δ_0=0, Δ_1=2,
Δ_2=2.5, Δ_3=3.1
Δ_ϕ=0.34
c_{Δ_3}
c_{Δ_2}
F
c_{Δ_1}
c_{Δ_0}

Conclusion

- The geometrization of unitarity and locality for observables in QFT
- These constraints imposed on "Factorization" (in terms of poles in the S-matrix or OPE in correlation function) $\bigoplus$ "Symmetry" (Lorentz or conformal), leads to the physical observable bounded by a cyclic polytope
- Cyclic polytope naturally leads to a RG like picture
- What other geometric structure lies beyond the real block ($z = \bar{z}$)?
- What does this imply for available models? Weak gravity conjecture ?