# Diagrammatic Approach to the Two-Body Dynamics @ 4PN - O(G^5)

Pierpaolo Mastrolia

QCD meets Gravity Bhaumik Institute UCLA, 13.12.2017

based on:

Foffa, PM, Sturani, Sturm, PRD95 (2017) 10, 104009

in collaboration with:

Foffa, Sturani, Sturm + Cristofoli, Torres-Bobadilla



#### Outline

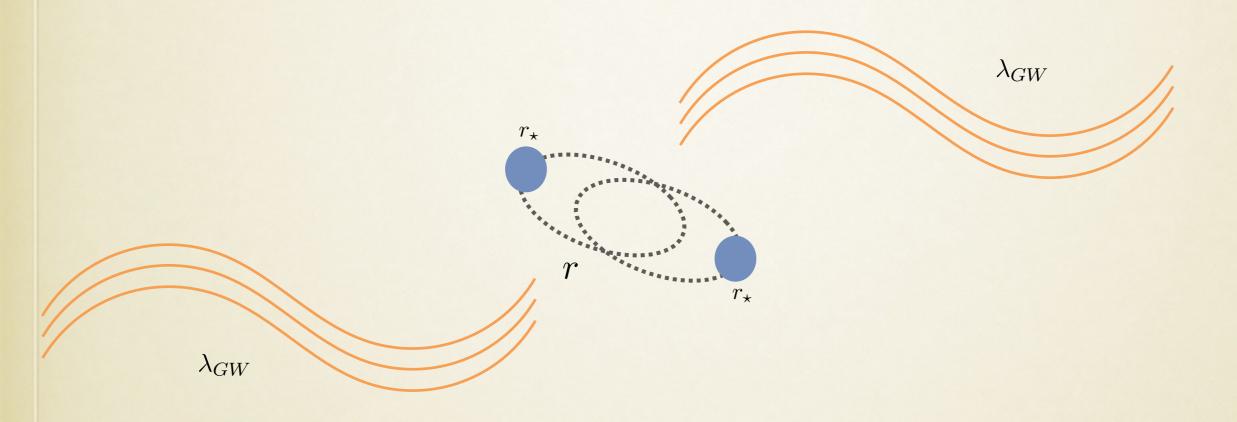
- **EFT-GR Feynman Rules**
- Amplitudes @ 4PN O(G^5)
- Feynman Integrals in Dimensional Regularisation
  - Integration-by-parts identities
  - Master Integrals
  - Dimensional Recurrence Relations
  - Differential Equations
- Results
- Conclusions/Outlook

## **Motivations**

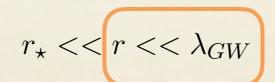
Describing the Two-body dynamics at high precision

- >> Rothstein's talk
- >> Buonanno's talk
- Providing an independent calculation of the 4PN-O(G^5) Lagrangean
- Broadening the application range of Multi-loop High Energy Computational Tools

# Length scales in a Binary System



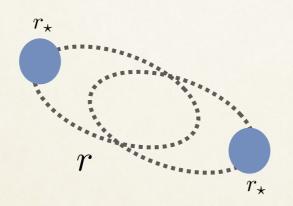
**Double Hierarchy** 



Dissipative system ::

GW emission

# Length scales in a Binary System



**Double Hierarchy** 

$$r_{\star} << r < \lambda_{GW}$$

Conservative system ::

GW emission

$$G_N \frac{m}{r} \sim v^2 << 1$$

expansion parameter

## **Effective Field Theory**

#### **Einstein-Hilbert**

$$S_{tot}[x,h] = S_g[h] + S_m[x,h]$$

$$e^{iS_{eff}[x]} = \int \mathcal{D}[h]e^{iS_{tot}[x,h]}$$

$$S_g = \frac{1}{32\pi G} \int \mathrm{d}^{(d+1)} x \sqrt{-g} \left[ R - g_{\mu\nu} g^{\alpha\beta} g^{\gamma\delta} \Gamma^{\mu}_{\alpha\beta} \Gamma^{\nu}_{\gamma\delta} \right]$$
 gauge fixing term

$$S_{m_a} \simeq S_{pp}^a = -m_a \int \sqrt{-g_{\mu\nu} dx_a^{\mu} dx_a^{\nu}} = -m_a \int dt \sqrt{-g_{\mu\nu} \dot{x}_a^{\mu} \dot{x}_a^{\nu}}$$
  $a = 1, 2$ 

#### Kaluza-Klein parametrisation Kol, Smolkin; Blanchet, Damour;

$$g_{\mu\nu} = e^{\frac{2\phi}{\Lambda}} \begin{pmatrix} -1 & \frac{A_i}{\Lambda} \\ \frac{A_i}{\Lambda} & e^{-c_d \frac{\phi}{\Lambda}} \gamma_{ij} - \frac{A_i A_j}{\Lambda^2} \end{pmatrix} \qquad \gamma_{ij} = \delta_{ij} + \frac{\sigma_{ij}}{\Lambda} \qquad c_d = 2\left(\frac{d-1}{d-2}\right) \\ \Lambda^{-1} = (32\pi G_N)^{\frac{1}{2}}.$$

$$\mathcal{S}_{g} = \int d^{d+1}x \sqrt{\gamma} \left[ \frac{1}{4} \left( (\nabla \sigma)^{2} - 2(\nabla \sigma_{ij})^{2} - (\dot{\sigma}^{2} - 2\dot{\sigma}^{ij}\dot{\sigma}_{ij}) \right) \right] - c_{d} \left[ (\nabla \phi)^{2} - \dot{\phi}^{2} \right] + \left[ \frac{1}{2} F_{ij} F^{ij} + (\vec{\nabla} \cdot \vec{A})^{2} - \dot{\vec{A}} \cdot \dot{\vec{A}} \right] + \dots$$

$$\mathcal{S}_{pp} = -m_{a} \int d\lambda_{a} e^{\frac{\phi}{\Lambda}} \sqrt{1 - e^{-(2+c_{d})\frac{\phi}{\Lambda}} v_{i} v_{j} \gamma^{ij}}$$

$$e^{i\mathcal{S}_{eff}(x_a)} = \int D\phi DA_i D\sigma_{nm} e^{i\mathcal{S}_g + i\mathcal{S}_{pp}}$$

## **Feynman Rules**

#### **Propagators**



$$\xrightarrow{\alpha\beta} \xrightarrow{\gamma\delta} \gamma\delta$$

$$D^{\alpha\beta\gamma\delta} = \frac{-i}{2k^2} \left( \frac{2}{2-D} \eta^{\alpha\beta} \eta^{\gamma\delta} + \eta^{\alpha\gamma} \eta^{\beta\delta} + \eta^{\alpha\delta} \eta^{\beta\gamma} \right)$$

#### **EFT NRG-fields**

$$\phi$$
-propagator,  $\rightarrow -\frac{1}{2c_d p^2}$ 

$$ightarrow -rac{\mathrm{i}}{2c_d p^2}$$

$$\sigma$$
-propagator,

$$\sigma$$
-propagator,  $\stackrel{\mathsf{rj}}{\qquad}_{\mathsf{P}}$   $\rightarrow \frac{\mathrm{i} P^{\sigma_{rj}\sigma_{kl}}}{2p^2}$ .

$$P^{\sigma_{ij}\sigma_{kl}} = -\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + (2 - c_d)\delta_{ij}\delta_{kl}\right)$$

$$A_i$$
-propagator

$$A_i$$
-propagator r t  $\rightarrow i \frac{\delta^{rt}}{2k^2}$ 

$$ightarrow$$
 i $rac{\delta^{rt}}{2k^2}$ 

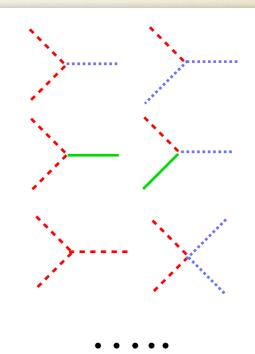
## **Feynman Rules**

#### **Interactions**



#### Self-interactions

$$\begin{array}{c} \underset{\mathbf{k},\mathbf{m}}{\text{p-ri}} & \rightarrow \mathrm{i} \frac{2c_d}{\Lambda} \left[ \frac{1}{2} (p-k) \cdot k \delta^{rj} - k^r (p-k)^j + \left(r \leftrightarrow j\right) \right], \\ \\ \underset{\mathbf{k},\mathbf{m}}{\text{p-q,rj}} & \rightarrow \mathrm{i} \frac{4c_d}{\Lambda^2} \left[ k^r p^l \delta^{jm} - \frac{1}{2} k^l p^m \delta^{rj} - \frac{1}{8} p \cdot k \mathcal{Q}^{rjlm} + \left(r \leftrightarrow j, l \leftrightarrow m\right) \right], \\ \\ \underset{\mathbf{k},\mathbf{m}}{\text{p-k,qr}} & \rightarrow \mathrm{i} \frac{1}{8\Lambda} \left\{ (p-k) \cdot k \left( \frac{1}{2} \delta^{tr} \mathcal{I}^{lmjq} - \frac{1}{4} \delta^{qr} \mathcal{I}^{tjlm} - \frac{1}{8} \delta^{tj} \mathcal{Q}^{qrlm} \right) + \\ & + \frac{1}{4} (p-k)^t k^j \mathcal{Q}^{qrlm} + \left[ \left( \frac{1}{2} \delta^{tj} \delta^{mr} - \delta^{tr} \delta^{jm} \right) (p-k)^q k^l - \left(l \leftrightarrow q\right) \right] + \\ & + \delta^{lm} \delta^{tr} (p-k)^q k^j - \delta^{tm} \delta^{qr} (p-k)^l k^j + \left(t \leftrightarrow j, l \leftrightarrow m, q \leftrightarrow r\right) \right\}, \end{array}$$





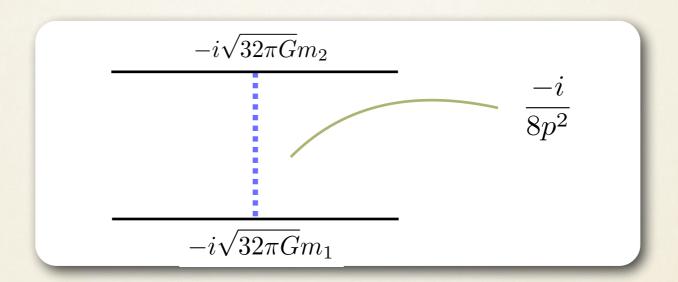
#### matter-interactions

$$= -\frac{im_a}{\Lambda^n n!} + \mathcal{O}(v^2) \qquad = -i\frac{m_a v_i}{\Lambda} + \mathcal{O}(v^3) \qquad = \frac{im_a}{2\Lambda} v^i v^j + \mathcal{O}(v^4)$$

$$\Lambda^{-1} = (32\pi G_N)^{\frac{1}{2}}.$$

#### **Newton Potential**

#### **⊌**d=3 space-dimension



#### **Fourier tfm**

$$V = i \int \frac{\mathrm{d}^3 p}{(2\pi)^3} (-i\sqrt{32\pi G} m_1) (-i\sqrt{32\pi G} m_2) \frac{-i}{8p^2} e^{ip.(x_1 - x_2)}$$

## **Post-Newtonian Corrections**

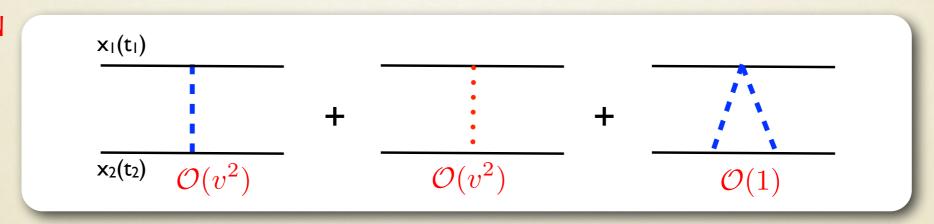
#### **Post-Newtonian expansion**

**n**-th order correction

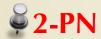
$$\mathcal{O}(G_N^k, v^{2m}), \qquad n = k + 2m - 1$$

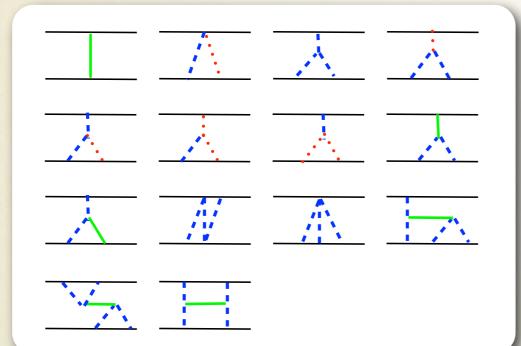
 $G_N \frac{m}{r} \sim v^2$ 

Virial



## **Post-Newtonian Corrections**





School on Gravitational Waves: from data to theory and back

UNESP Sao Paulo, August 3-11, 2015

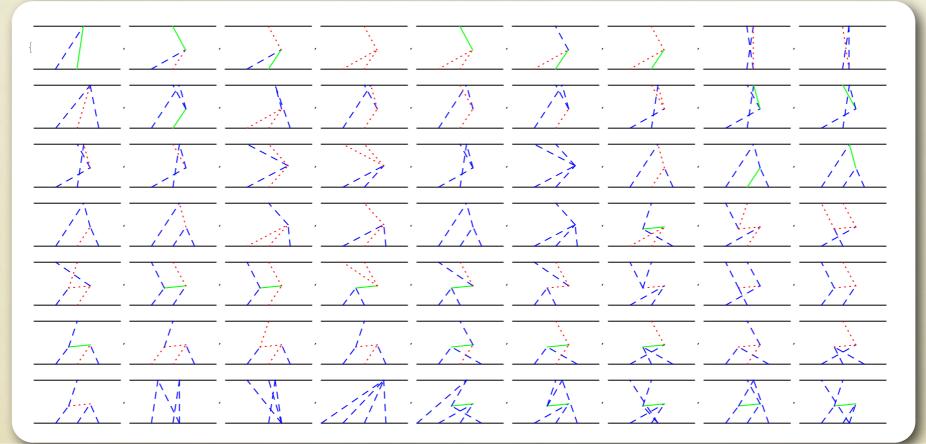
Effective field theory methods to model astrophysical binaries

Stefano Foffa

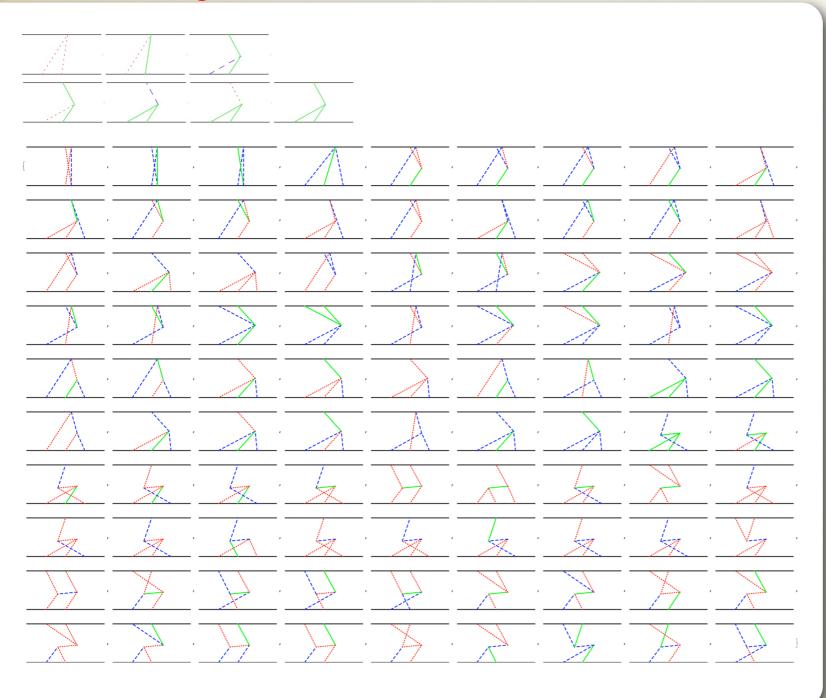
additional material

Goldberger Rothstein Porto













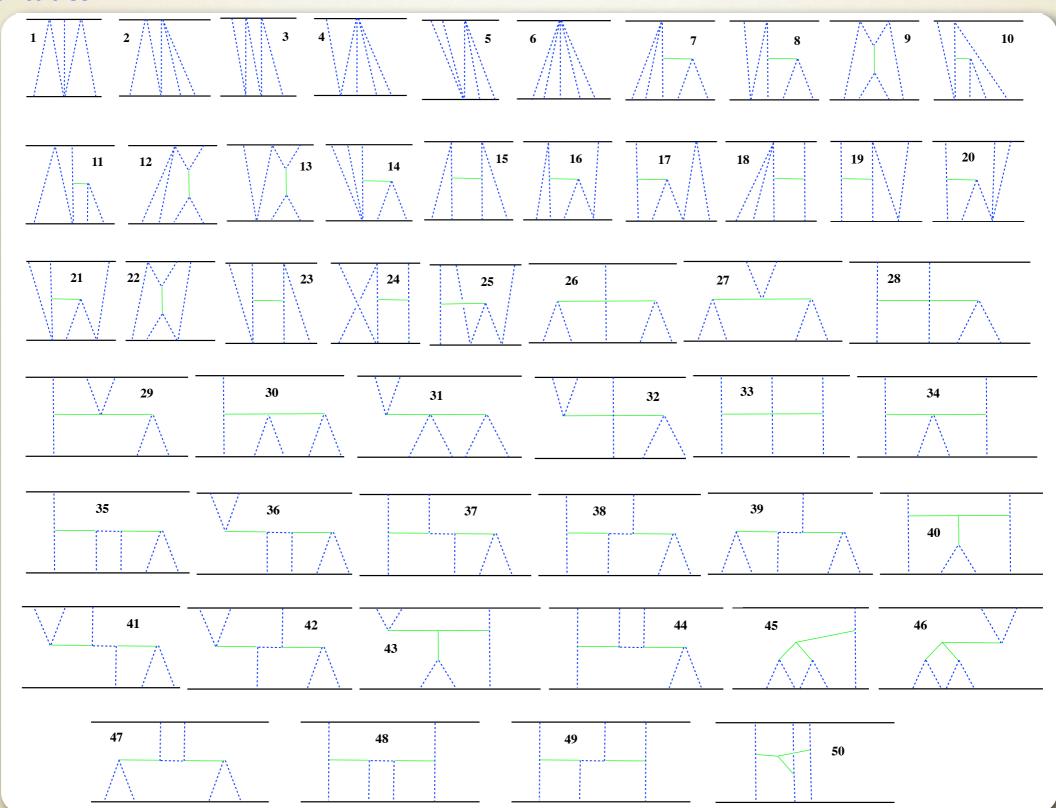




## Amplitudes @ 4PN - O(G^5)

Foffa, Sturani, Sturm, & P.M.

**50** Amplitudes



## From Amplitudes to Lagrangian



$$\mathcal{L}_a = -i \lim_{d \to 3} \int \frac{d^d p}{(2\pi)^d} e^{ip \cdot r}$$

$$a = 1, \dots, 50$$

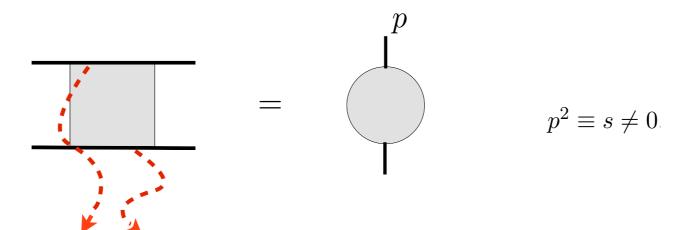
# From Amplitudes to Lagrangian



$$\mathcal{L}_a = -i \lim_{d \to 3} \int \frac{d^d p}{(2\pi)^d} e^{ip \cdot r}$$
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# EFT-GR Diagrams vs 2-point QFT Diagrams

**key observation** 



static sources <==> non propagating <==> pinching lines

# From Amplitudes to Lagrangian

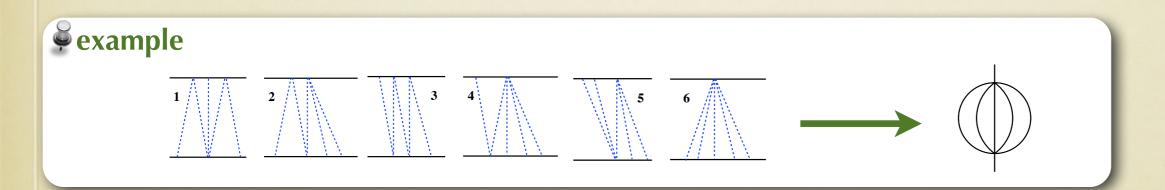


$$\mathcal{L}_a = -i \lim_{d \to 3} \int \frac{\mathrm{d}^d p}{(2\pi)^d} e^{\mathrm{i}p \cdot r}$$

$$a = 1, \dots, 50$$

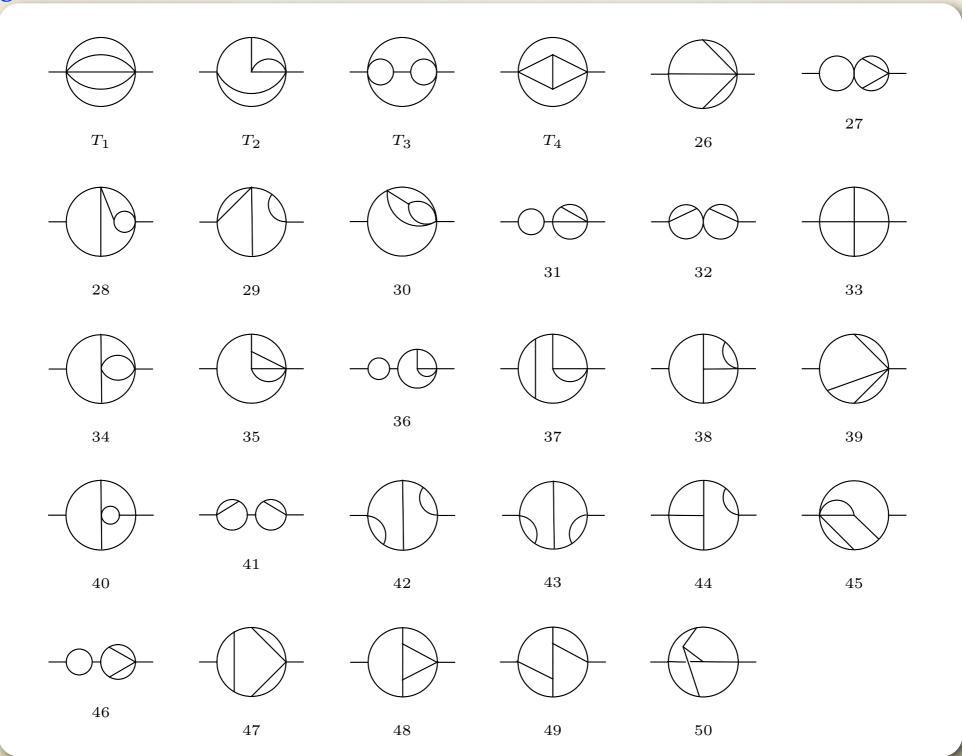
# EFT-GR Diagrams vs 2-point QFT Diagrams

$$= \qquad \qquad p^2 \equiv s \neq 0$$

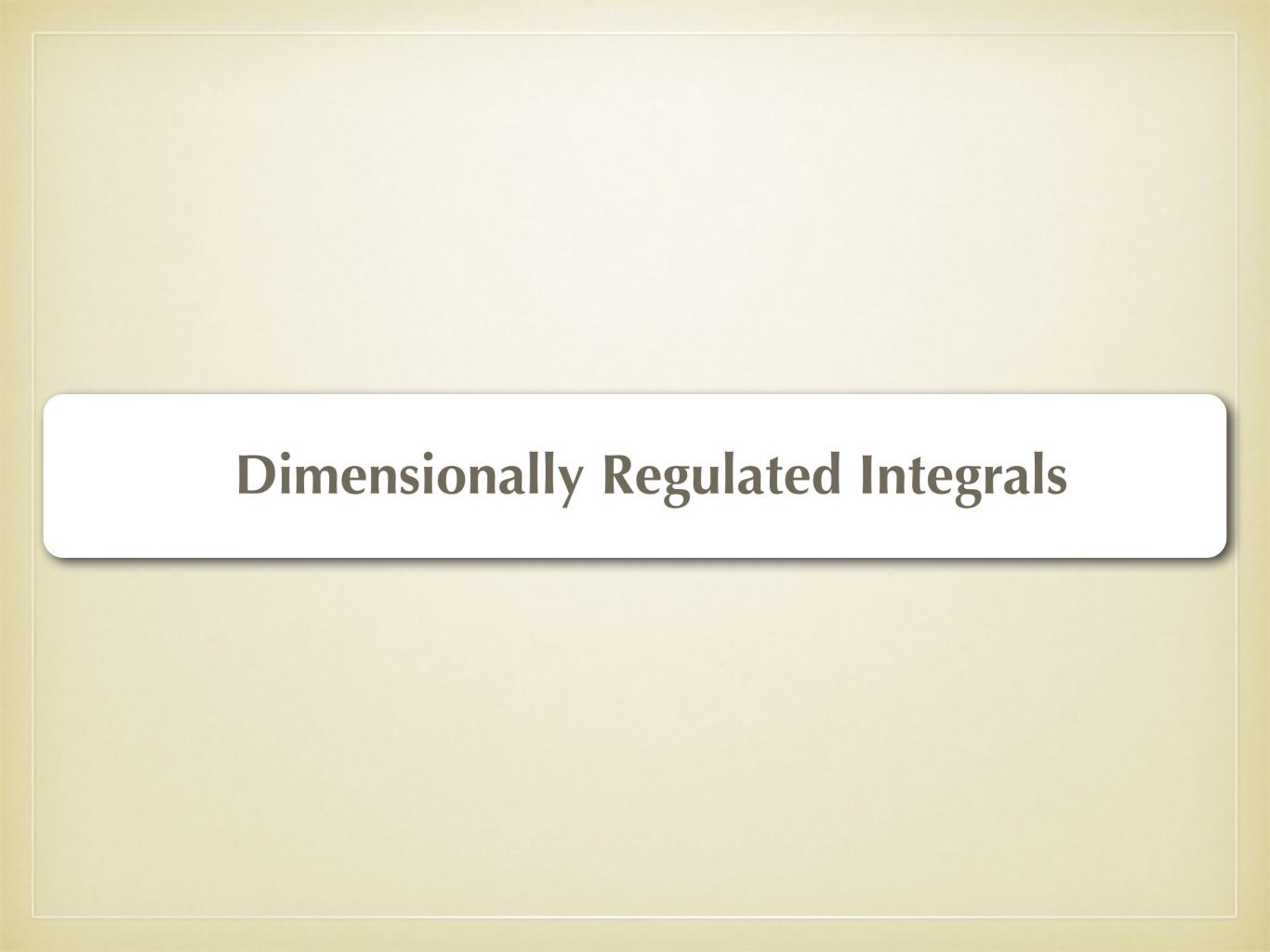


# Four-Loop QFT-like Graphs

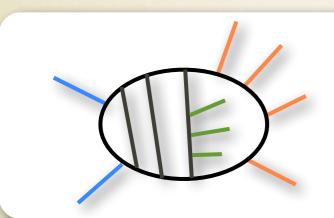
#### **29 Topologies**



 $T_1 = \{1, 2, 3, 4, 5, 6\}, T_2 = \{7, 8, 10, 11, 14, 16, 17, 20, 21, 25\}, T_3 = \{9, 12, 13, 22\}, T_4 = \{15, 18, 19, 23, 24\}$ 



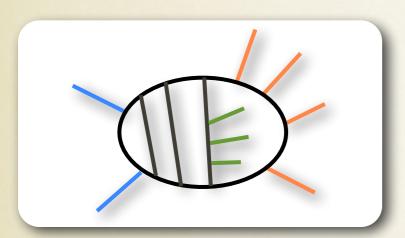
# **Graph Topology & Integrals**



$$e = \# \text{ legs} :: p_i, \quad (i = 1, ..., e);$$
  
 $\ell = \# \text{ loops} :: q_i \quad (i = 1, ..., \ell);$   
 $n = \# \text{ denominators} :: D_i \quad (i = 1, ..., n);$ 

```
N= # scalar products (of types q_i \cdot p_j and q_i \cdot q_j) N=\ell(e-1)+\frac{\ell(\ell+1)}{2} n= # reducible scalar products (expressed in terms of denominators); m= # irreducible scalar products =N-n::S_i (i=1,\ldots,m)
```

# **Graph Topology & Integrals**



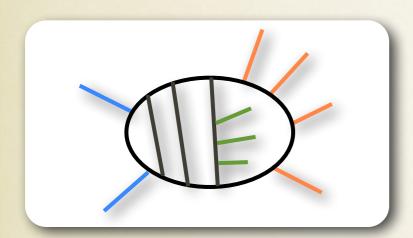
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# **Graph Topology & Integrals**



$$e = \# \text{ legs} :: p_i, \quad (i = 1, \dots, e);$$
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$$N = \#$$
 scalar products (of types  $q_i \cdot p_j$  and  $q_i \cdot q_j$ ) 
$$N = \ell(e-1) + \frac{\ell(\ell+1)}{2}$$

n = # reducible scalar products (expressed in terms of denominators);

$$m = \#$$
 irreducible scalar products  $= N - n :: S_i \quad (i = 1, ..., m)$ 

#### Associated Integrals ::

$$F_{n,m}^{[d]}(\mathbf{x},\mathbf{y}) \equiv \int_{q_1...q_\ell} f_{n,m}(\mathbf{x},\mathbf{y}) , \qquad \int_{q_1...q_\ell} \equiv \int \frac{\mathrm{d}^d q_1}{(2\pi)^d} \cdots \frac{\mathrm{d}^d q_\ell}{(2\pi)^d}$$

$$f_{n,m}(\mathbf{x}, \mathbf{y}) = \frac{S_1^{y_1} \cdots S_m^{y_m}}{D_1^{x_1} \cdots D_n^{x_n}} \leftarrow$$

## Integration-by-parts Identities (IBPs)

Tkachov; Chetyrkin, Tkachov; Laporta;

$$\int_{q_1...q_\ell} \frac{\partial}{\partial q_i^{\mu}} \left( v^{\mu} f_{n,m}(\mathbf{x}, \mathbf{y}) \right) = 0 , \qquad v = q_1, \ldots, q_\ell, \ p_1, \ldots, p_{e-1}.$$

 $\forall (n,m), N_{\text{IBP}} = \# \text{ of IBP relations} = \ell(\ell + e - 1)$ 

Relations between integrals associated to the same topology (or subtopologies)

$$c_0 F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) + \sum_{i,j} c_{i,j} F_{n,m}^{[d]}(\mathbf{x_i}, \mathbf{y_j}) = 0$$
,

$$\mathbf{x_i} = \{x_1, \dots, x_i \pm 1, \dots, x_n\}$$

$$\mathbf{y_j} = \{y_1, \dots, y_j \pm 1, \dots y_n\}$$

public codes :: AIR; Reduze2; FIRE; LiteRed; private codes :: ... many authors ... Sturm ...

## Master Integrals (MIs)

Independent set of integrals  $M_i^{[d]}$ ,

$$M_i^{[d]} \equiv \int_{q_1...q_\ell} m_i(\mathbf{\bar{x}},\mathbf{\bar{y}}) ,$$

with a definite set of powers  $\bar{\mathbf{x}}, \bar{\mathbf{y}}$  such that

$$F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \stackrel{\text{IBP}}{=} \sum_{k} c_k M_k^{[d]}, \quad \forall (n, m)$$

They form a basis for the integrals of the corresponding topology.

#### Two special cases

Two types of integrals generated from the master integrands

• Polynomial insertion:

$$\int_{q_1...q_\ell} P(q_i \cdot p_j, q_i \cdot q_j) \ m_i(\mathbf{\bar{x}}, \mathbf{\bar{y}}) = \sum_{n,m} \alpha_{n,m} \ F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \stackrel{\text{IBP}}{=} \sum_i c_i \ M_i^{[d]}$$

• External-leg derivatives:

$$p_i^{\mu} \frac{\partial}{\partial p_j^{\mu}} M_k^{[d]} = \int_{q_1 \dots q_\ell} p_i^{\mu} \frac{\partial}{\partial p_j^{\mu}} \ m_k(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = \sum_{n,m} \beta_{n,m} \ F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \stackrel{\text{IBP}}{=} \sum_i c_i \ M_i^{[d]}$$

Gram determinant

$$P(q_i \cdot p_j, q_i \cdot q_j) = \mathbf{G}(q_i, p_j) = \begin{vmatrix} q_1^2 & \dots & (q_1 \cdot p_{e-1}) \\ \vdots & \ddots & \vdots \\ (p_{e-1} \cdot q_1) & \dots & p_{e-1}^2 \end{vmatrix}$$

Bern, Dixon, Kosower Tarasov; Baikov; Lee; Gluza, Kajda, Kosower

#### Dimension-shifted integrals

$$F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \equiv \int_{q_1...q_\ell} f_{n,m}(\mathbf{x}, \mathbf{y})$$

Bern, Dixon, Kosower Tarasov; Baikov; Lee; Gluza, Kajda, Kosower

Gram determinant

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$$F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \equiv \int_{q_1...q_\ell} f_{n,m}(\mathbf{x}, \mathbf{y}) \qquad \Rightarrow \int_{q_1...q_\ell} \mathbf{G} f_{n,m}(\mathbf{x}, \mathbf{y}) = \Omega(d, p_i) F_{n,m}^{[d+2]}(\mathbf{x}, \mathbf{y})$$

Bern, Dixon, Kosower Tarasov; Baikov; Lee; Gluza, Kajda, Kosower

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Dimension-shifted integrals

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G-insertion generates shifted dim. integrals: d --> d+2

Bern, Dixon, Kosower Tarasov; Baikov; Lee; Gluza, Kajda, Kosower

Gram determinant

$$P(q_i \cdot p_j, q_i \cdot q_j) = \mathbf{G}(q_i, p_j) = \begin{vmatrix} q_1^2 & \dots & (q_1 \cdot p_{e-1}) \\ \vdots & \ddots & \vdots \\ (p_{e-1} \cdot q_1) & \dots & p_{e-1}^2 \end{vmatrix}$$

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$$F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \equiv \int_{q_1 \dots q_\ell} f_{n,m}(\mathbf{x}, \mathbf{y}) \qquad \Rightarrow \int_{q_1 \dots q_\ell} \mathbf{G} \ f_{n,m}(\mathbf{x}, \mathbf{y}) = \Omega(d, p_i) \ F_{n,m}^{[d+2]}(\mathbf{x}, \mathbf{y})$$

In the case of Master integrals

$$M_k^{[d+2]} = \Omega(d, p_i)^{-1} \int_{q_1 \dots q_\ell} \mathbf{G} \ m_k(\mathbf{\bar{x}}, \mathbf{\bar{y}}) \stackrel{\text{IBP}}{=} \sum_i c_{k,i} \ M_i^{[d]}$$

Bern, Dixon, Kosower Tarasov; Baikov; Lee; Gluza, Kajda, Kosower

$$P(q_i \cdot p_j, q_i \cdot q_j) = \mathbf{G}(q_i, p_j) = \begin{vmatrix} q_1^2 & \dots & (q_1 \cdot p_{e-1}) \\ \vdots & \ddots & \vdots \\ (p_{e-1} \cdot q_1) & \dots & p_{e-1}^2 \end{vmatrix}$$

#### Dimension-shifted integrals

$$F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \equiv \int_{q_1...q_\ell} f_{n,m}(\mathbf{x}, \mathbf{y}) \qquad \Rightarrow \int_{q_1...q_\ell} \mathbf{G} \ f_{n,m}(\mathbf{x}, \mathbf{y}) = \Omega(d, p_i) \ F_{n,m}^{[d+2]}(\mathbf{x}, \mathbf{y})$$

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$$\underbrace{M_k^{[d+2]}} = \Omega(d, p_i)^{-1} \int_{q_1 \dots q_\ell} \mathbf{G} \ m_k(\mathbf{\bar{x}}, \mathbf{\bar{y}}) \stackrel{\text{IBP}}{=} \sum_i c_{k,i} \ M_i^{[d]}$$

which can be seen as a **Dimensional recurrence relation** 

In general, n MIs obey a system of Dimensional recurrence relations

$$\mathbf{M}^{[d]} \equiv egin{pmatrix} M_1^{[d]} \ dots \ M_n^{[d]} \end{pmatrix} \qquad \qquad \mathbf{M}^{[d+2]} = \mathbb{C}(d) \,\, \mathbf{M}^{[d]}$$

## Differential Equations for MIs

Bern, Dixon, Kosower Kotikov; Remiddi; Gehrmann, Remiddi Argeri, Bonciani, Ferroglia, Remiddi, **P.M**.

..

Henn; Henn, Smirnov;

Lee; Papadopoulos;

Argeri, diVita, Mirabella, Schlenk, Schubert, Tancredi, P.M.

diVita, Schubert, Yundin, P.M.

Zeng

Primo, Tancredi

.

$$p^{2} \frac{\partial}{\partial p^{2}} \left\{ p - p \right\} = \frac{1}{2} p_{\mu} \frac{\partial}{\partial p_{\mu}} \left\{ p - p \right\}$$

$$P^{2} \frac{\partial}{\partial P^{2}} \left\{ \begin{array}{c} p_{1} \\ p_{2} \end{array} \right\} = \left[ A \left( p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} + p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} \right) + B \left( p_{1,\mu} \frac{\partial}{\partial p_{2,\mu}} + p_{2,\mu} \frac{\partial}{\partial p_{1,\mu}} \right) \right] \left\{ \begin{array}{c} p_{1} \\ p_{2} \end{array} \right\}$$

$$P = p_1 + p_2,$$

$$P^2 \frac{\partial}{\partial P^2} \left\{ \sum_{p_2}^{p_1} \sum_{p_4}^{p_3} \right\} = \left[ C \left( p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} - p_{3,\mu} \frac{\partial}{\partial p_{3,\mu}} \right) + D p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} + E (p_{1,\mu} + p_{3,\mu}) \left( \frac{\partial}{\partial p_{3,\mu}} - \frac{\partial}{\partial p_{1,\mu}} + \frac{\partial}{\partial p_{2,\mu}} \right) \right] \left\{ \sum_{p_2}^{p_3} \sum_{p_4}^{p_3} \left( \frac{\partial}{\partial p_{3,\mu}} - \frac{\partial}{\partial p_{3,\mu}} \right) \right\} \left\{ \sum_{p_4}^{p_4} \sum_{p_4}^{p_5} \left( \frac{\partial}{\partial p_{3,\mu}} - \frac{\partial}{\partial p_{4,\mu}} \right) \right\} \left\{ \sum_{p_4}^{p_5} \sum_{p_4}^{p_5} \left( \frac{\partial}{\partial p_{4,\mu}} - \frac{\partial}{\partial p_{4,\mu}} \right) \right\} \left\{ \sum_{p_4}^{p_5} \sum_{p_4}^{p_5} \left( \frac{\partial}{\partial p_{4,\mu}} - \frac{\partial}{\partial p_{4,\mu}} \right) \right\} \left\{ \sum_{p_4}^{p_5} \sum_{p_4}^{p_5} \left( \frac{\partial}{\partial p_{4,\mu}} - \frac{\partial}{\partial p_{4,\mu}} \right) \right\} \left\{ \sum_{p_4}^{p_5} \sum_{p_4}^{p_5} \left( \frac{\partial}{\partial p_{4,\mu}} - \frac{\partial}{\partial p_{4,\mu}} \right) \right\} \left\{ \sum_{p_4}^{p_5} \sum_{p_4}^{p_5} \sum_{p_4}^{p_5} \left( \frac{\partial}{\partial p_{4,\mu}} - \frac{\partial}{\partial p_{4,\mu}} \right) \right\} \left\{ \sum_{p_4}^{p_5} \sum_{p_4}^{p_5} \sum_{p_4}^{p_5} \left( \frac{\partial}{\partial p_{4,\mu}} - \frac{\partial}{\partial p_{4,\mu}} \right) \right\} \left\{ \sum_{p_4}^{p_5} \sum_{p_5}^{p_5} \sum_{p_$$

In general, n MIs obey a system of 1st ODE

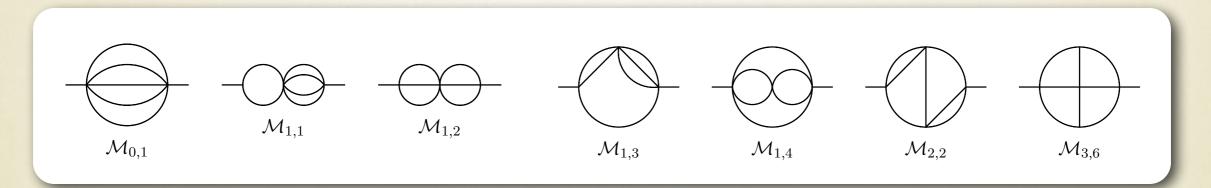
$$\partial_z \mathbf{M}^{[d]} = \mathbb{A}(d, z) \ \mathbf{M}^{[d]}$$

back to EFT-GR @ 4PN - O(G^5)

## 7 Master Integrals

- **IBP Reduction (i.** *in-house* code + ii. Reduze2)
- 50 EFT Integrals ==> 29 Topologies ==> 7 MIs

$$\bar{\mathbf{x}} = (1, \dots, 1) , \quad \bar{\mathbf{y}} = (0, \dots, 0)$$



$$\mathcal{M}_{0,1} = \int_{k_{1...4}} \frac{1}{D_{1...4}D_{14}},$$

$$\mathcal{M}_{1,2} = \int_{k_{1...4}} \frac{1}{D_{1...4}D_{10}D_{11}},$$

$$\mathcal{M}_{1,4} = \int_{k_{1...4}} \frac{1}{D_{1...4}D_{7}D_{13}},$$

$$\mathcal{M}_{3,6} = \int_{k_{1...4}} \frac{1}{D_{1...4}D_{5}D_{6}D_{10}D_{14}},$$

$$\mathcal{M}_{1,1} = \int_{k_{1...4}} \frac{1}{D_{1...4}D_9D_{12}},$$

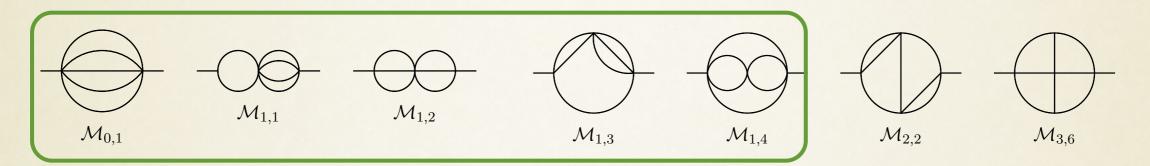
$$\mathcal{M}_{1,3} = \int_{k_{1...4}} \frac{1}{D_{1...4}D_8D_{10}},$$

$$\mathcal{M}_{2,2} = \int_{k_{1...4}} \frac{1}{D_{1...4}D_{10}D_{15}D_{16}},$$

$$D_{1...4} = k_1^2 k_2^2 k_3^2 k_4^2,$$
  $D_5 = (k_2 - k_3)^2,$   $D_6 = (k_1 - k_4)^2,$   $D_7 = (k_2 + k_3 - k_4)^2,$   $D_8 = (k_1 + k_2 + k_3 - k_4)^2,$   $D_9 = (k_1 - p)^2,$ 

$$D_{10} = (k_1 + k_2 - p)^2$$
,  $D_{11} = (k_3 + k_4 + p)^2$ ,  $D_{12} = (k_2 - k_3 - k_4 + p)^2$ ,  $D_{13} = (k_1 - k_2 - k_3 + p)^2$ ,  $D_{14} = (k_1 + k_2 - k_3 - k_4 - p)^2$ ,  $D_{15} = (k_1 + k_4 - p)^2$ ,  $D_{16} = (k_2 + k_3 - p)^2$ .

# 7 Master Integrals



≥ 5 easy MIs

$$\mathcal{M}_{0,1} = (4\pi)^{-2d} s^{2d-5} \frac{\Gamma(5-2d)\Gamma(\frac{d}{2}-1)^5}{\Gamma(\frac{5}{2}d-5)}$$

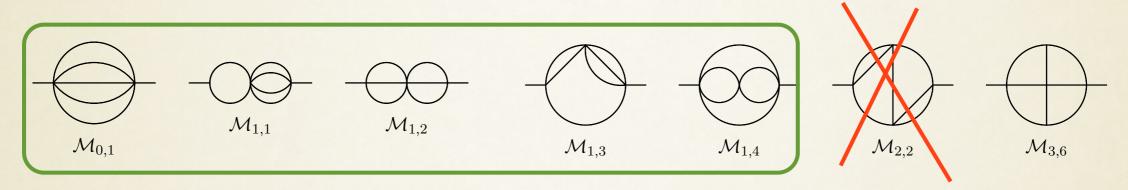
$$\mathcal{M}_{1,1} = (4\pi)^{-2d} s^{2d-6} \frac{\Gamma\left(4 - \frac{3}{2}d\right) \Gamma\left(2 - \frac{d}{2}\right) \Gamma\left(\frac{d}{2} - 1\right)^{6}}{\Gamma(d-2)\Gamma(2d-4)}$$

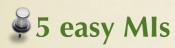
$$\mathcal{M}_{1,2} = (4\pi)^{-2d} s^{2d-6} \frac{\Gamma(3-d)^2 \Gamma(\frac{d}{2}-1)^6}{\Gamma(\frac{3}{2}d-3)^2}$$

$$\mathcal{M}_{1,3} = (4\pi)^{-2d} s^{2d-6} \frac{\Gamma(6-2d)\Gamma(3-d)\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2}-1)^{6}\Gamma(2d-5)}{\Gamma(5-\frac{3}{2}d)\Gamma(d-2)\Gamma(\frac{3}{2}d-3)\Gamma(\frac{5}{2}d-6)}$$

$$\mathcal{M}_{1,4} = (4\pi)^{-2d} s^{2d-6} \frac{\Gamma(6-2d)\Gamma(2-\frac{d}{2})^2 \Gamma(\frac{d}{2}-1)^6 \Gamma(\frac{3}{2}d-4)}{\Gamma(4-d)\Gamma(d-2)^2 \Gamma(\frac{5}{2}d-6)}$$

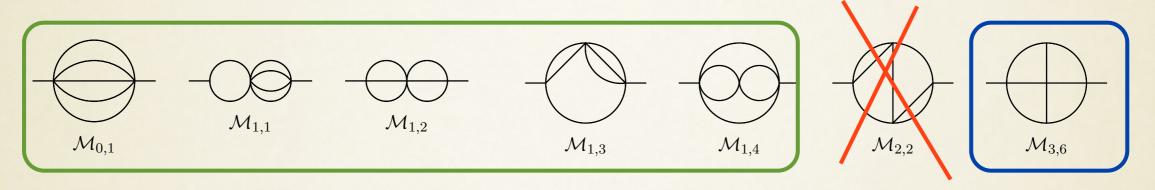
# 7 Master Integrals





M<sub>2,2</sub> drops out in the d --> 3 limit

# 7 Master Integrals



- ≥ 5 easy MIs
- M<sub>2,2</sub> drops out in the d --> 3 limit
- M<sub>3,6</sub> non-trivial

$$\frac{1}{(4\pi)^4} \cdot \boxed{ } = a_1 \overline{ } + a_2 \overline{ } + a_3 \overline{ } + a_4 \overline{ } + a_5 \overline{ }$$

$$\frac{1}{(4\pi)^4} \cdot \boxed{ } = a_1 \overline{ } + a_2 \overline{ } + a_3 \overline{ } + a_4 \overline{ } + a_5 \overline{ }$$

**☑** d=3 from d=5

Numerical Solution of Dim. Rec. Rel.

SUMMERTIME Lee, Mingulov

 $\mathcal{M}_{3,6} = s^{2\varepsilon-2} \begin{bmatrix} 0.00002005074659118034216631402981859119949575742549538723187/\varepsilon^2 \\ -0.00009840138812460833249783740685543350373855084153798514138/\varepsilon \\ +0.00008751790270929812451430800595930715306389454769730505664 \\ +0.00083640896480242565453996588706281367341758130837556548495\varepsilon \\ +\mathcal{O}\left(\varepsilon^2\right) \end{bmatrix}$ 

$$\frac{1}{(4\pi)^4} \cdot \frac{1}{(4\pi)^4} \cdot \frac{1}$$

#### **☑** d=3 from d=5

Numerical Solution of Dim. Rec. Rel.

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Analytic ansatz :: experimental mathematics ::

$$= s^{2\varepsilon - 2} (4\pi)^{-4 - 2\varepsilon} e^{2\varepsilon \gamma_E} \frac{1}{2} \left[ \frac{1}{\varepsilon^2} - \frac{1}{\varepsilon} - 8 + \frac{\pi^2}{12} - \varepsilon \left( 18 - \pi^2 \left( \frac{13}{4} - 2\log 2 \right) - \frac{77}{3} \zeta_3 \right) + \mathcal{O}\left(\varepsilon^2\right) \right] \quad \boxed{\square}$$

confirmed by Damour, Jaranowski (analytical)

$$\frac{1}{(4\pi)^4} \cdot \frac{1}{(4\pi)^4} \cdot \frac{1}$$

#### **☑** d=3 from d=5

Numerical Solution of Dim. Rec. Rel. SUMMERTIME

**SUMMERTIME** Lee, Mingulov

$$\mathcal{M}_{3,6} = s^{2\varepsilon-2} \begin{bmatrix} 0.00002005074659118034216631402981859119949575742549538723187/\varepsilon^2 \\ -0.00009840138812460833249783740685543350373855084153798514138/\varepsilon \\ +0.00008751790270929812451430800595930715306389454769730505664 \\ +0.00083640896480242565453996588706281367341758130837556548495\varepsilon \\ +\mathcal{O}\left(\varepsilon^2\right) \end{bmatrix}$$

Analytic ansatz :: experimental mathematics ::

$$= s^{2\varepsilon - 2} (4\pi)^{-4 - 2\varepsilon} e^{2\varepsilon \gamma_E} \frac{1}{2} \left[ \frac{1}{\varepsilon^2} - \frac{1}{\varepsilon} - 8 + \frac{\pi^2}{12} \right] - \varepsilon \left( 18 - \pi^2 \left( \frac{13}{4} - 2\log 2 \right) - \frac{77}{3} \zeta_3 \right) + \mathcal{O}\left(\varepsilon^2\right) \right]$$
confirmed by
Damour, Jaranowski (analytical)

important impact

# **Computational Algorithm**

#### **Amplitudes**

$$\mathcal{A}_{49} = \boxed{ } = -2 i (8\pi G_N)^5 \left(\frac{(d-2)}{(d-1)} m_1 m_2\right)^3 - \boxed{ } [N_{49}]$$

$$- [N_{49}] \equiv \int_{k_1, k_2, k_3, k_4} \frac{N_{49}}{k_1^2 p_2^2 k_3^2 p_4^2 k_{12}^2 k_{13}^2 k_{23}^2 k_{24}^2 k_{34}^2} ,$$

$$N_{49} \equiv (k_1 \cdot k_3 \ k_{12} \cdot k_{23} - k_1 \cdot k_{12} \ k_3 \cdot k_{23} - k_1 \cdot k_{23} \ k_3 \cdot k_{12}) \times (p_2 \cdot k_{23} \ p_4 \cdot k_{34} + p_4 \cdot k_{23} \ p_2 \cdot k_{34} - p_2 \cdot p_4 \ k_{23} \cdot k_{34}) ,$$

$$=c_1$$
  $+c_2$   $+c_3$   $+c_4$   $+c_5$ 

$$\mathcal{A}_{49} = -\mathrm{i}(8\pi G_N)^5 (m_1 m_2)^3 2^{-4} (4\pi)^{-(4+2\varepsilon)} e^{2\varepsilon \gamma_E} s^{(1+2\varepsilon)} \left[ \frac{1}{\varepsilon} \left( \frac{\pi^2}{16} - \frac{2}{3} \right) + \frac{29}{18} - \frac{13}{144} \pi^2 - \frac{\pi^2}{8} \log 2 + \mathcal{O}(\varepsilon^1) \right]$$



#### **Lagrangian**

$$\mathcal{L}_{49} = -i \lim_{d \to 3} \int_{p} e^{ip \cdot r} \mathcal{A}_{49} = (32 - 3\pi^{2}) \frac{G_{N}^{5} m_{1}^{3} m_{2}^{3}}{r^{5}}$$

**Fourier Tfm** 

#### Individual terms

Foffa, Sturani, Sturm, & P.M.

Damour, Jaranowski

$$0 = \mathcal{L}_9 = \mathcal{L}_{12} = \mathcal{L}_{13} = \mathcal{L}_{22} = \mathcal{L}_{26} = \mathcal{L}_{27} = \mathcal{L}_{31} = \mathcal{L}_{36} = \mathcal{L}_{46} = \mathcal{L}_{47},$$

$$\frac{1}{2} \frac{G_N^5 m_1^3 m_2^3}{r^5} = \mathcal{L}_1 = \mathcal{L}_3 = 4\mathcal{L}_5 = 3\mathcal{L}_{14} = \frac{\mathcal{L}_{19}}{8} = \frac{3\mathcal{L}_{20}}{2} = \frac{3\mathcal{L}_{21}}{4} = \frac{\mathcal{L}_{23}}{4} = \frac{\mathcal{L}_{24}}{4} = \frac{3\mathcal{L}_{25}}{2},$$

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$$\frac{1}{120} \frac{G_N^5 m_1^5 m_2}{r^5} = \mathcal{L}_6 = \frac{\mathcal{L}_7}{20} = \frac{3\mathcal{L}_{30}}{20} = -\frac{3\mathcal{L}_{35}}{56} = \frac{\mathcal{L}_{39}}{24} = \frac{\mathcal{L}_{45}}{12},$$

$$\mathcal{L}_{28} = \frac{G_N^5 m_1^4 m_2^2}{r^5} \left[ \frac{428}{75} + \frac{4}{15} \mathcal{P} \right] ,$$

$$\mathcal{L}_{32} = \frac{G_N^5 m_1^3 m_2^3}{r^5} \left[ -\frac{91}{450} + \frac{1}{15} \mathcal{P} \right] ,$$

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$$\mathcal{L}_{37} = -\frac{G_N^5 m_1^4 m_2^2}{r^5} \left[ 17 + 2 \mathcal{P} \right] ,$$

$$\mathcal{L}_{40} = \frac{G_N^5 m_1^4 m_2^2}{r^5} \left[ -\frac{39}{25} + \frac{4}{15} \mathcal{P} \right] ,$$

$$\mathcal{L}_{42} = -\frac{G_N^5 m_1^3 m_2^3}{r^5} \left[ \frac{97}{225} + \frac{1}{15} \mathcal{P} \right] ,$$

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$$\mathcal{L}_{50} = \frac{G_N^5 m_1^3 m_2^3}{r^5} \left( 4\pi^2 - \frac{124}{3} \right)$$

$$\mathcal{P} \equiv \frac{1}{\varepsilon} - 5 \log \frac{r}{L_0}$$
 
$$L = \sqrt{4\pi e^{\gamma_E}} L_0$$
 
$$\Lambda^{-2} \equiv 32\pi G_N L^{d-3}$$

#### Individual terms

Foffa, Sturani, Sturm, & **P.M.**Damour, Jaranowski

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diverge @ d=3

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$$\mathcal{L}_{29} = \frac{G_N^5 m_1^3 m_2^3}{r^5} \left[ -\frac{409}{450} + \frac{1}{5} \mathcal{P} \right],$$

$$\mathcal{L}_{33} = \frac{G_N^5 m_1^3 m_2^3}{r^5} \left( 16 - \pi^2 \right),$$

$$\mathcal{L}_{37} = -\frac{G_N^5 m_1^4 m_2^2}{r^5} \left[ 17 + 2 \mathcal{P} \right],$$

$$\mathcal{L}_{40} = \frac{G_N^5 m_1^4 m_2^2}{r^5} \left[ -\frac{39}{25} + \frac{4}{15} \mathcal{P} \right],$$

$$\mathcal{L}_{42} = -\frac{G_N^5 m_1^3 m_2^3}{r^5} \left[ \frac{97}{225} + \frac{1}{15} \mathcal{P} \right],$$

$$\mathcal{L}_{44} = -\frac{G_N^5 m_1^3 m_2^3}{r^5} \left[ \frac{37}{75} + \frac{2}{5} \mathcal{P} \right],$$

$$\mathcal{L}_{49} = \frac{G_N^5 m_1^3 m_2^3}{r^5} \left( 32 - 3\pi^2 \right),$$

$$\mathcal{L}_{50} = \frac{G_N^5 m_1^3 m_2^3}{r^5} \left( 4\pi^2 - \frac{124}{3} \right)$$

$$\mathcal{P} \equiv \frac{1}{\varepsilon} - 5 \log \frac{r}{L_0}$$
 
$$L = \sqrt{4\pi e^{\gamma_E}} L_0$$
 
$$\Lambda^{-2} \equiv 32\pi G_N L^{d-3}$$

finite @ d=3

Foffa, Sturani, Sturm, & P.M.

Damour, Jaranowski

**Total contribution** 

$$\sum_{a=1}^{50} \mathcal{L}_a = \frac{3}{8} \frac{G_N^5 m_1^5 m_2}{r^5} + \frac{31}{3} \frac{G_N^5 m_1^4 m_2^2}{r^5} + \frac{141}{8} \frac{G_N^5 m_1^3 m_2^3}{r^5} \,.$$

finite @ d=3

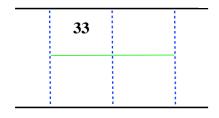
Foffa, Sturani, Sturm, & P.M.

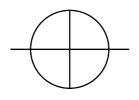
Damour, Jaranowski

#### **Total contribution**

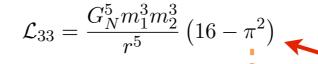
$$\sum_{a=1}^{50} \mathcal{L}_a = \frac{3}{8} \frac{G_N^5 m_1^5 m_2}{r^5} + \frac{31}{3} \frac{G_N^5 m_1^4 m_2^2}{r^5} + \frac{141}{8} \frac{G_N^5 m_1^3 m_2^3}{r^5}.$$

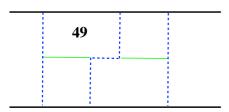
#### Recap :: Pi^2-Terms

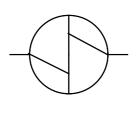




33

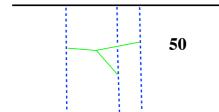


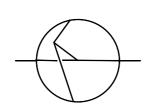




49

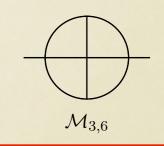
$$\mathcal{L}_{49} = \frac{G_N^5 m_1^3 m_2^3}{r^5} \left( 32 - 3\pi^2 \right)$$





$$\mathcal{L}_{50} = \frac{G_N^5 m_1^3 m_2^3}{r^5} \left( 4\pi^2 - \frac{124}{3} \right)$$

### M<sub>3,6</sub> contribution

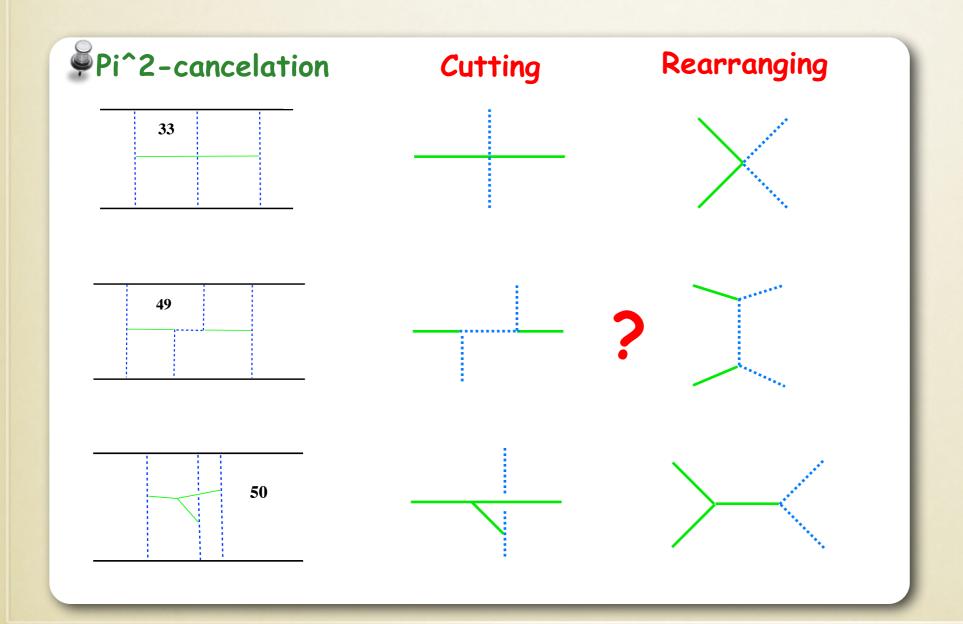


Damour, Jaranowski Foffa, Sturani, Sturm, & P.M.

Foffa, Sturani, Sturm, & P.M.

#### **Total contribution**

$$\sum_{n=1}^{50} \mathcal{L}_a = \frac{3}{8} \frac{G_N^5 m_1^5 m_2}{r^5} + \frac{31}{3} \frac{G_N^5 m_1^4 m_2^2}{r^5} + \frac{141}{8} \frac{G_N^5 m_1^3 m_2^3}{r^5}.$$



### Summary ...

- **Multi-Loop Diagrammatic Techniques** 
  - Powerful tools for General Relativity
  - **□ IBPs + Difference & Differential Equations**
- Application :: Coalescent Binaries Dynamics @ 4PN-O(G^5)
  - Basic idea :: EFT-GR Diagrams ~ 2-point QFT Diagrams
  - 4-loop EFT-GR Diagrams mapped into 4-loop QFT Problem
- **QCD** meets Gravity :: QCD beta-function vs PN-expansion in GR-EFT

### ... and Outlook

- M<sub>3,6</sub> analytically via systems of Differential Equations and/or R-star operation
- How about higher-loops?
- How about higher-legs (diagrams with radiation)?
- GR & GW-physics EFT vs HEP & Amplitudes
- Unitarity-based methods, Multi-loop Integrals and Integrands decomposition, Amplitudes-inspired dualities, BCJ/Double-copy
- Finite integrals in position-space



# Status of PN Corrections Porto

Newton 1687

	no spin		spin
4PN	Damour, Jaranowski Foffa, Sturani, Sturm, & PM Bernard, Blanchet, Bohe, Faye, Marsat Damour, Jaranowski, Schaefer Jaranowski, Schaefer Le Tiec, Blanchet, Whiting Jaranowski, Schaefer Foffa, Sturani Blanchet, Damour '88		
3PN	De Andrade, Esposito-Farese, Itho, Futamase Blanchet, Faye Damour, Jaranowski, Schaefer	NNLO	Steinhoff Levi
2PN	Damour, Schaefer '85 Damour, Deruelle '81-'83	NLO	Steinhoff, Hergt, Schaefer Damour, Jaranowski, Schaefer Buonanno, Faye, Blanchet Porto, Rothstein
1PN	Einstein-Infeld-Hoffman 1938	LO	Barker, O'Connel '75
ODNI			

### Experimental Mathematics for M<sub>3,6</sub>

- Numerical Reconstruction
- **from SUMMERTIME**

```
ln[1]:= nM36 = (
        0.00002005074659118034216631402981859119949575742549538723187 / \epsilon^2
           -0.00009840138812460833249783740685543350373855084153798514138 / \epsilon
          + 0.00008751790270929812451430800595930715306389454769730505664
          + 0.00083640896480242565453996588706281367341758130837556548495 * \epsilon
          +\epsilon^2 * Help[\epsilon, 2]
       );
ln[2]:= pref = (4 * Pi) ^ (-4 - 2 * \epsilon) * Exp[2 * \epsilon * EulerGamma]
Out[2]= e^{2\gamma\epsilon} (4\pi)^{-2\epsilon-4}
ln[3]:= npref = N[Series[pref, {\epsilon, 0, 2}], 50] // Chop
0.00015670128306685598066304675407368460848558683208520 \epsilon +
      0.00030616431167705224971803922217880567378514178260532 \epsilon^2 + O(\epsilon^3)
In[4]:= nBexp = nM36 / npref
     Out[4]=
      3.58876648328794339088189620833849370269526252470 + O(\epsilon^{1})
```

## Experimental Mathematics for M<sub>3,6</sub>

Numerical Reconstructionfrom SUMMERTIME

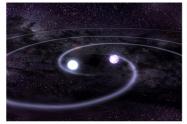
```
ln[1]:= nM36 = (
        0.00002005074659118034216631402981859119949575742549538723187 / \epsilon^2
          -0.00009840138812460833249783740685543350373855084153798514138 / \epsilon
         + 0.00008751790270929812451430800595930715306389454769730505664
         + 0.00083640896480242565453996588706281367341758130837556548495 * \epsilon
         +\epsilon^2 * Help[\epsilon, 2]
       );
ln[2]:= pref = (4 * Pi) ^ (-4 - 2 * \epsilon) * Exp[2 * \epsilon * EulerGamma]
Out[2]= e^{2\gamma\epsilon} (4\pi)^{-2\epsilon-4}
ln[3]:= npref = N[Series[pref, {\epsilon, 0, 2}], 50] // Chop
0.00015670128306685598066304675407368460848558683208520 \epsilon +
      0.00030616431167705224971803922217880567378514178260532 \epsilon^2 + O(\epsilon^3)
|n[4]:= nBexp = nM36 / npref
                                                        ———— (1/2) :: double pole
     Out[4]=
                                                          ——— (-1/2) :: single pole
      3.58876648328794339088189620833849370269526252470 \leftarrow O(\epsilon^1) :: finite term
```

# Experimental Mathematics for M<sub>3,6</sub>

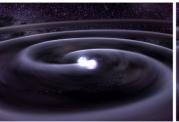
### Finite term O(1)

```
■ 50 digits
ln[5]:= test = N[Coefficient[nBexp, \epsilon, 0], 50]
Out[5]= -3.58876648328794339088189620833849370269526252470
In[6]:= vars = {1, Pi, Log[2], Zeta[2], Log[2]^2, Zeta[3], Log[2] * Zeta[2], Pi * Zeta[2]}
                                                                                     :: transcendental constants ::
Out[6]= \left\{1, \pi, \log(2), \frac{\pi^2}{6}, \log^2(2), \zeta(3), \frac{1}{6}\pi^2 \log(2), \frac{\pi^3}{6}\right\}
                                                                                             :: educated guess ::
|n|7|:= quess = {test, vars} // Flatten
Out[7]= \left\{-3.58876648328794339088189620833849370269526252470, 1, \pi, \log(2), \frac{\pi^2}{6}, \log^2(2), \zeta(3), \frac{1}{6}\pi^2 \log(2), \frac{\pi^3}{6}\right\}
In[8]:= fit = FindIntegerNullVector[guess]
Out[8]= \{4, 16, 0, 0, -1, 0, 0, 0, 0\}
In[9]:= check = Sum[fit[[i]] * guess[[i]], {i, 1, Length[guess]}]
Out[9]= 0.\times 10^{-48}
    result
ln[10]:= res = -1 / fit[[1]] * Sum[fit[[i]] * guess[[i]], {i, 2, Length[guess]}] // Expand
                                           ----- :: finite term
```

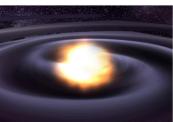
#### Binary coalescence: a tale made of three stories



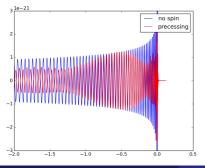
Inspiral phase post-Newtonian approximation: v/c



Merger: fully non-perturbative: Nu- Perturbed merical Relativity



Ring-down: Kerr Black Hole



Spin can induce precession and change the amplitude (and phase) of the waveform due to  $cos(\theta_{LN})$  factors in  $h_{+,\times}$ 

#### Was it necessary to build a detector? The Hulse-Taylor binary pulsar

GW's first observed in the NS-NS binary system PSR B1913+16 Observation of orbital parameters  $(a_p \sin \iota, e, P, \dot{\theta}, \gamma, \dot{P})$ 

determination of  $m_p$ ,  $m_c$  (1PN physics, GR)

Energy dissipation in GW's  $\rightarrow \dot{P}^{(GR)}(m_p,m_c,P,e)$  vs.  $\dot{P}^{(obs)}$ 

$$\frac{1}{2\pi}\phi = \int_0^T \frac{1}{P(t)} dt \simeq \frac{T}{P_0} - \frac{\dot{P_0}}{P_0^2} \frac{T^2}{2}$$

• Test of the 1PN conservative

$$E(v) = -\frac{1}{2}\nu M v^2 \left(1 + \#(\nu)v^2 + \#(\nu)v^4 + \ldots\right)$$

leading order dissipative dynamics

$$F(v) \equiv -\frac{dE}{dt} = \frac{32}{5G_N}v^{10}\left(1 + \#(\nu)v^2 + \#(\nu)v^3 + \ldots\right)$$

#### Modeling the inspiral

 $\frac{A}{A} \ll \dot{\phi}$ Inspiral  $h = A\cos(\phi(t))$ Virial relation:

$$v \equiv (G_N M \pi f_{GW})^{1/3}$$
  $v = \frac{m_1 m_2}{(m_1 + m_2)^2}$ 

$$E(v) = -\frac{1}{2}\nu M v^{2} \left(1 + \#(\nu)v^{2} + \#(\nu)v^{4} + \ldots\right)$$

$$P(v) \equiv -\frac{dE}{dt} = \frac{32}{5G_{N}}v^{10} \left(1 + \#(\nu)v^{2} + \#(\nu)v^{3} + \ldots\right)$$

E(v)(P(v)) known up to 3(3.5)PN

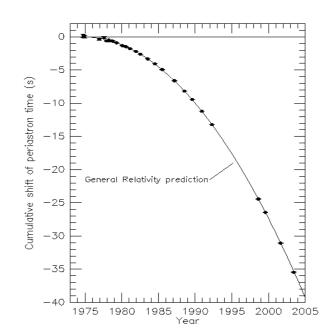
$$\frac{1}{2\pi}\phi(T) = \frac{1}{2\pi}\int^{T}\omega(t)dt = -\int^{v(T)}\frac{\omega(v)dE/dv}{P(v)}dv$$

$$\sim \int \left(1 + \#(\nu)v^{2} + \ldots + \#(\nu)v^{6} + \ldots\right)\frac{dv}{v^{6}}$$

Post-Newtonian Coefficients

Riccardo Sturani (IFT-UNESP/ICTP-SAIFR)

#### Weisberg and Taylor (2004)



 $\frac{P_{GR}-P_{exp}}{\dot{p}}\sim 10^{-3}$ 

10 pulsars in NS-NS, still  $\sim$  100Myr for coalescence