

# KLT Relations from Intersection Theory

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Sebastian Mizera

*Perimeter Institute*

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Mathematicians studying hypergeometric functions developed a unified formalism behind such identities, called *twisted de Rham theory*

The goal of today's talk is to explain how the same mathematical tools can be used to study KLT relations and scattering amplitudes more generally

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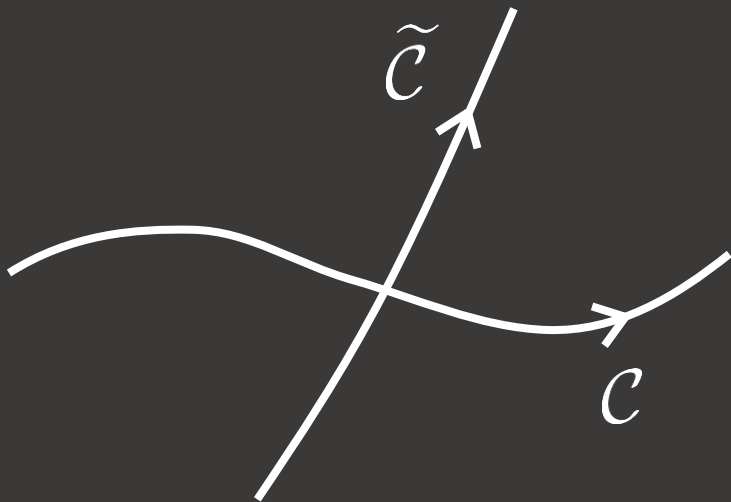
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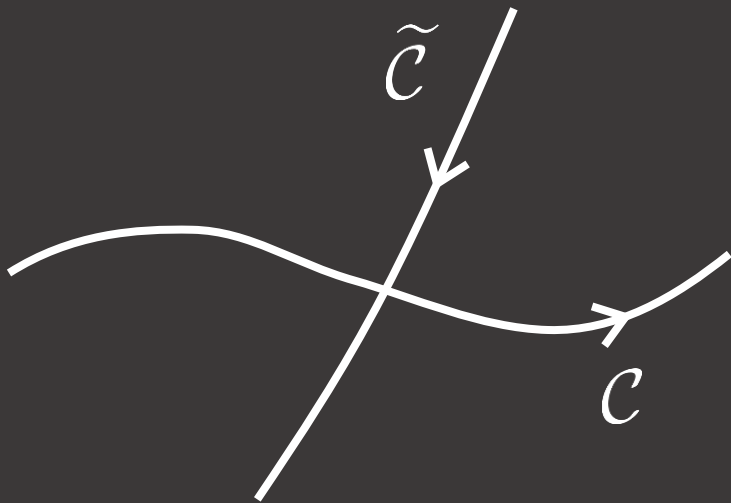
$$\langle c, \tilde{c} \rangle =$$


The diagram illustrates the intersection of two cycles. A white curve, representing a cycle  $c$ , starts on the left, curves upwards, then downwards, and ends on the right. A second white curve, representing another cycle, starts from the right side of the first curve, crosses it from above to below, and then continues upwards and to the right. The intersection point is marked with a small white arrow pointing to the right, and the label  $c$  is placed below this intersection point.

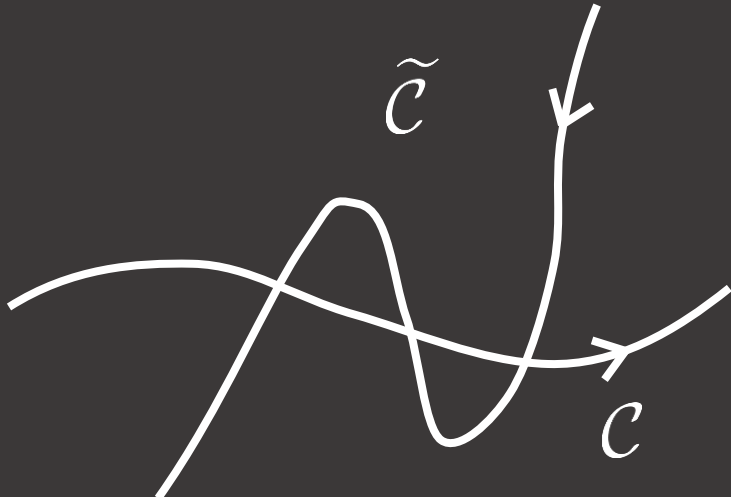
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The diagram illustrates the intersection of two oriented curves,  $c$  and  $\tilde{c}$ . Curve  $c$  is a wavy line with an arrow pointing to the right. Curve  $\tilde{c}$  is a straight line with an arrow pointing upwards and to the right. They intersect at a single point. The intersection number is shown to be  $+1$ .

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$$\langle c, \tilde{c} \rangle = \begin{array}{c} \tilde{c} \\ \downarrow \\ \text{---} \\ \uparrow \\ c \end{array} = -1$$


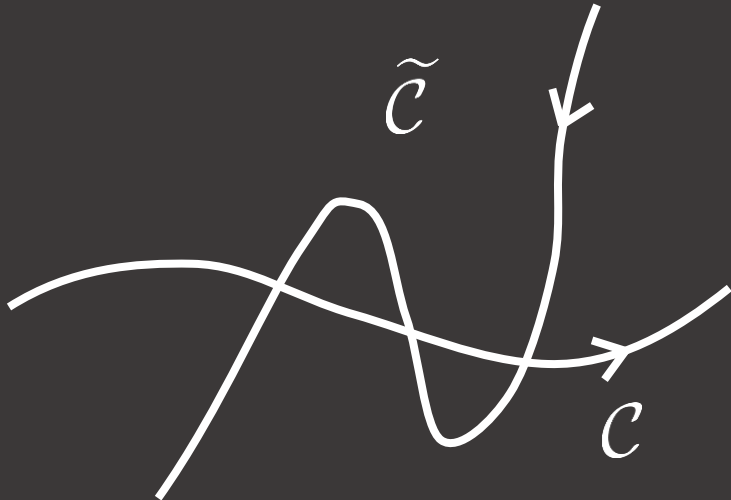
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The diagram illustrates the intersection of two oriented curves,  $c$  and  $\tilde{c}$ . Curve  $c$  is a smooth curve with an arrow pointing to the right at its right end. Curve  $\tilde{c}$  is a curve that crosses  $c$  at two points. At the first intersection point,  $\tilde{c}$  crosses  $c$  from below to above. At the second intersection point,  $\tilde{c}$  crosses  $c$  from above to below. The orientation of  $\tilde{c}$  is indicated by an arrow pointing downwards at its top end.



Intersection numbers of cycles:

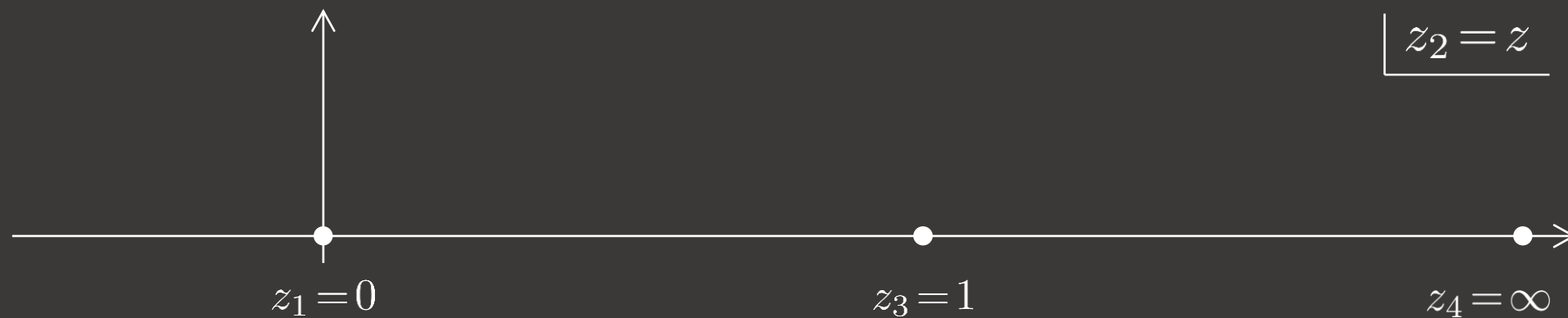
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In general, intersection numbers of *twisted* cycles will be non-integer once monodromies are taken into account

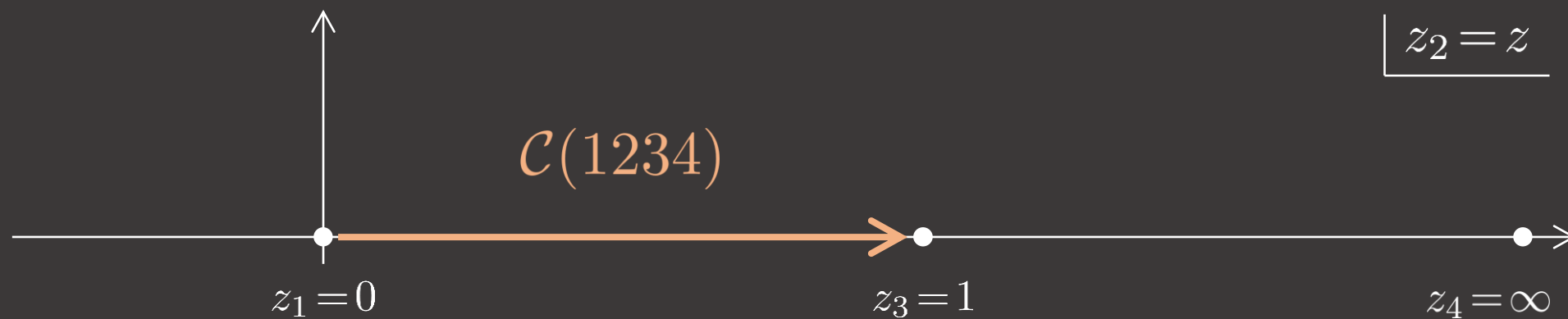
[Kita & Yoshida '92]

String theory 4-pt twisted cycles in  $\mathcal{M}_{0,4} \simeq \mathbb{CP}^1 \setminus \{0, 1, \infty\}$ :

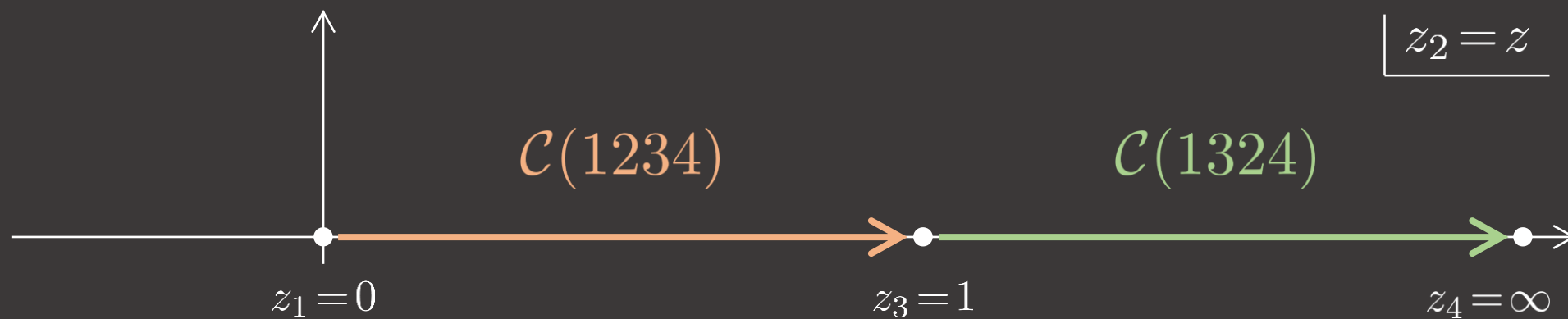
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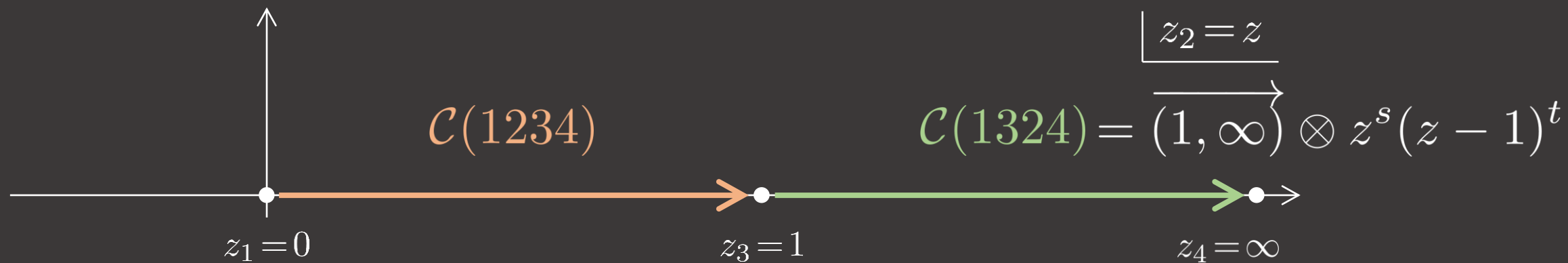
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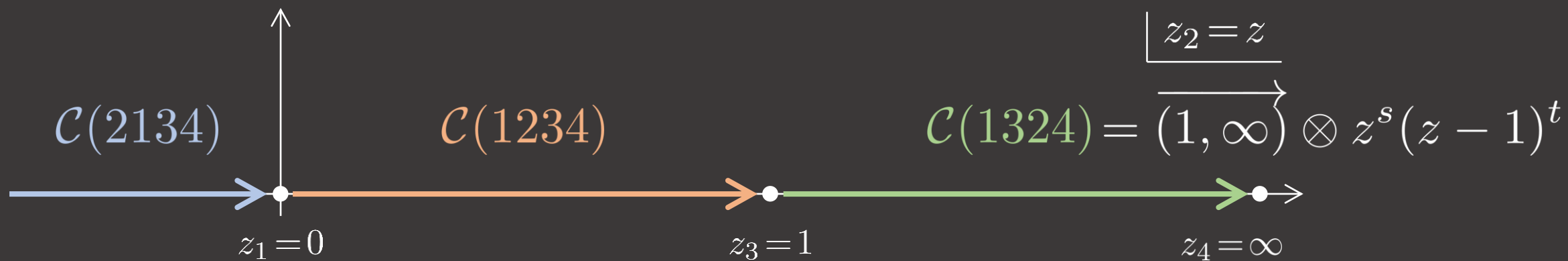
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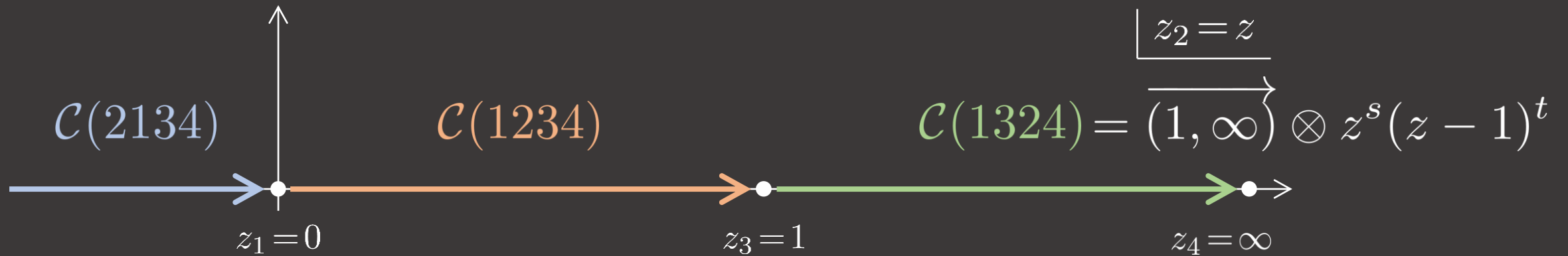
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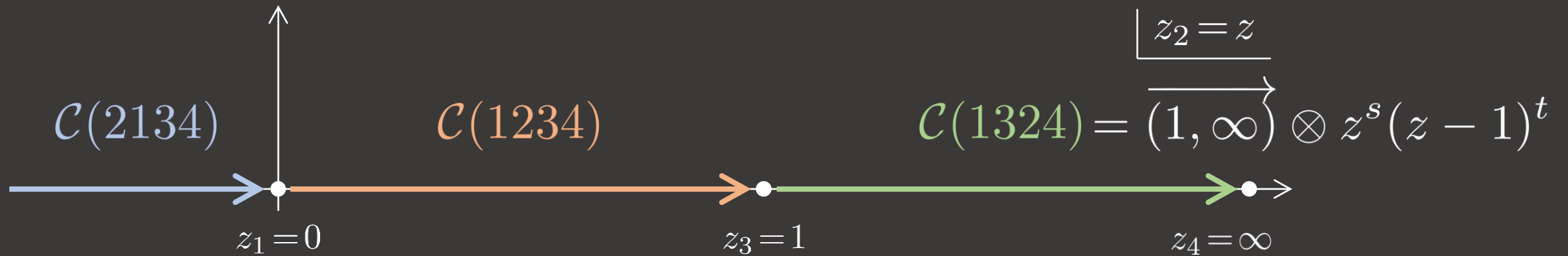
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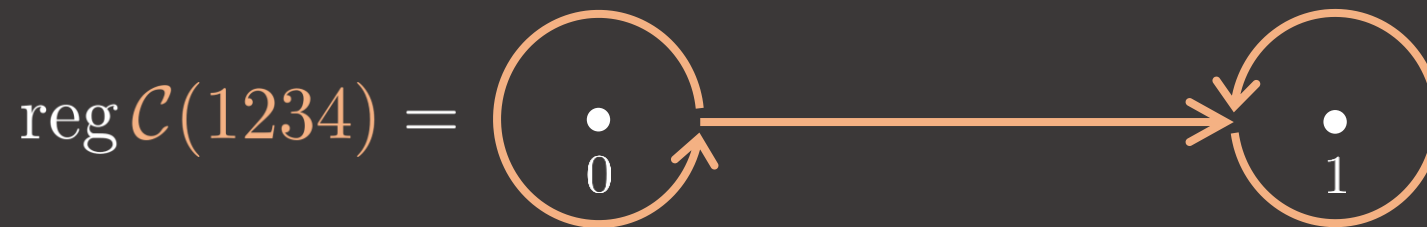
Intersections are ill-defined—need to regularize twisted cycles using the Pochhammer contour:



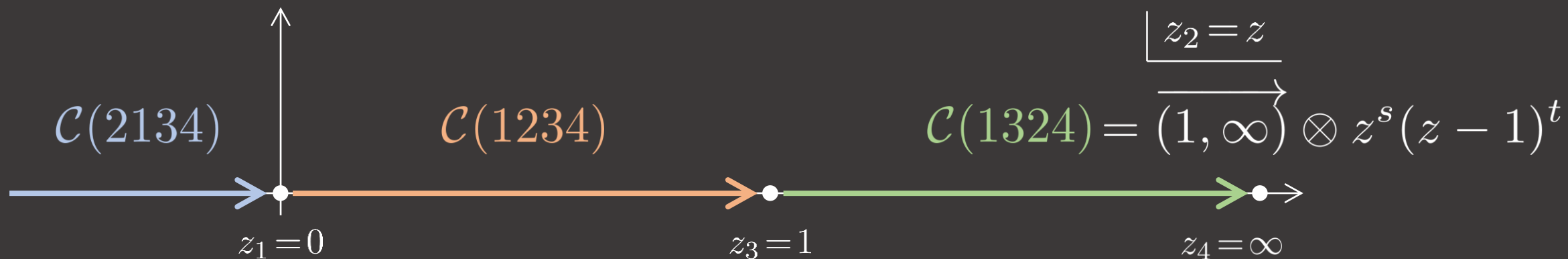
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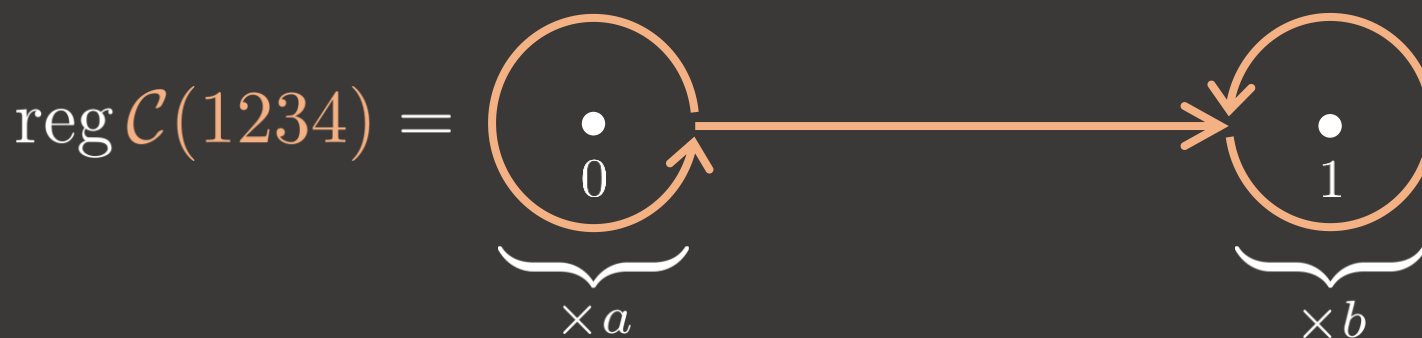
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The diagram consists of two nodes, 0 and 1, arranged horizontally. Node 0 is on the left and node 1 is on the right. Each node is represented by a circle with a central dot. Node 0 has a self-loop arrow pointing clockwise. Node 1 also has a self-loop arrow pointing clockwise. A horizontal arrow points from node 0 to node 1. Below node 0 is a white curly brace labeled 'x a'. Below node 1 is a white curly brace labeled 'x b'. The entire diagram is enclosed in large parentheses, with a partial derivative symbol '∂' to the left of the opening parenthesis.



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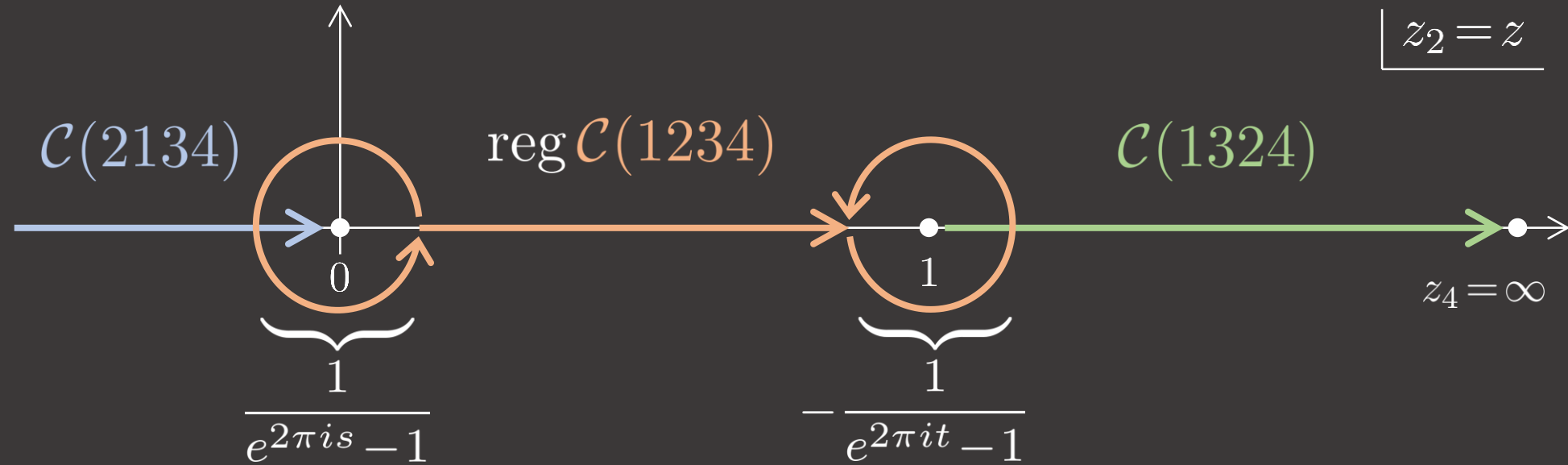
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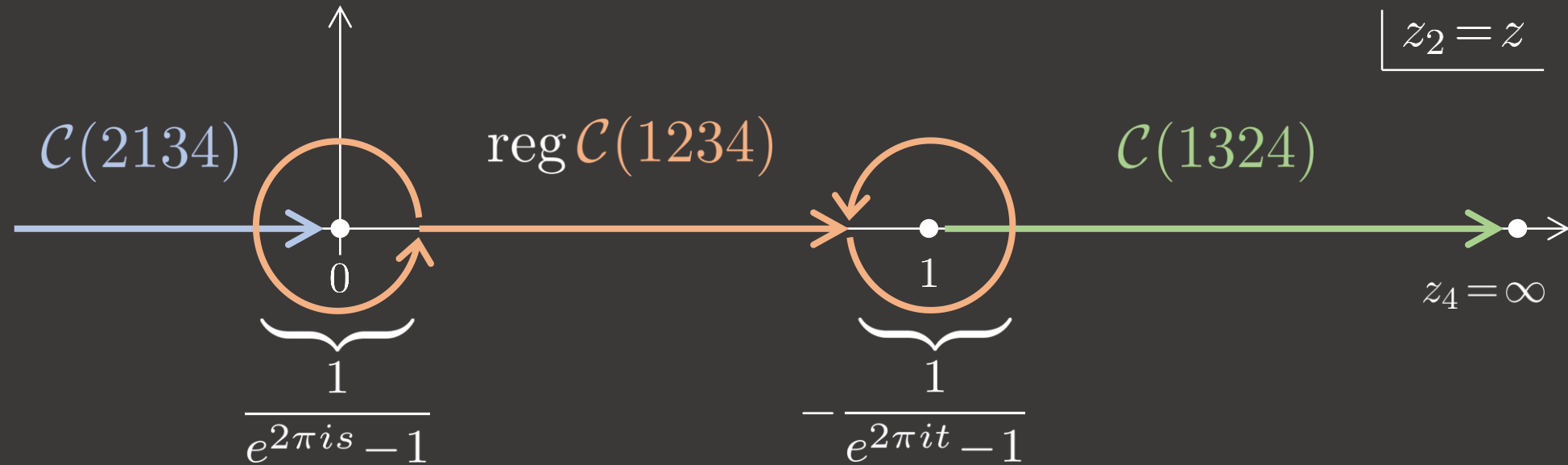
Manifests factorization channels corresponding to boundaries of the moduli space

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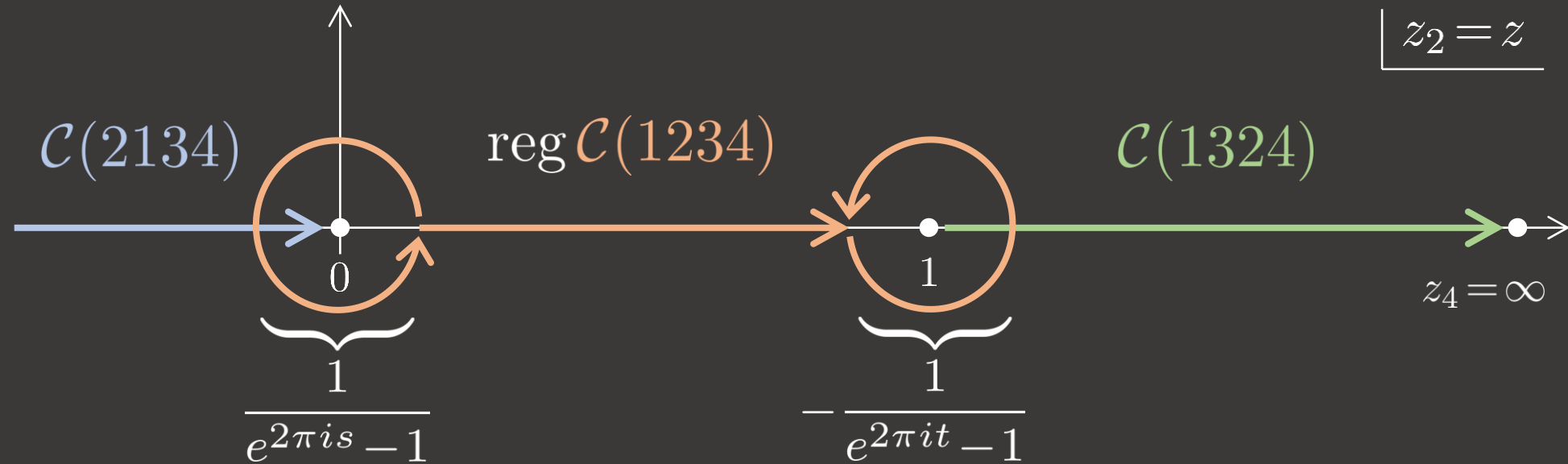


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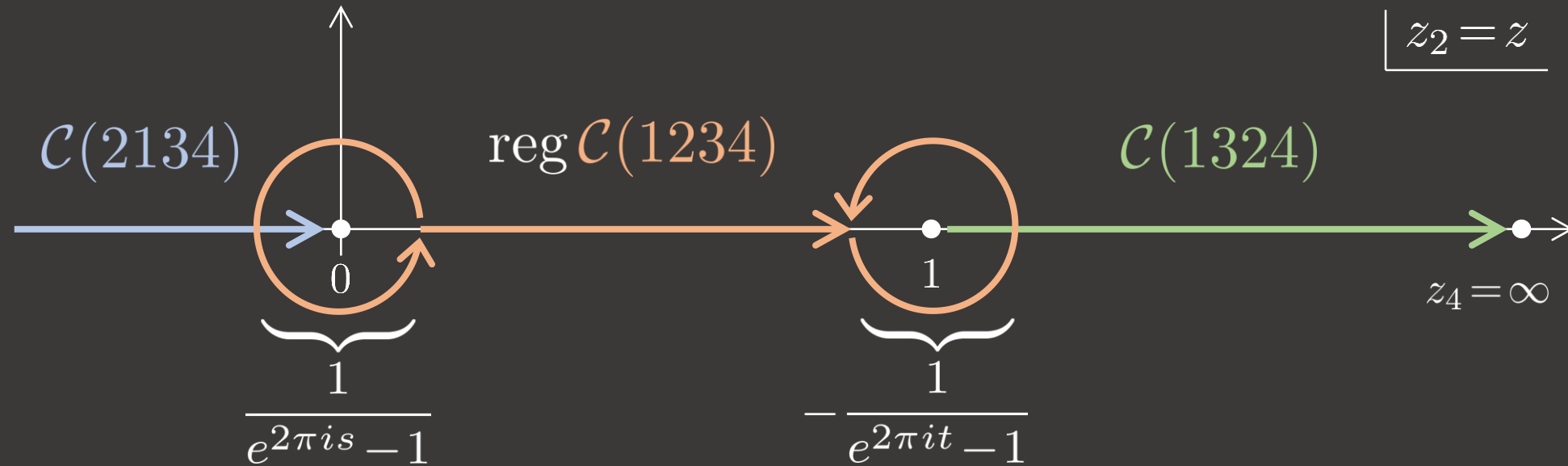
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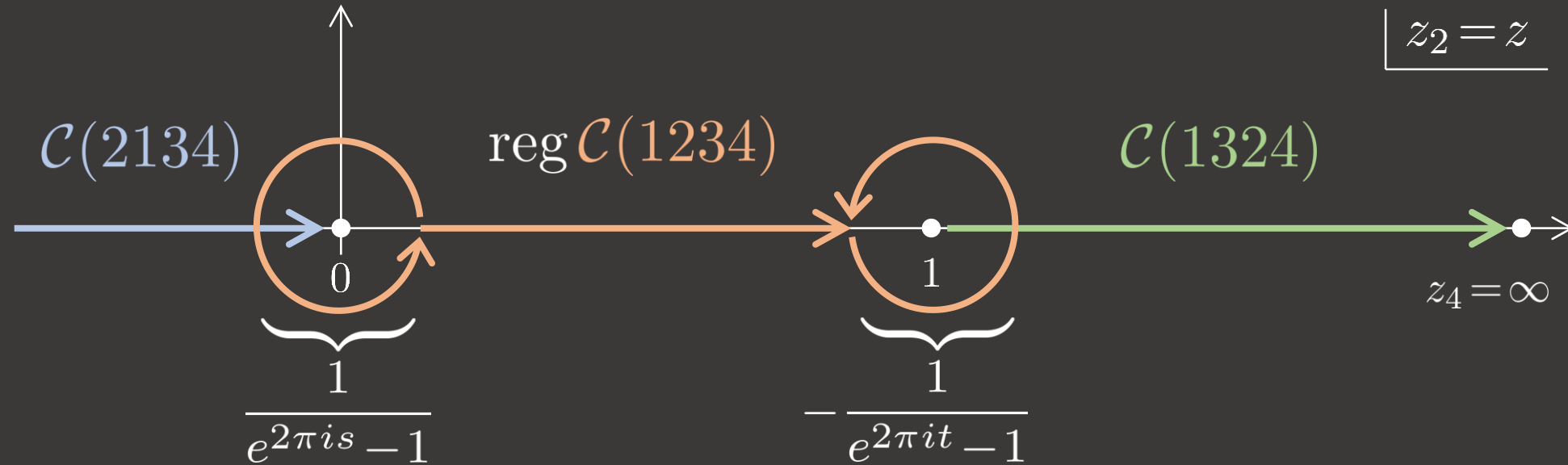


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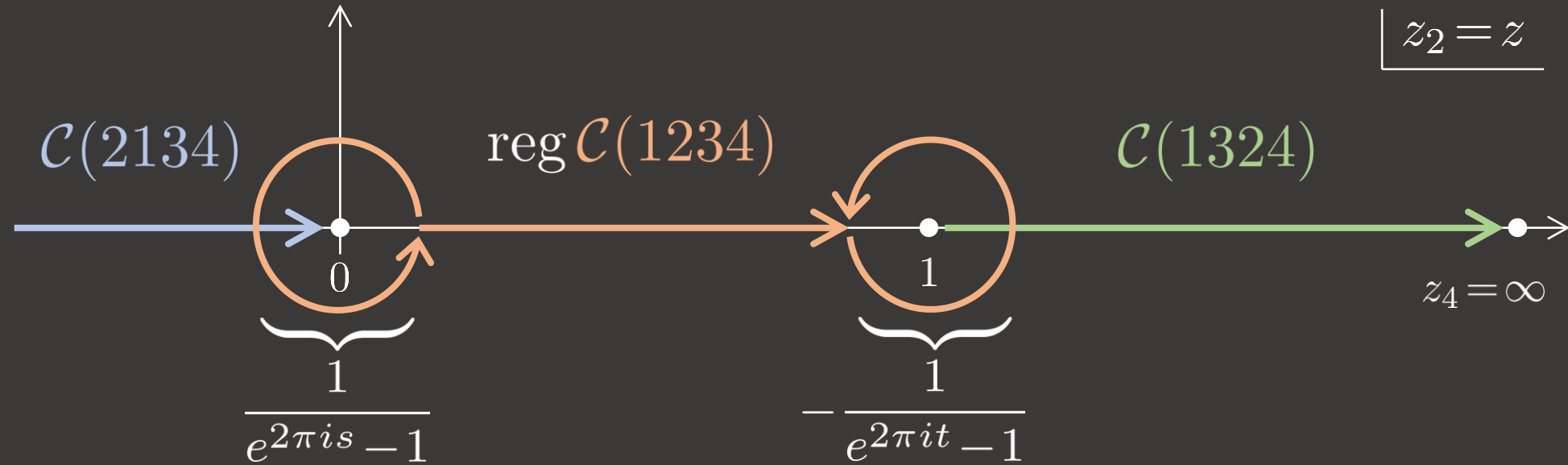
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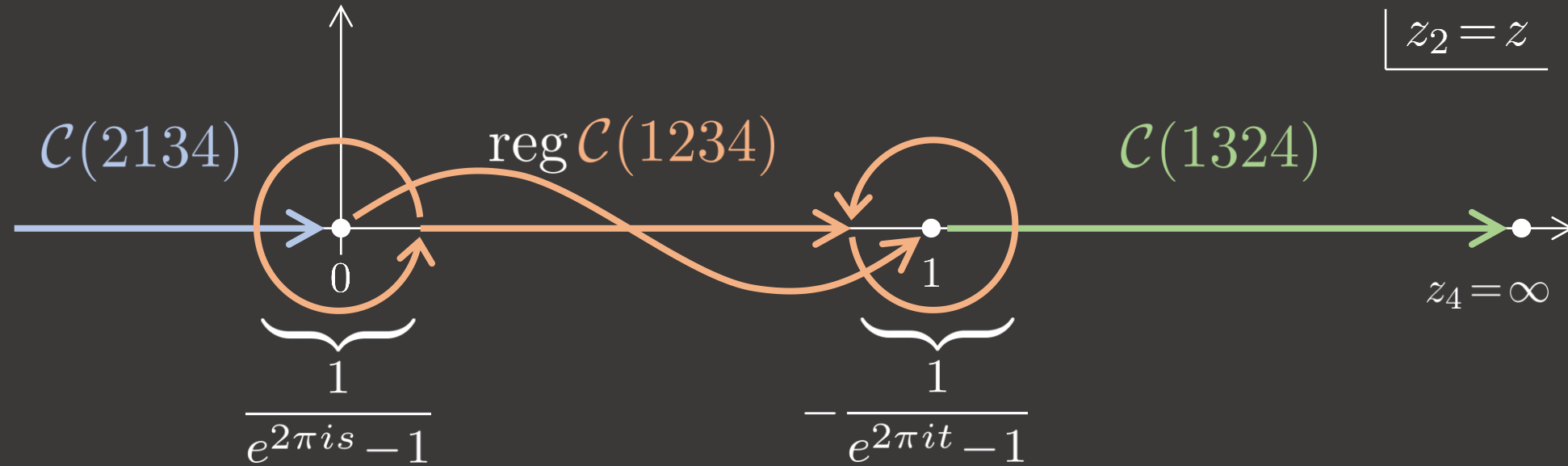


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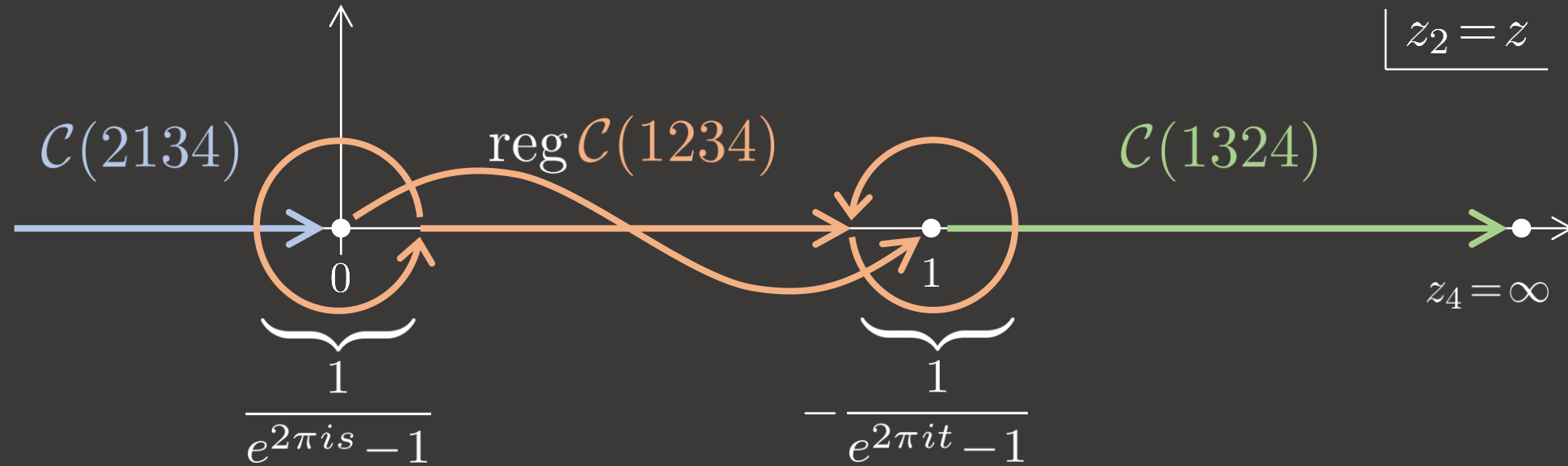


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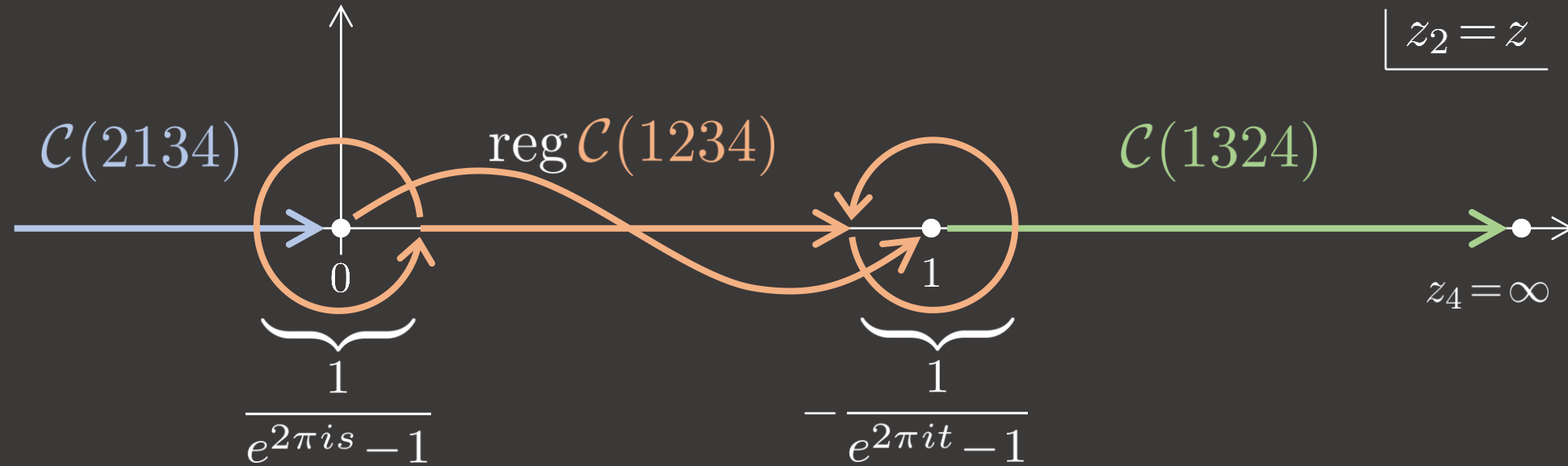


$$\langle \text{reg } \mathcal{C}(1234), \mathcal{C}(2134) \rangle = \frac{+e^{\pi i s}}{e^{2\pi i s} - 1}$$

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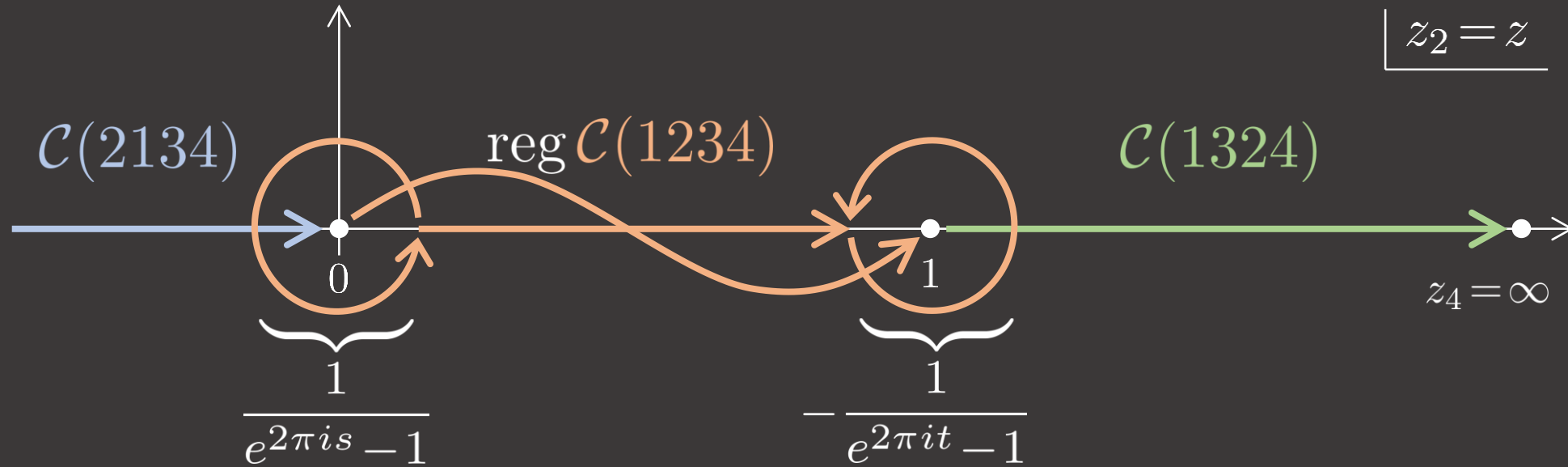


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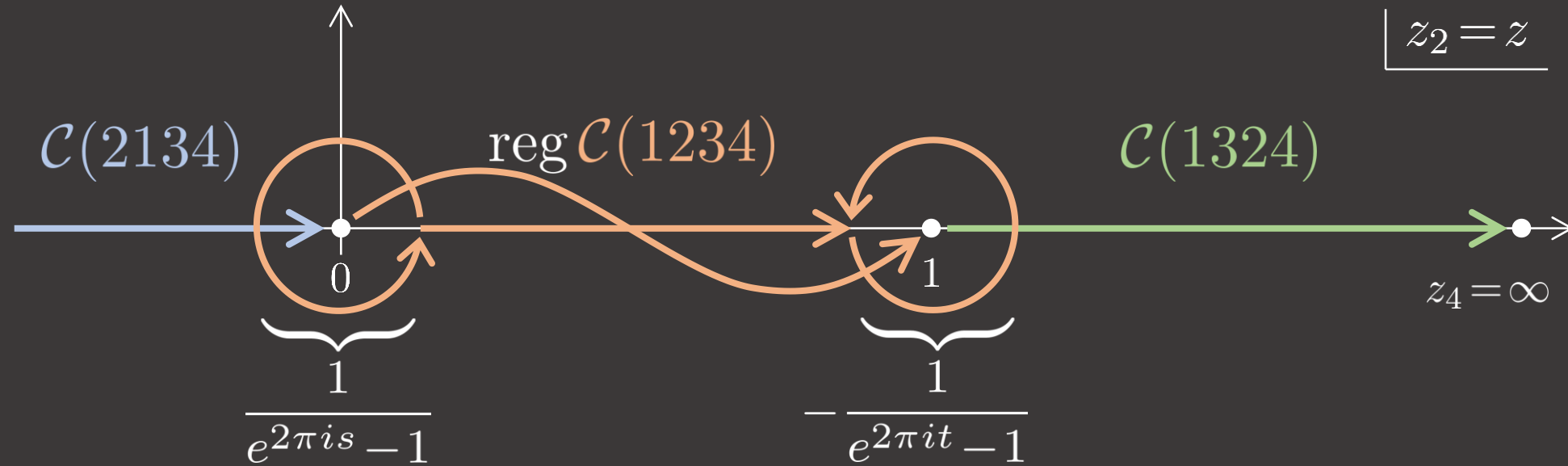


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These are the Kawai–Lewellen–Tye relations at 4-pt!

[KLT '85]

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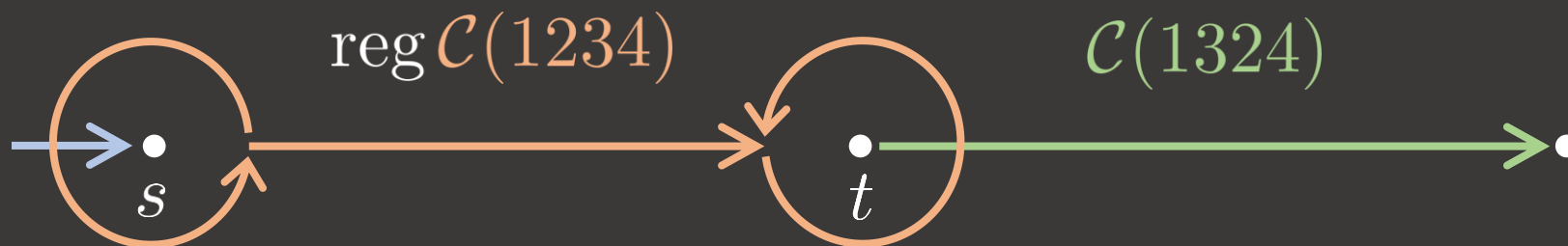
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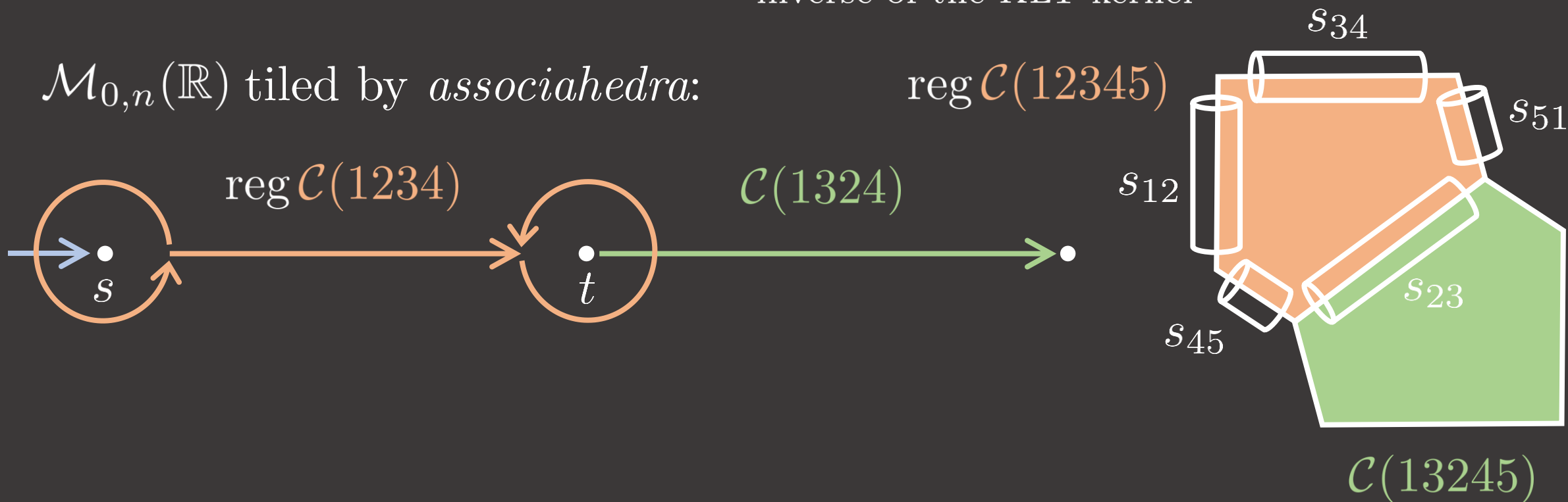


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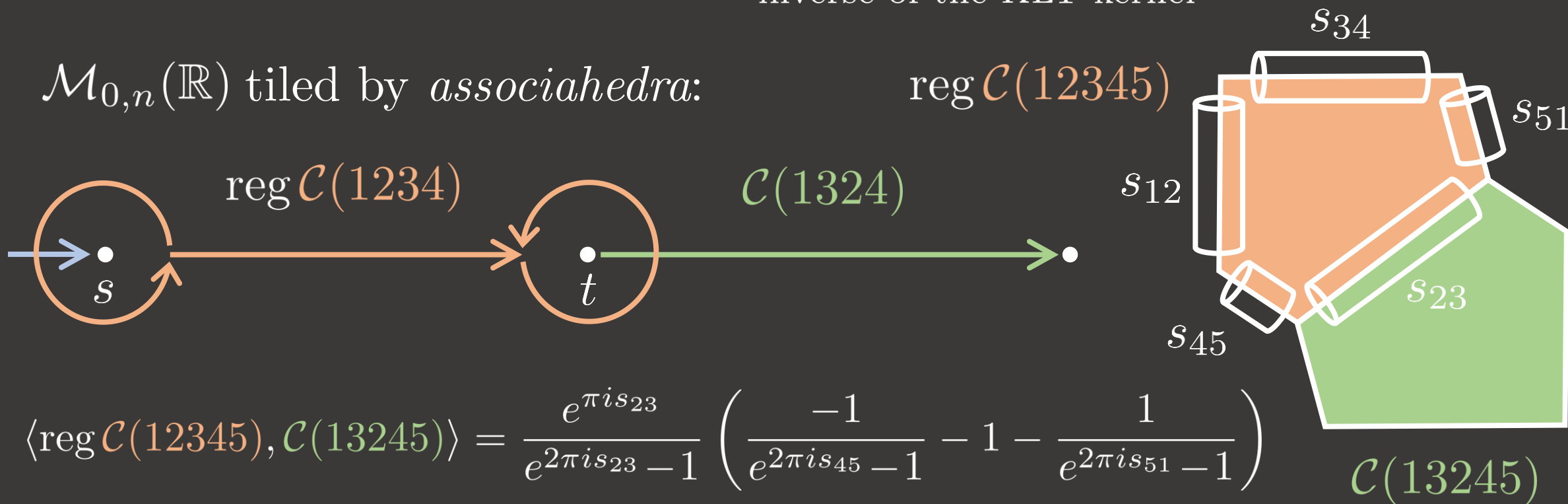


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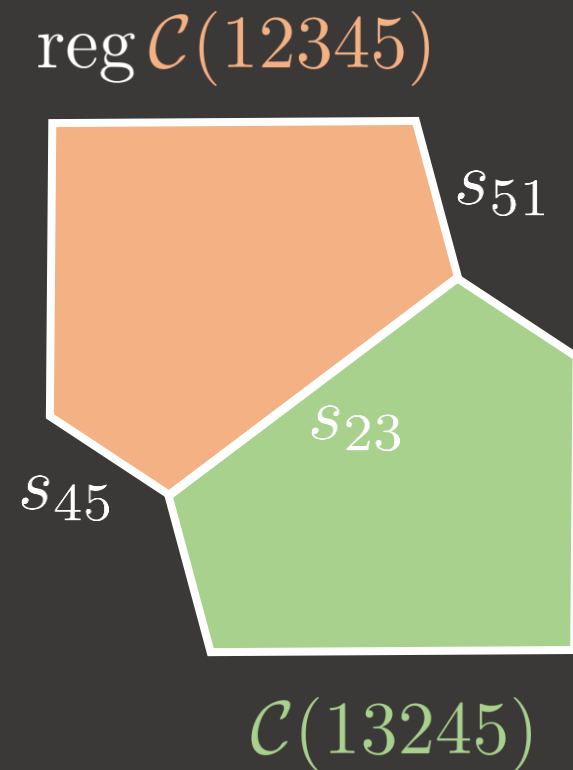
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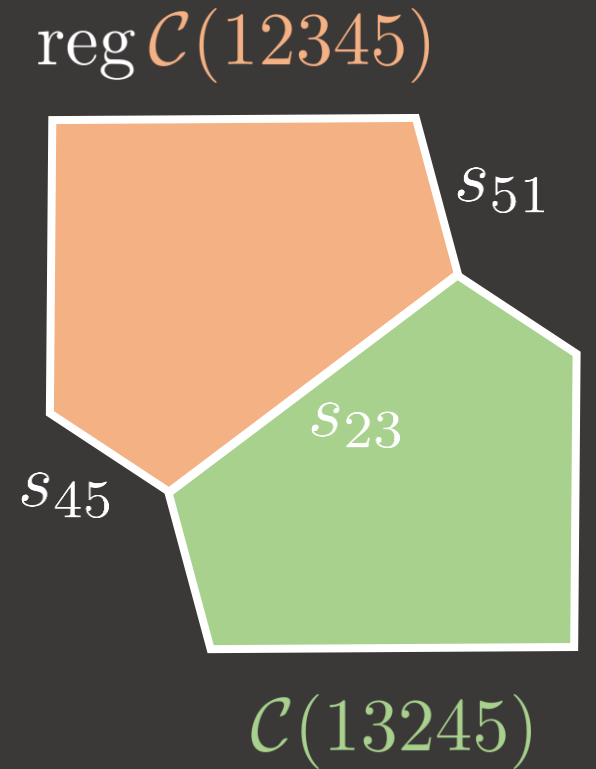
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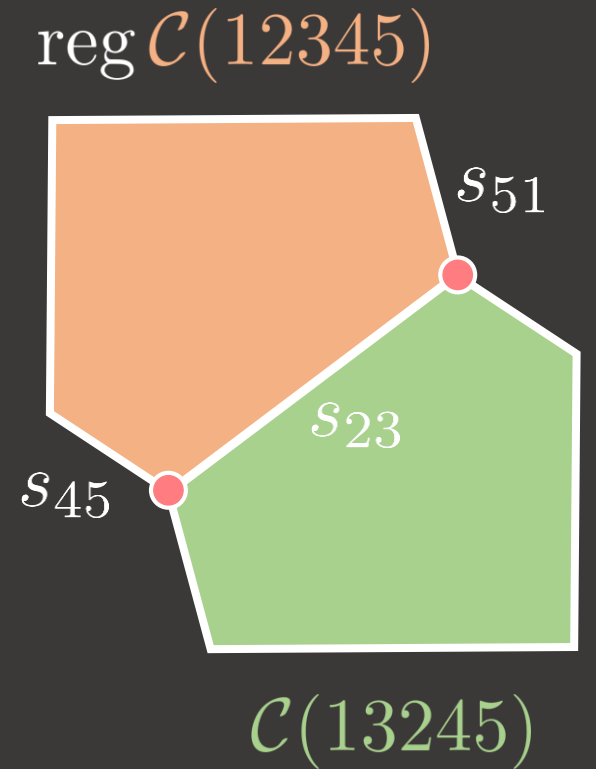
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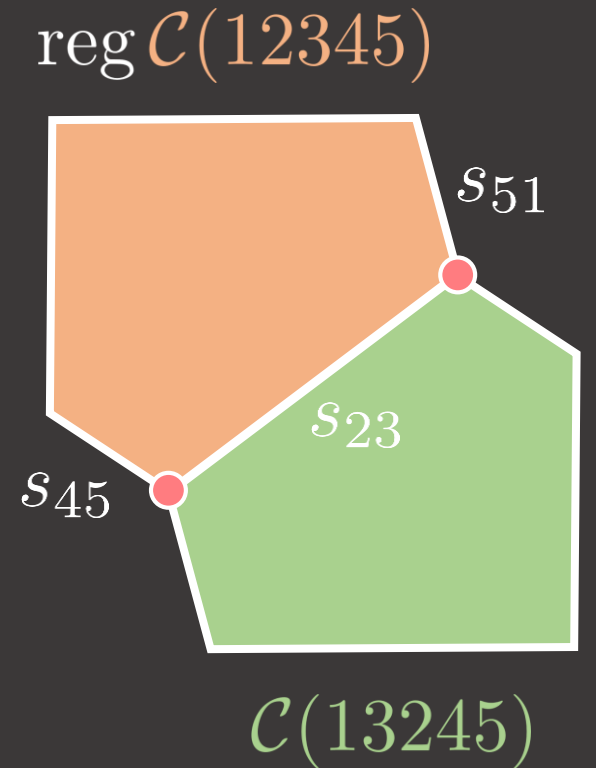
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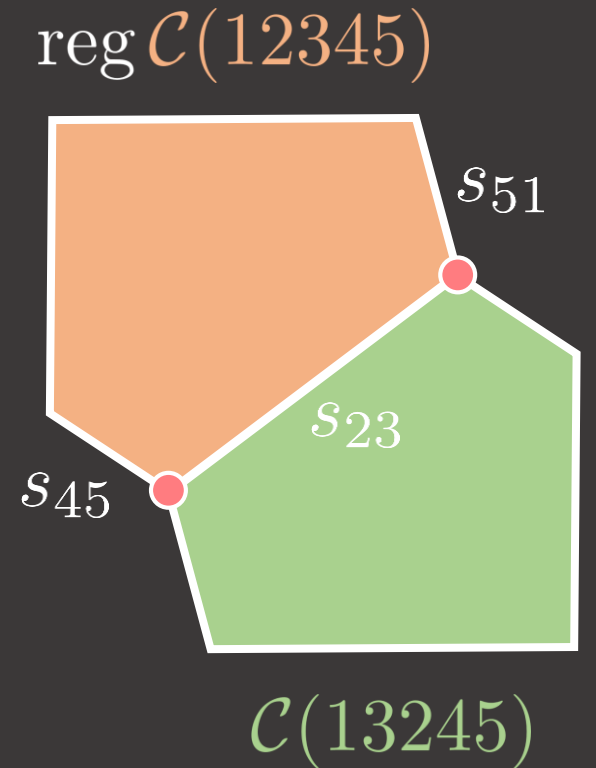


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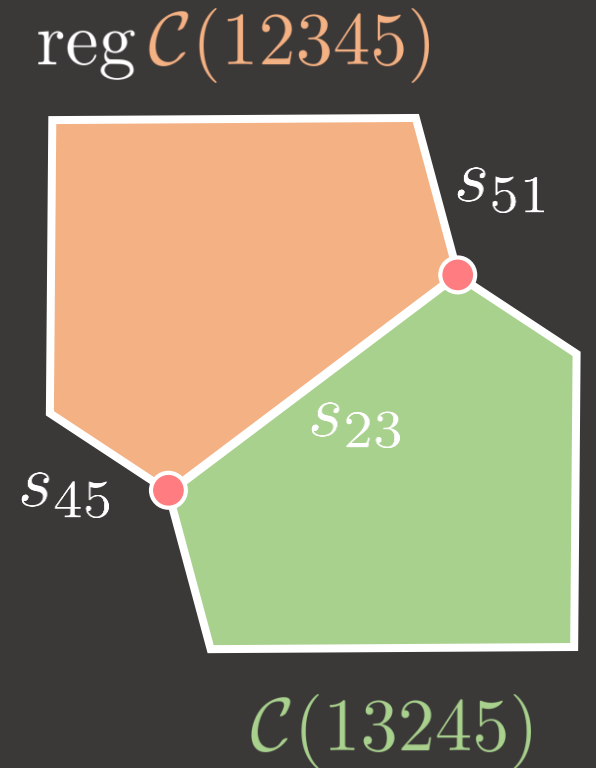


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For details see:

- “Combinatorics and Topology of Kawai–Lewellen–Tye Relations”, SM, [hep-th/1706.08527]
- “Inverse of the String Theory KLT Kernel”, SM, [hep-th/1610.04230]

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Inverse of the KLT kernel describes how different associahedra intersect one another in the moduli space

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This is the reason why bi-adjoint scalar amplitudes appear in the KLT relations

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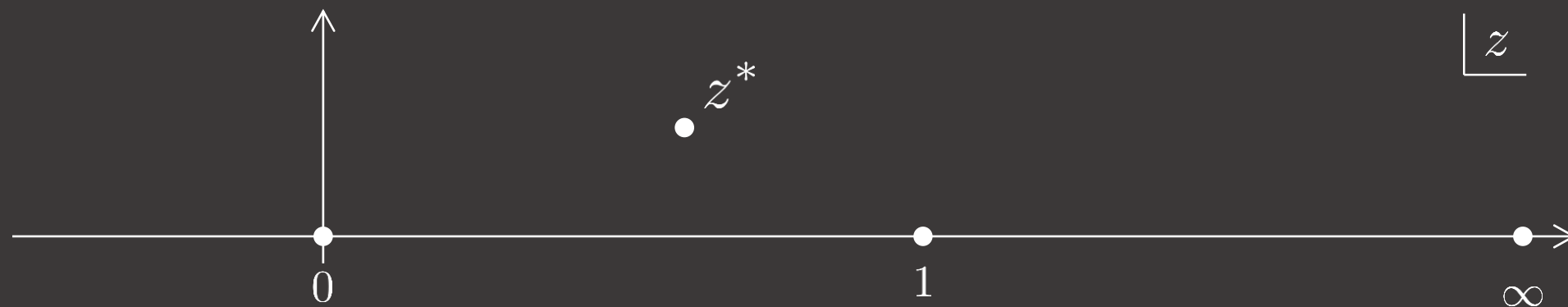
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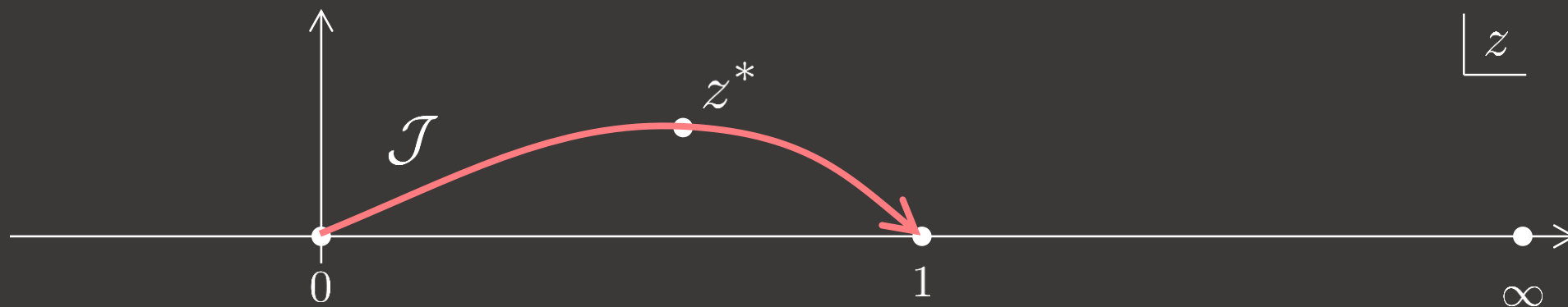
Here we illustrate what intersection theory has to say about this connection at the example of 4-pt massless amplitude

We use paths of *steepest descent*  $\mathcal{J}$  and *steepest ascent*  $\mathcal{K}$  intersecting at the saddle point of  $z^{\alpha's}(1-z)^{\alpha't}$  denoted by  $z^*$ :

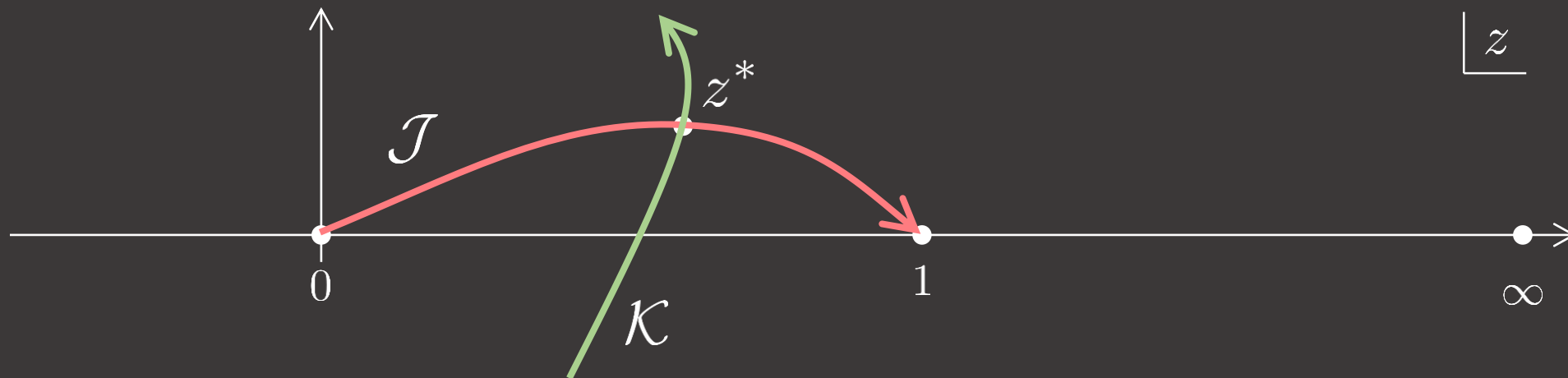
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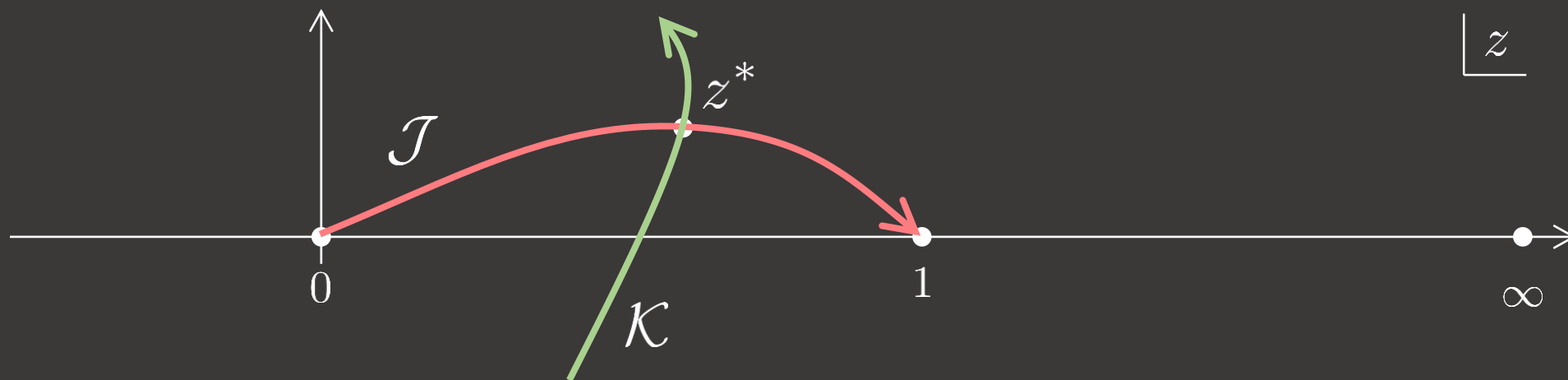
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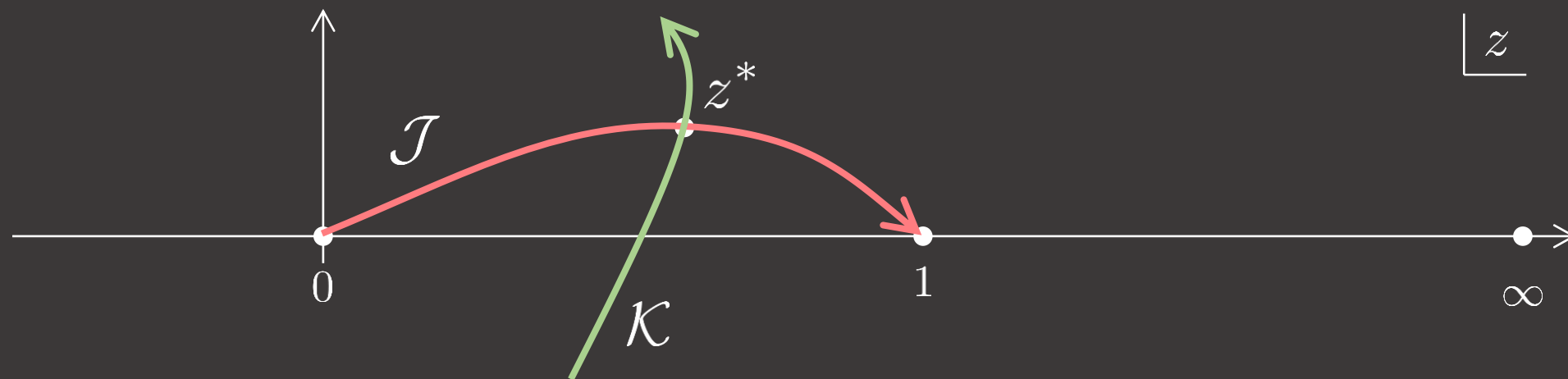
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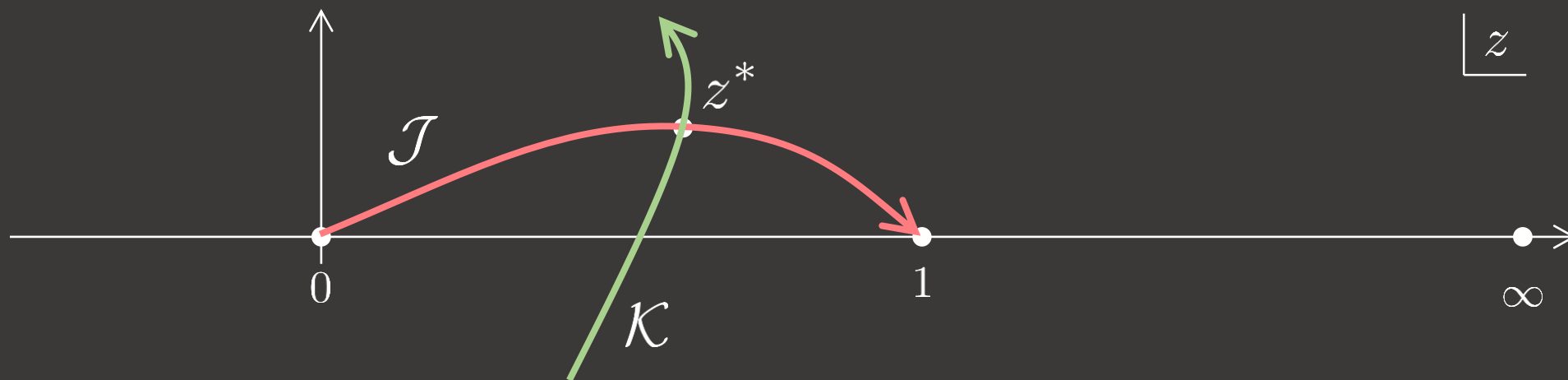


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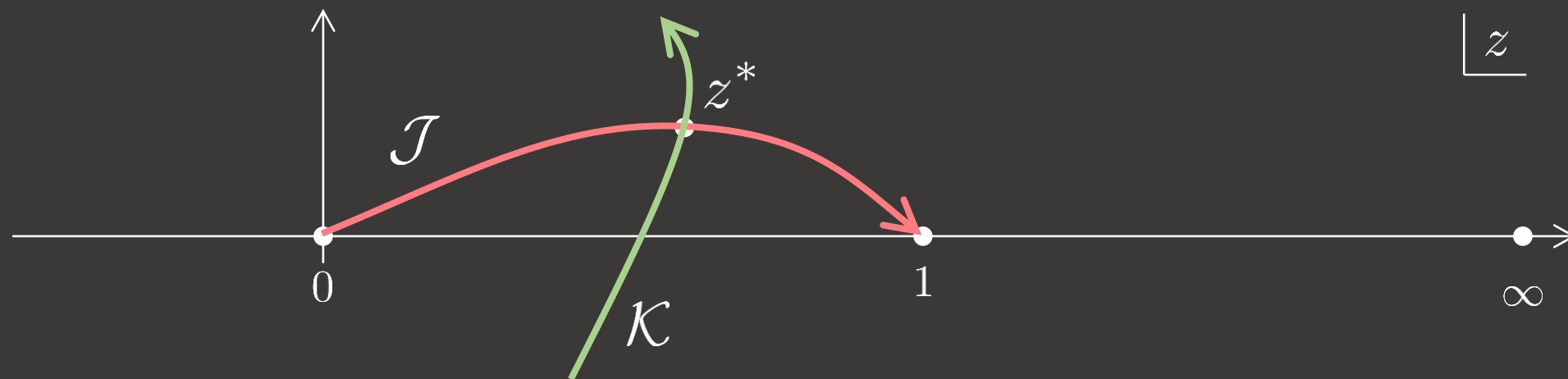
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$$\xrightarrow{\alpha' \rightarrow \infty} -\frac{2\pi i}{\alpha'} \frac{\varphi_L \varphi_R}{\frac{\partial}{\partial z} \left( \frac{s}{z} + \frac{t}{z-1} \right)} \Big|_{z=z^*}$$



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Since we're only interested in the leading term, we can invert one KN factor: [Aomoto, Deligne, Kita, ...; Siegel, Huang, Yuan, Casali, Tourkine, Azevedo, Jusinkas, ...]

$$\begin{aligned}
 -\frac{1}{\alpha'} \mathcal{A}_4^{\text{ft}} + \cancel{\dots} &= \underbrace{\left( \int_{\mathcal{J}} z^{\alpha's} (1-z)^{\alpha't} \varphi_L \right)}_{\text{most convergent}} \underbrace{\langle \mathcal{J}, \mathcal{K} \rangle^{-1}}_{+1} \underbrace{\left( \int_{\mathcal{K}} z^{-\alpha's} (1-z)^{-\alpha't} \varphi_R \right)}_{\text{most convergent}} \\
 \xrightarrow{\alpha' \rightarrow \infty} & -\frac{2\pi i}{\alpha'} \frac{\varphi_L \varphi_R}{\frac{\partial}{\partial z} \left( \frac{s}{z} + \frac{t}{z-1} \right)} \Big|_{z=z^*} = -\frac{2\pi i}{\alpha'} \underbrace{\int \delta \left( \frac{s}{z} + \frac{t}{z-1} \right) \varphi_L \varphi_R}_{\mathcal{A}_4^{\text{CHY}}}
 \end{aligned}$$

The result is exact in  $\alpha'$ , so  $\lim_{\alpha' \rightarrow 0} \mathcal{A}_4^{\text{closed}} = \frac{2\pi i}{\alpha'} \mathcal{A}_4^{\text{CHY}} + \dots$

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For details see:

- “*Scattering Amplitudes from Intersection Theory*”, SM, [hep-th/1711.00469]

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We've also seen evidence that intersection theory is a useful tool for the study of the connections between string theory amplitudes and CHY formulae

Thank you!