KLT Relations from Intersection Theory

Sebastian Mizera

Perimeter Institute

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

$$B(s,t)B(s,u) = -\frac{\pi}{s \sin \pi s} \frac{\Gamma(s)\Gamma(t)\Gamma(u)}{\Gamma(-s)\Gamma(-t)\Gamma(-u)}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

$$B(s,t)B(s,u) = -\frac{\pi}{s \sin \pi s} \frac{\Gamma(s)\Gamma(t)\Gamma(u)}{\Gamma(-s)\Gamma(-t)\Gamma(-u)}$$

How are such identities proved?

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

$$B(s,t)B(s,u) = -\frac{\pi}{s \sin \pi s} \frac{\Gamma(s)\Gamma(t)\Gamma(u)}{\Gamma(-s)\Gamma(-t)\Gamma(-u)}$$

How are such identities proved?

Mathematicians studying hypergeometric functions developed a unified formalism behind such identities, called *twisted de Rham theory*

The goal of today's talk is to
explain how the same mathematical tools
can be used to study KLT relations
and scattering amplitudes more generally

Objects of our interest are called *twisted cycles* and *twisted cocycles:*

[Aomoto, Cho, Kita, Matsumoto, Mimachi, Yoshida, ...]

Objects of our interest are called *twisted cycles* and *twisted cocycles:*

[Aomoto, Cho, Kita, Matsumoto, Mimachi, Yoshida, ...]

$$B(s,t) = \int_0^1 z^s (1-z)^t \, \frac{dz}{z(1-z)}$$

Objects of our interest are called *twisted cycles* and *twisted cocycles:*

[Aomoto, Cho, Kita, Matsumoto, Mimachi, Yoshida, ...]

$$B(s,t) = \int_0^1 \overbrace{z^s(1-z)^t}^{\text{multi-valued}} \frac{dz}{z(1-z)}$$

Objects of our interest are called *twisted cycles* and *twisted cocycles:*

[Aomoto, Cho, Kita, Matsumoto, Mimachi, Yoshida, ...]

$$B(s,t) = \int_0^1 \overbrace{z^s(1-z)^t}^{\text{multi-valued}} \overbrace{\frac{dz}{z(1-z)}}^{\text{single-valued}}$$

Objects of our interest are called *twisted cycles* and *twisted cocycles:*

[Aomoto, Cho, Kita, Matsumoto, Mimachi, Yoshida, ...]

$$B(s,t) = \underbrace{\int_0^1 \overbrace{z^s (1-z)^t}^{\text{multi-valued}}}_{\text{twisted cycle } \mathcal{C}} \overbrace{\frac{dz}{z(1-z)}}^{\text{single-valued}}$$

Objects of our interest are called *twisted cycles* and *twisted cocycles:*

[Aomoto, Cho, Kita, Matsumoto, Mimachi, Yoshida, ...]

$$B(s,t) = \underbrace{\int_0^1 \overbrace{z^s(1-z)^t}^{\text{multi-valued}}}_{\text{twisted cycle } \mathcal{C}} \; \underbrace{\overbrace{\frac{dz}{z(1-z)}}^{\text{single-valued}}}_{\text{twisted cocycle } \varphi \ (d\log \text{ form})}$$

Objects of our interest are called *twisted cycles* and *twisted cocycles:*

[Aomoto, Cho, Kita, Matsumoto, Mimachi, Yoshida, ...]

$$B(s,t) = \underbrace{\int_0^1 \overbrace{z^s(1-z)^t}^{\text{multi-valued}}}_{\text{twisted cycle } \mathcal{C}} \underbrace{\overbrace{\frac{dz}{z(1-z)}}^{\text{single-valued}}}_{\text{twisted cocycle } \varphi \text{ (}d\text{log form)}}$$

In this example: $\mathcal{C} = \overrightarrow{(0,1)} \otimes z^s(1-z)^t$

Objects of our interest are called *twisted cycles* and *twisted cocycles:*

[Aomoto, Cho, Kita, Matsumoto, Mimachi, Yoshida, ...]

$$B(s,t) = \underbrace{\int_0^1 \overbrace{z^s(1-z)^t}^{\text{multi-valued}}}_{\text{twisted cycle } \mathcal{C}} \underbrace{\overbrace{\frac{dz}{z(1-z)}}^{\text{single-valued}}}_{\text{twisted cocycle } \varphi \ (d\log \text{ form})}$$

In this example: $\mathcal{C} = \underbrace{\overrightarrow{(0,1)}}_{\text{topological cycle}} \otimes z^s(1-z)^t$

Objects of our interest are called *twisted cycles* and *twisted cocycles:*

[Aomoto, Cho, Kita, Matsumoto, Mimachi, Yoshida, ...]

$$B(s,t) = \underbrace{\int_0^1 \overbrace{z^s(1-z)^t}^{\text{multi-valued}}}_{\text{twisted cycle } \mathcal{C}} \underbrace{\overbrace{\frac{dz}{z(1-z)}}^{\text{single-valued}}}_{\text{twisted cocycle } \varphi \ (d\log \text{ form})}$$

In this example: $\mathcal{C} = \underbrace{\overrightarrow{(0,1)}}_{\text{topological cycle}} \otimes \overbrace{z^s(1-z)^t}^{\text{branch specification}}$

Objects of our interest are called *twisted cycles* and *twisted cocycles:*

[Aomoto, Cho, Kita, Matsumoto, Mimachi, Yoshida, ...]

$$B(s,t) = \underbrace{\int_0^1 \overbrace{z^s(1-z)^t}^{\text{multi-valued}}}_{\text{twisted cycle } \mathcal{C}} \underbrace{\overbrace{\frac{dz}{z(1-z)}}^{\text{single-valued}}}_{\text{twisted cocycle } \varphi \text{ (}d\text{log form)}}$$

In this example: $\mathcal{C} = \underbrace{\overrightarrow{(0,1)}}_{\text{topological cycle}} \otimes \overbrace{z^s(1-z)^t}^{\text{branch specification}} \neq \overrightarrow{(0,1)} \otimes z^s(z-1)^t$

Objects of our interest are called *twisted cycles* and *twisted cocycles:*

[Aomoto, Cho, Kita, Matsumoto, Mimachi, Yoshida, ...]

$$B(s,t) = \underbrace{\int_0^1 \overbrace{z^s(1-z)^t}^{\text{multi-valued}}}_{\text{twisted cycle } \mathcal{C}} \underbrace{\overbrace{\frac{dz}{z(1-z)}}^{\text{single-valued}}}_{\text{twisted cocycle } \varphi \ (d\log \text{ form})}$$

In this example: $\mathcal{C} = \underbrace{\overrightarrow{(0,1)}}_{\text{topological cycle}} \otimes \overbrace{z^s(1-z)^t}^{\text{branch specification}} \neq \overrightarrow{(0,1)} \otimes z^s(z-1)^t$

Monodromies are properties of the integration cycles, not integrands

Invariant pairings of twisted cycles and cocycles:

Invariant pairings of twisted cycles and cocycles:

$$\langle \mathcal{C}, \varphi \rangle = \int_0^1 z^s (1-z)^t \, \varphi$$

Invariant pairings of twisted cycles and cocycles:

$$\langle \mathcal{C}, \varphi \rangle = \int_0^1 z^s (1-z)^t \, \varphi$$

$$\langle \varphi, \widetilde{\varphi} \rangle = \int_{\mathbb{C}} |z|^{2s} |1-z|^{2t} \, \varphi \wedge \widetilde{\varphi}$$

Invariant pairings of twisted cycles and cocycles:

$$\langle \mathcal{C}, \varphi \rangle = \int_0^1 z^s (1-z)^t \, \varphi$$

$$\langle \varphi, \widetilde{\varphi} \rangle = \int_{\mathbb{C}} |z|^{2s} |1-z|^{2t} \, \varphi \wedge \widetilde{\varphi}$$

$$\langle \mathcal{C}, \widetilde{\mathcal{C}} \rangle = \text{intersection number}$$

Invariant pairings of twisted cycles and cocycles:

$$\langle \mathcal{C}, \varphi \rangle = \int_0^1 z^s (1-z)^t \, \varphi$$

$$\langle \varphi, \widetilde{\varphi} \rangle = \int_{\mathbb{C}} |z|^{2s} |1-z|^{2t} \, \varphi \wedge \widetilde{\varphi}$$

$$\langle \mathcal{C}, \widetilde{\mathcal{C}} \rangle = \text{intersection number}$$

Connected by *twisted period relations*: [Cho & Matsumoto '94]

Invariant pairings of twisted cycles and cocycles:

$$\langle \mathcal{C}, \varphi \rangle = \int_0^1 z^s (1-z)^t \, \varphi$$

$$\langle \varphi, \widetilde{\varphi} \rangle = \int_{\mathbb{C}} |z|^{2s} |1-z|^{2t} \, \varphi \wedge \widetilde{\varphi}$$

$$\langle \mathcal{C}, \widetilde{\mathcal{C}} \rangle = \text{intersection number}$$

Connected by *twisted period relations*: [Cho & Matsumoto '94]

$$\langle \varphi, \widetilde{\varphi} \rangle = \langle \varphi, \mathcal{C} \rangle \, \langle \mathcal{C}, \widetilde{\mathcal{C}} \rangle^{-1} \langle \widetilde{\mathcal{C}}, \widetilde{\varphi} \rangle$$

Invariant pairings of twisted cycles and cocycles:

$$\langle \mathcal{C}, \varphi \rangle = \int_0^1 z^s (1-z)^t \, \varphi \qquad \rightsquigarrow \mathcal{A}^{\text{open string}}$$

$$\langle \varphi, \widetilde{\varphi} \rangle = \int_{\mathbb{C}} |z|^{2s} |1-z|^{2t} \, \varphi \wedge \widetilde{\varphi}$$

$$\langle \mathcal{C}, \widetilde{\mathcal{C}} \rangle = \text{intersection number}$$

Connected by *twisted period relations*: [Cho & Matsumoto '94]

$$\langle \varphi, \widetilde{\varphi} \rangle = \langle \varphi, \mathcal{C} \rangle \, \langle \mathcal{C}, \widetilde{\mathcal{C}} \rangle^{-1} \langle \widetilde{\mathcal{C}}, \widetilde{\varphi} \rangle$$

Invariant pairings of twisted cycles and cocycles:

$$\langle \mathcal{C}, \varphi \rangle = \int_0^1 z^s (1-z)^t \, \varphi \qquad \rightsquigarrow \mathcal{A}^{\text{open string}}$$

$$\langle \varphi, \widetilde{\varphi} \rangle = \int_{\mathbb{C}} |z|^{2s} |1-z|^{2t} \, \varphi \wedge \widetilde{\varphi} \qquad \rightsquigarrow \mathcal{A}^{\text{closed string}}$$

$$\langle \mathcal{C}, \widetilde{\mathcal{C}} \rangle = \text{intersection number}$$

Connected by *twisted period relations*: [Cho & Matsumoto '94]

$$\langle \varphi, \widetilde{\varphi} \rangle = \langle \varphi, \mathcal{C} \rangle \, \langle \mathcal{C}, \widetilde{\mathcal{C}} \rangle^{-1} \langle \widetilde{\mathcal{C}}, \widetilde{\varphi} \rangle$$

Invariant pairings of twisted cycles and cocycles:

$$\langle \mathcal{C}, \varphi \rangle = \int_0^1 z^s (1-z)^t \, \varphi \qquad \rightsquigarrow \mathcal{A}^{\text{open string}}$$

$$\langle \varphi, \widetilde{\varphi} \rangle = \int_{\mathbb{C}} |z|^{2s} |1-z|^{2t} \, \varphi \wedge \widetilde{\varphi} \qquad \rightsquigarrow \mathcal{A}^{\text{closed string}}$$

$$\langle \mathcal{C}, \widetilde{\mathcal{C}} \rangle = \text{intersection number} \qquad \rightsquigarrow \text{let's see}$$

Connected by *twisted period relations*: [Cho & Matsumoto '94]

$$\langle \varphi, \widetilde{\varphi} \rangle = \langle \varphi, \mathcal{C} \rangle \, \langle \mathcal{C}, \widetilde{\mathcal{C}} \rangle^{-1} \langle \widetilde{\mathcal{C}}, \widetilde{\varphi} \rangle$$

Intersection numbers of cycles:

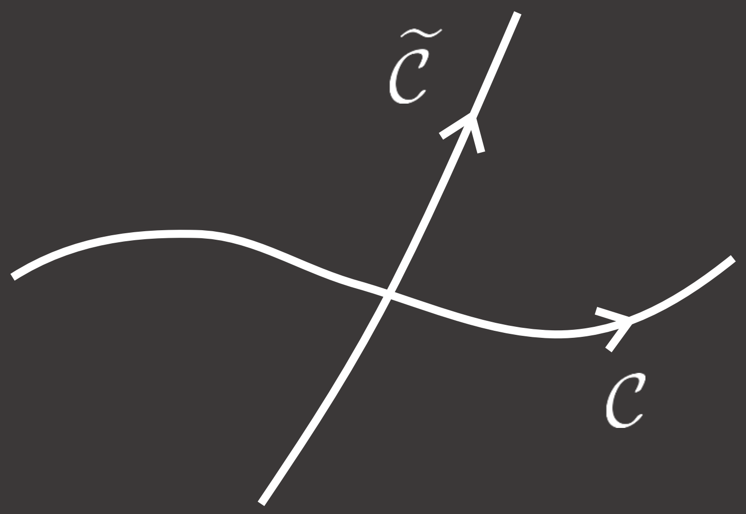
Intersection numbers of cycles:

$$\langle \mathcal{C}, \widetilde{\mathcal{C}} \rangle =$$

Intersection numbers of cycles:

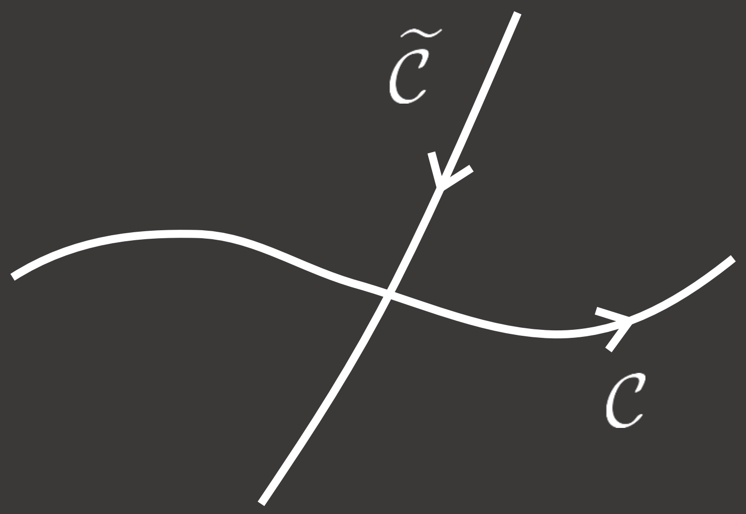
$$\langle \mathcal{C}, \widetilde{\mathcal{C}} \rangle =$$

Intersection numbers of cycles:

$\langle \mathcal{C}, \widetilde{\mathcal{C}} \rangle =$  $= +1$

Intersection numbers of cycles:

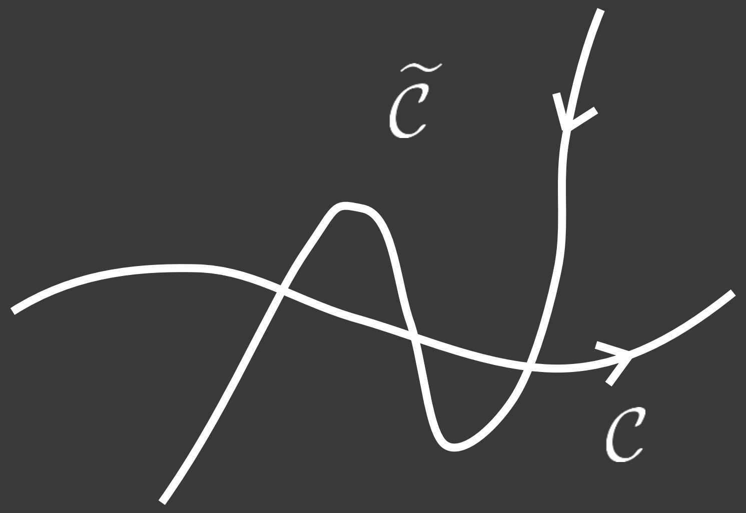
$$\langle \mathcal{C}, \widetilde{\mathcal{C}} \rangle =$$



$$= -1$$

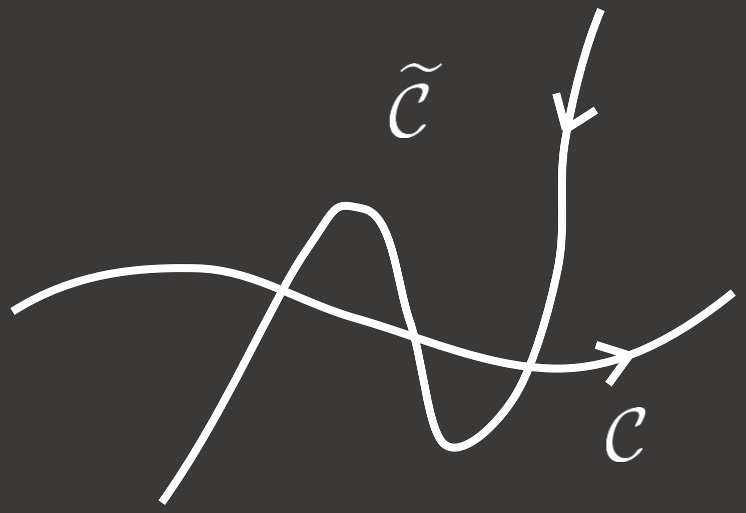
Intersection numbers of cycles:

$\langle \mathcal{C}, \widetilde{\mathcal{C}} \rangle =$



$= -1$

Intersection numbers of cycles:

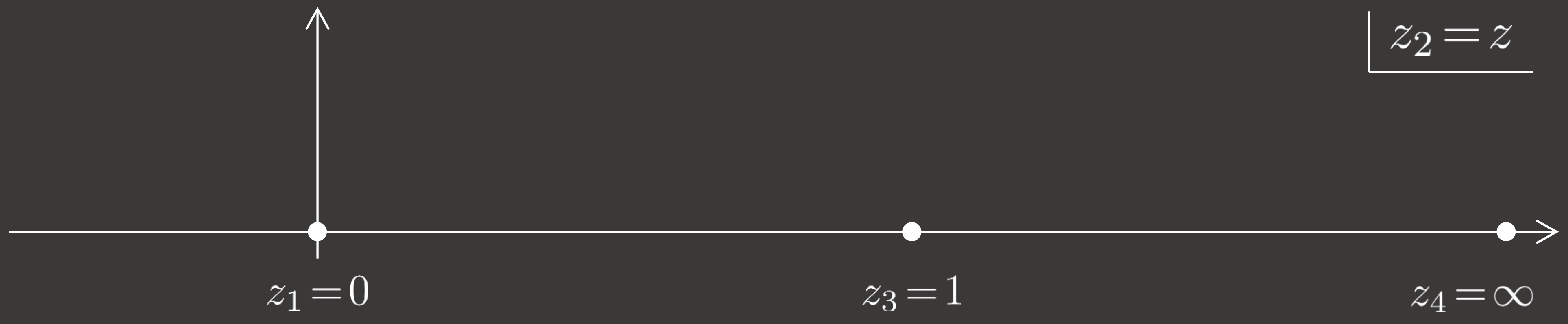
$\langle \mathcal{C}, \widetilde{\mathcal{C}} \rangle =$  $= -1$

In general, intersection numbers of *twisted* cycles will be non-integer once monodromies are taken into account

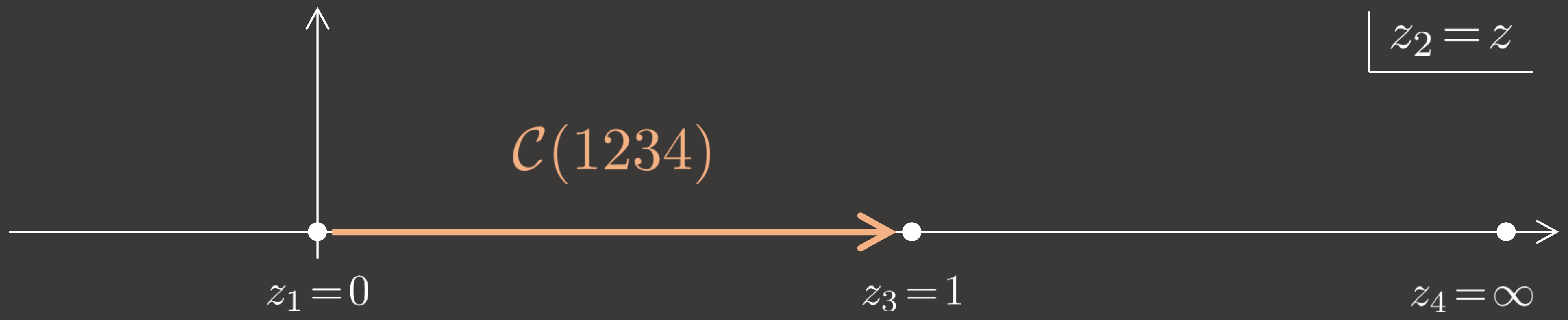
[Kita & Yoshida '92]

String theory 4-pt twisted cycles in $\mathcal{M}_{0,4} \simeq \mathbb{CP}^1 \setminus \{0, 1, \infty\}$:

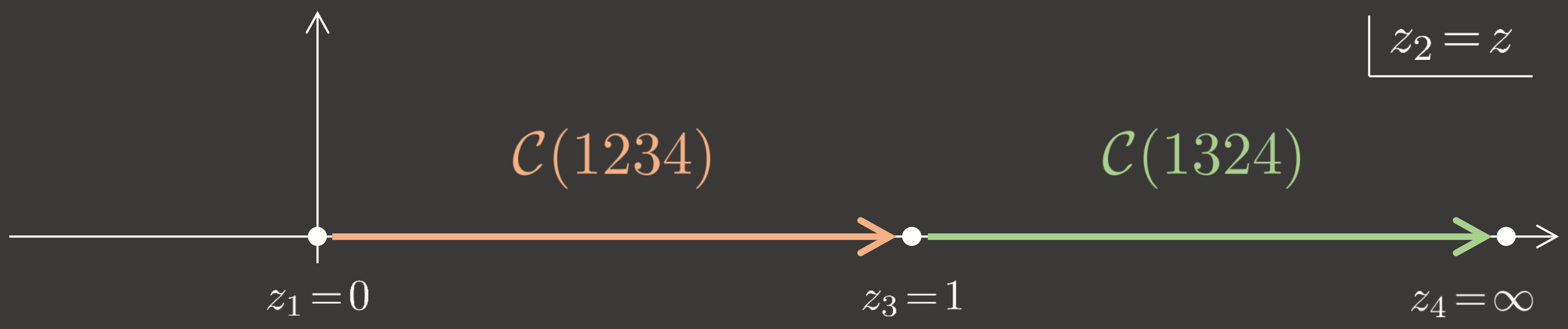
String theory 4-pt twisted cycles in $\mathcal{M}_{0,4} \simeq \mathbb{CP}^1 \setminus \{0, 1, \infty\}$:



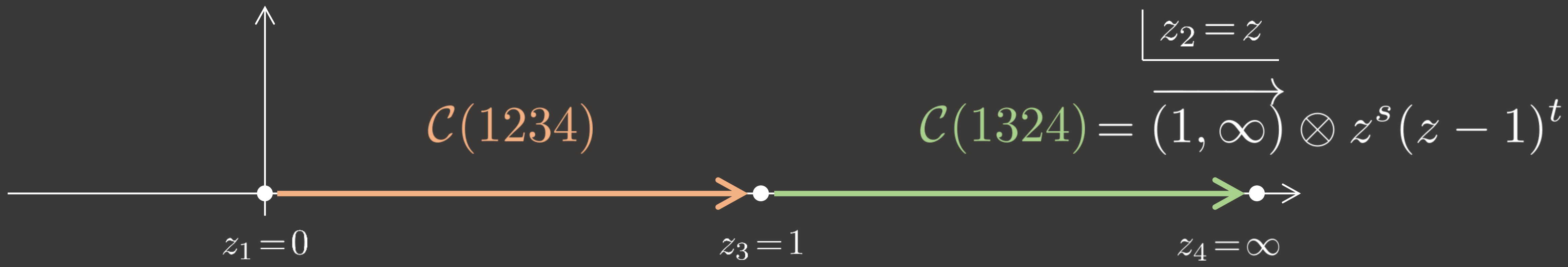
String theory 4-pt twisted cycles in $\mathcal{M}_{0,4} \simeq \mathbb{CP}^1 \setminus \{0, 1, \infty\}$:



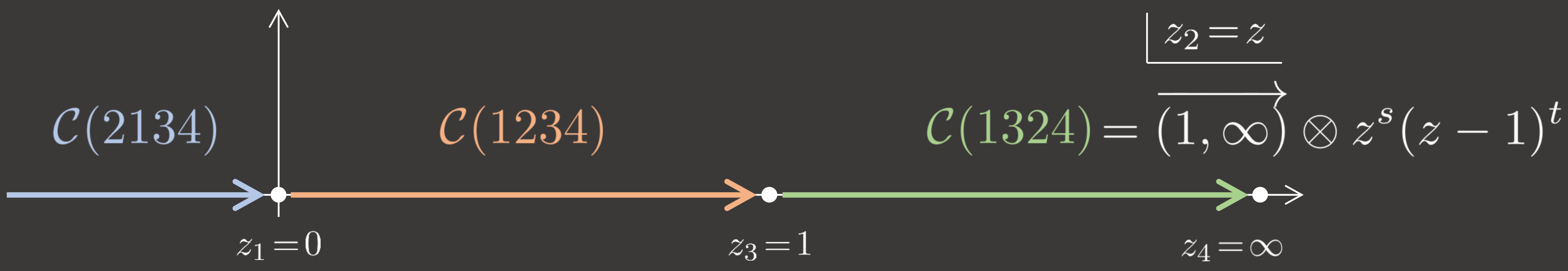
String theory 4-pt twisted cycles in $\mathcal{M}_{0,4} \simeq \mathbb{CP}^1 \setminus \{0, 1, \infty\}$:



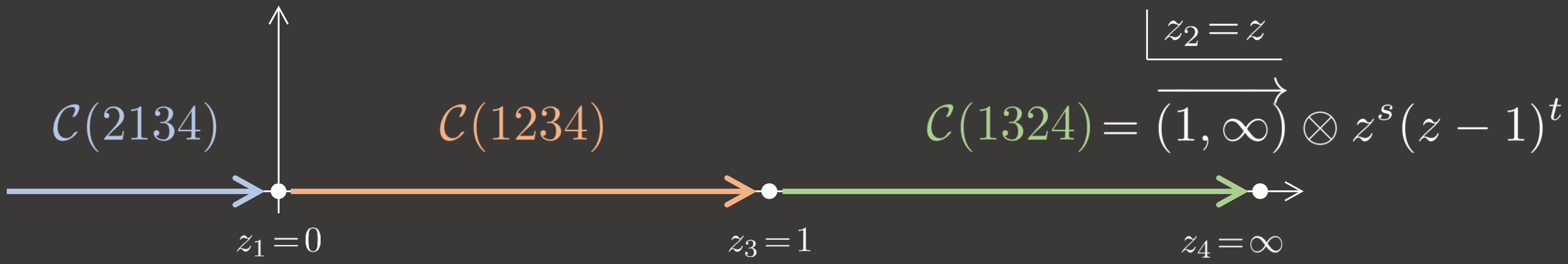
String theory 4-pt twisted cycles in $\mathcal{M}_{0,4} \simeq \mathbb{CP}^1 \setminus \{0, 1, \infty\}$:



String theory 4-pt twisted cycles in $\mathcal{M}_{0,4} \simeq \mathbb{CP}^1 \setminus \{0, 1, \infty\}$:

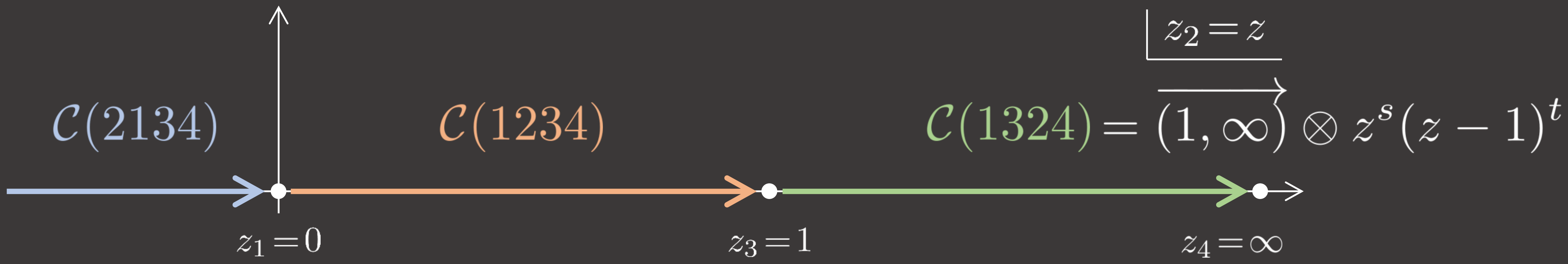


String theory 4-pt twisted cycles in $\mathcal{M}_{0,4} \simeq \mathbb{CP}^1 \setminus \{0,1,\infty\}$:



Intersections are ill-defined—need to regularize twisted cycles using the Pochhammer contour:

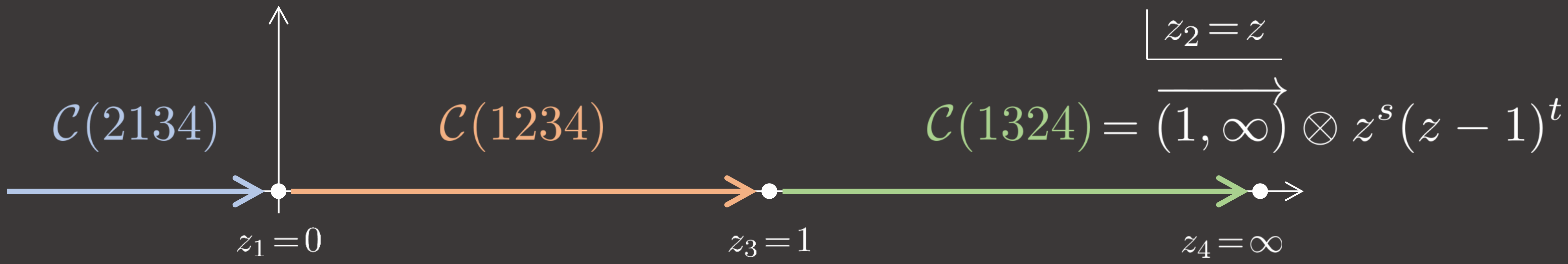
String theory 4-pt twisted cycles in $\mathcal{M}_{0,4} \simeq \mathbb{CP}^1 \setminus \{0, 1, \infty\}$:



Intersections are ill-defined—need to regularize twisted cycles using the Pochhammer contour:

reg $\mathcal{C}(1234)$ =

String theory 4-pt twisted cycles in $\mathcal{M}_{0,4} \simeq \mathbb{CP}^1 \setminus \{0, 1, \infty\}$:



Intersections are ill-defined—need to regularize twisted cycles using the Pochhammer contour:

$\operatorname{reg} \mathcal{C}(1234) =$

Factors a & b fixed by consistency condition that $\partial \, \mathrm{reg} \, \mathcal{C}(1234) = \varnothing$:

Factors a & b fixed by consistency condition that $\partial\,\mathrm{reg}\,\mathcal{C}(1234) = \varnothing$:

Factors a & b fixed by consistency condition that $\partial\,\mathrm{reg}\,\mathcal{C}(1234) = \varnothing$:

$$= \{\varepsilon\}\Big(\qquad \Big) + \{1-\varepsilon\}\Big(\qquad \Big)$$

Factors a & b fixed by consistency condition that $\partial\,\mathrm{reg}\,\mathcal{C}(1234) = \varnothing$:

$$= \{\varepsilon\}\Big(-1 \qquad\qquad\qquad \Big)$$

$$+\{1-\varepsilon\}\Big(+1 \qquad\qquad\qquad \Big)$$

Factors a & b fixed by consistency condition that $\partial\,\mathrm{reg}\,\mathcal{C}(1234) = \varnothing$:

$\partial\Big($

$\Big)$

$$= \{\varepsilon\}\left(-1 - a + a e^{2\pi i s}\right)$$

$$+ \{1-\varepsilon\}\left(+1 \qquad\qquad\qquad\right)$$

Factors a & b fixed by consistency condition that $\partial\,\mathrm{reg}\,\mathcal{C}(1234) = \varnothing$:

$$= \{\varepsilon\}\left(-1-a+ae^{2\pi i s}\right)$$

$$+\{1-\varepsilon\}\left(+1-b+be^{2\pi i t}\right)$$

Factors a & b fixed by consistency condition that $\partial\,\mathrm{reg}\,\mathcal{C}(1234) = \varnothing$:

$\partial\Big($ 

$\Big)$

$$= \{\varepsilon\}\left(-1 - a + a e^{2\pi i s}\right) + \{1-\varepsilon\}\left(+1 - b + b e^{2\pi i t}\right) = \varnothing$$

Factors a & b fixed by consistency condition that $\partial\,\mathrm{reg}\,\mathcal{C}(1234) = \varnothing$:

∂ 

$$= \{\varepsilon\}\left(-1 - a + a e^{2\pi i s}\right)$$

$$+\{1-\varepsilon\}\left(+1 - b + b e^{2\pi i t}\right) = \varnothing$$

$$\Longrightarrow \quad a = \frac{1}{e^{2\pi i s} - 1}, \qquad b = -\frac{1}{e^{2\pi i t} - 1}$$

Factors a & b fixed by consistency condition that $\partial\,\mathrm{reg}\,\mathcal{C}(1234) = \varnothing$:

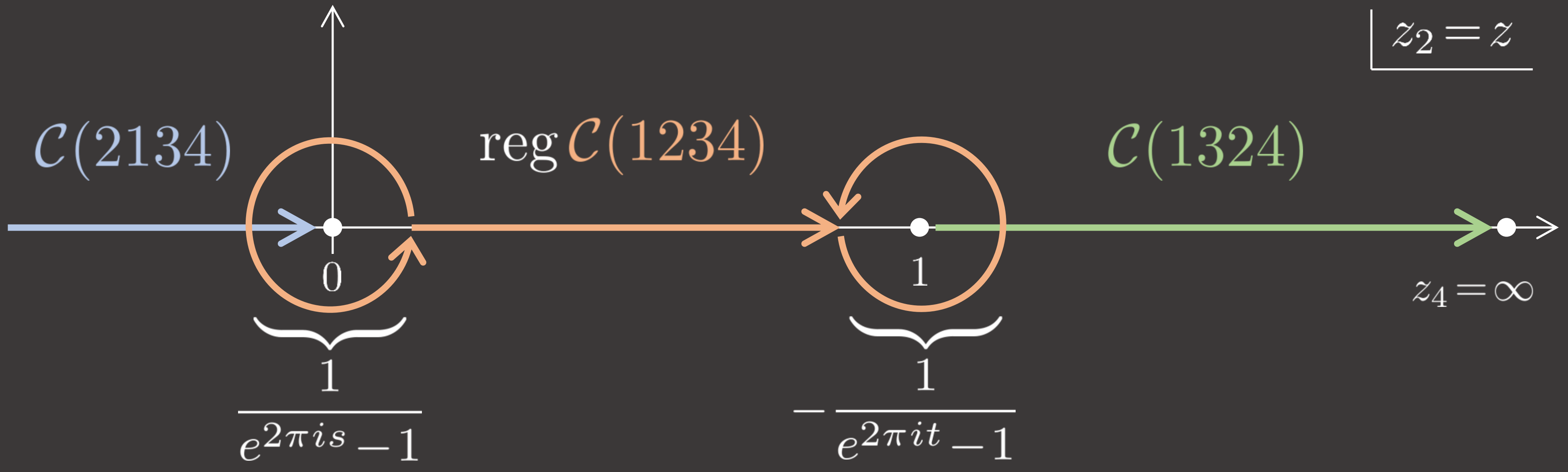
$$\partial\left(\cdots \right) = \{\varepsilon\}\left(-1-a+ae^{2\pi i s}\right)$$
$$+\{1-\varepsilon\}\left(+1-b+be^{2\pi i t}\right) = \varnothing$$

$$\implies a = \frac{1}{e^{2\pi i s}-1}, \quad b = -\frac{1}{e^{2\pi i t}-1}$$

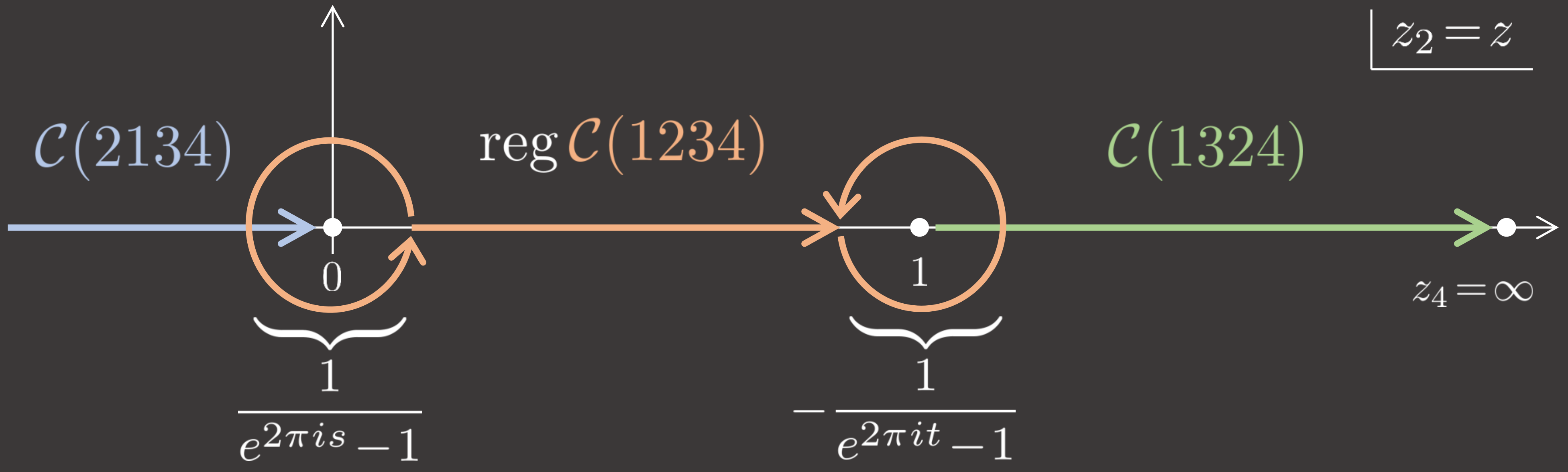
Manifests factorization channels corresponding to boundaries of the moduli space

We can now compute intersection numbers:

We can now compute intersection numbers:

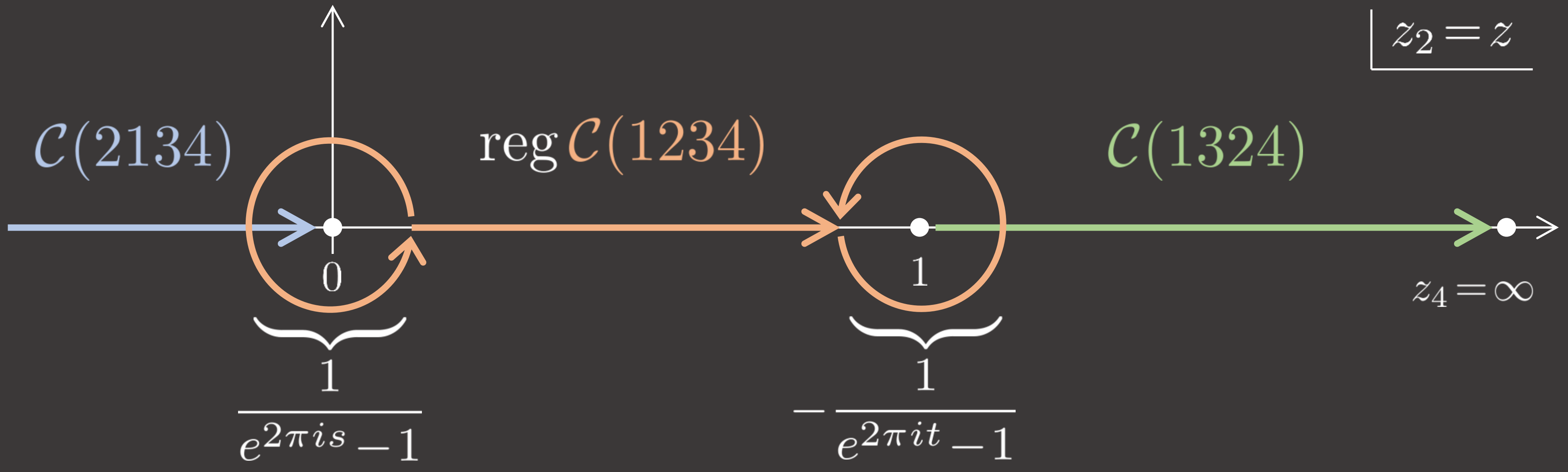


We can now compute intersection numbers:



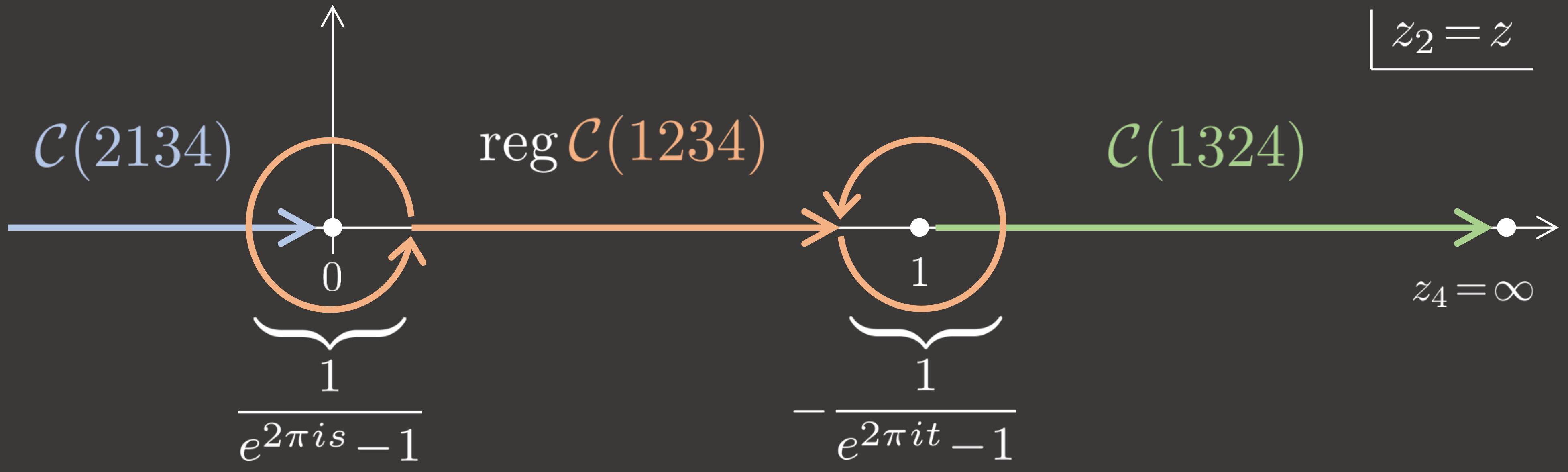
$$\langle \mathrm{reg}\, \mathcal{C}(1234), \mathcal{C}(2134) \rangle =$$

We can now compute intersection numbers:



$$\langle \mathrm{reg}\, \mathcal{C}(1234), \mathcal{C}(2134) \rangle = \frac{+e^{\pi i s}}{e^{2\pi i s} - 1}$$

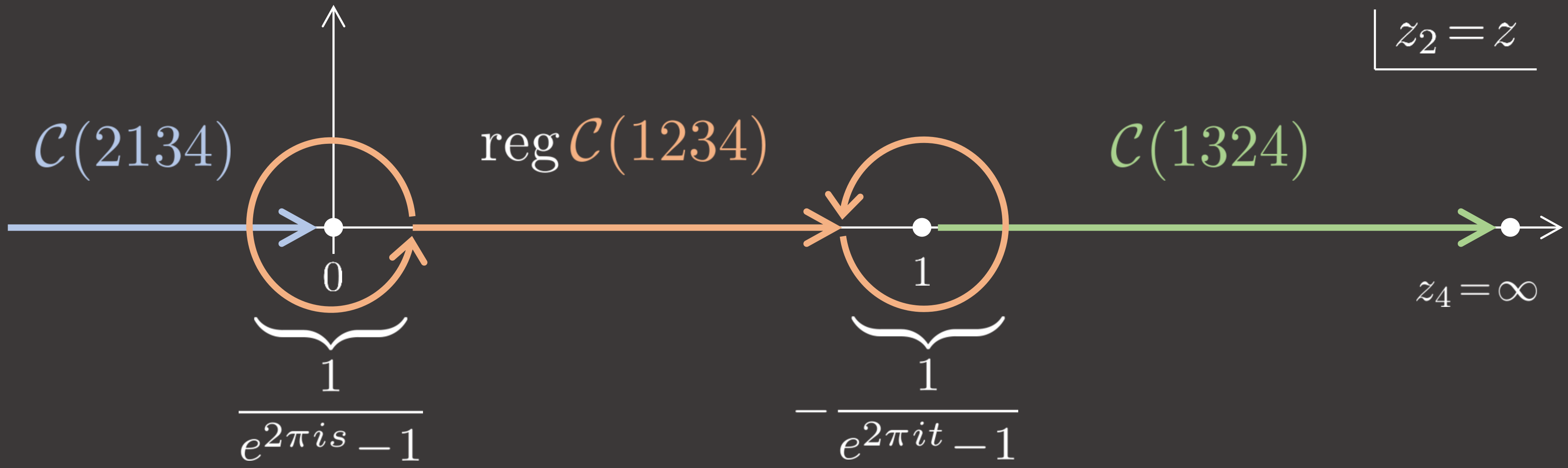
We can now compute intersection numbers:



$$\langle \mathrm{reg}\,\mathcal{C}(1234), \mathcal{C}(2134) \rangle = \frac{+e^{\pi i s}}{e^{2\pi i s} - 1}$$

$$\langle \mathrm{reg}\,\mathcal{C}(1234), \mathcal{C}(1324) \rangle =$$

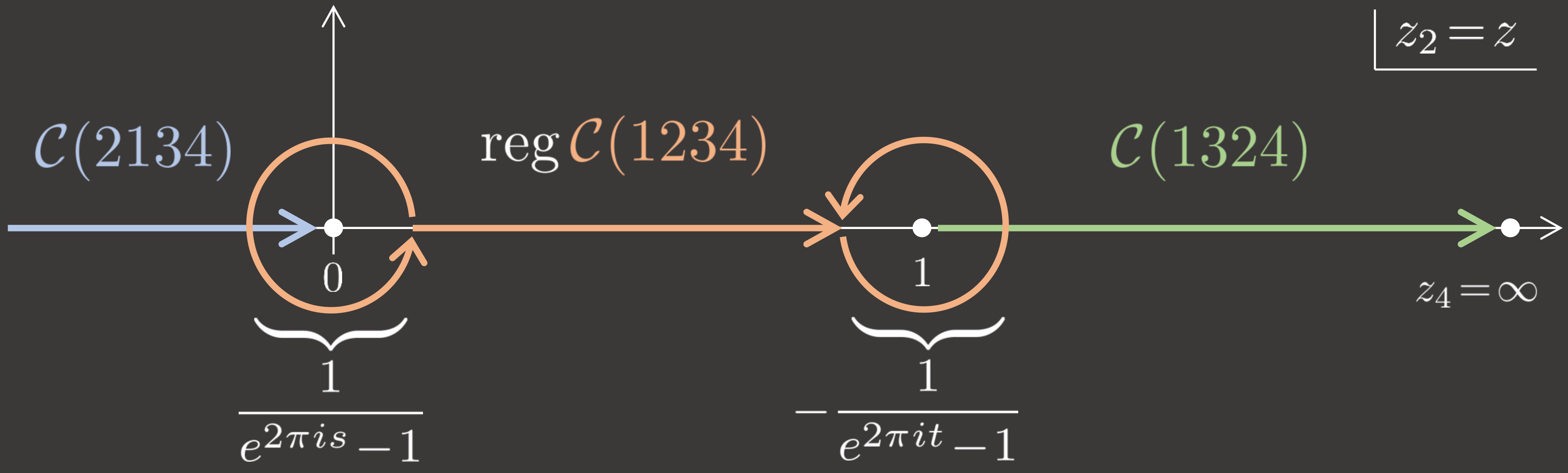
We can now compute intersection numbers:



$$\langle \operatorname{reg}\mathcal{C}(1234), \mathcal{C}(2134)\rangle = \frac{+e^{\pi i s}}{e^{2\pi i s}-1}$$

$$\langle \operatorname{reg}\mathcal{C}(1234), \mathcal{C}(1324)\rangle = -\frac{-e^{\pi i t}}{e^{2\pi i t}-1}$$

We can now compute intersection numbers:

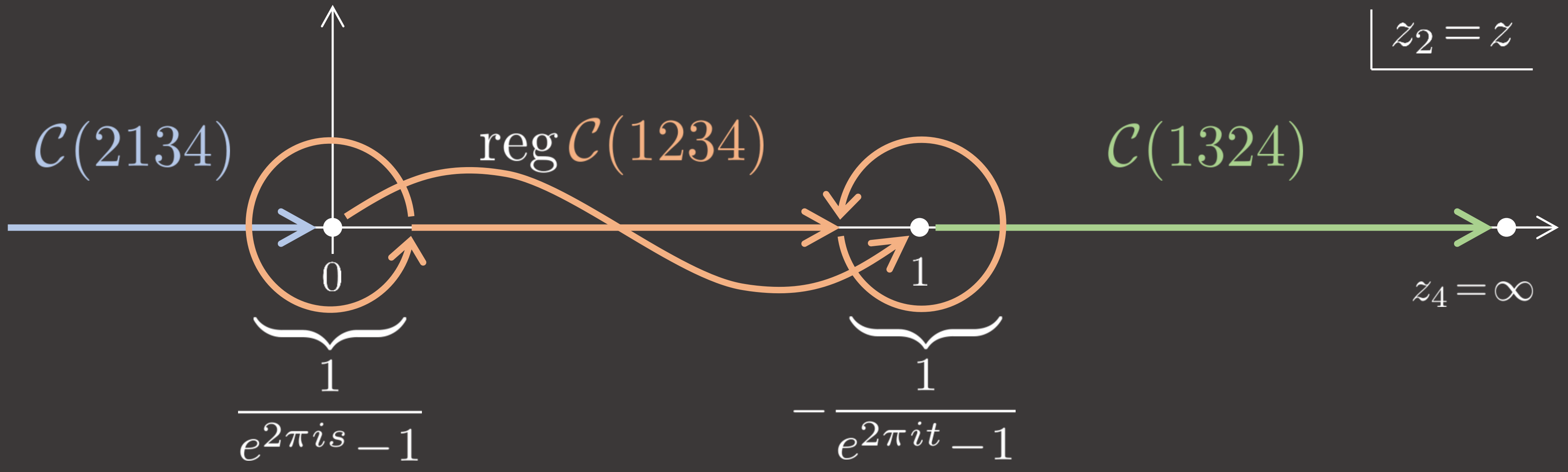


$$\langle \text{reg}\,\mathcal{C}(1234), \mathcal{C}(2134) \rangle = \frac{+e^{\pi i s}}{e^{2\pi i s} - 1}$$

$$\langle \text{reg}\,\mathcal{C}(1234), \mathcal{C}(1324) \rangle = -\frac{-e^{\pi i t}}{e^{2\pi i t} - 1}$$

$$\langle \text{reg}\,\mathcal{C}(1234), \mathcal{C}(1234) \rangle =$$

We can now compute intersection numbers:

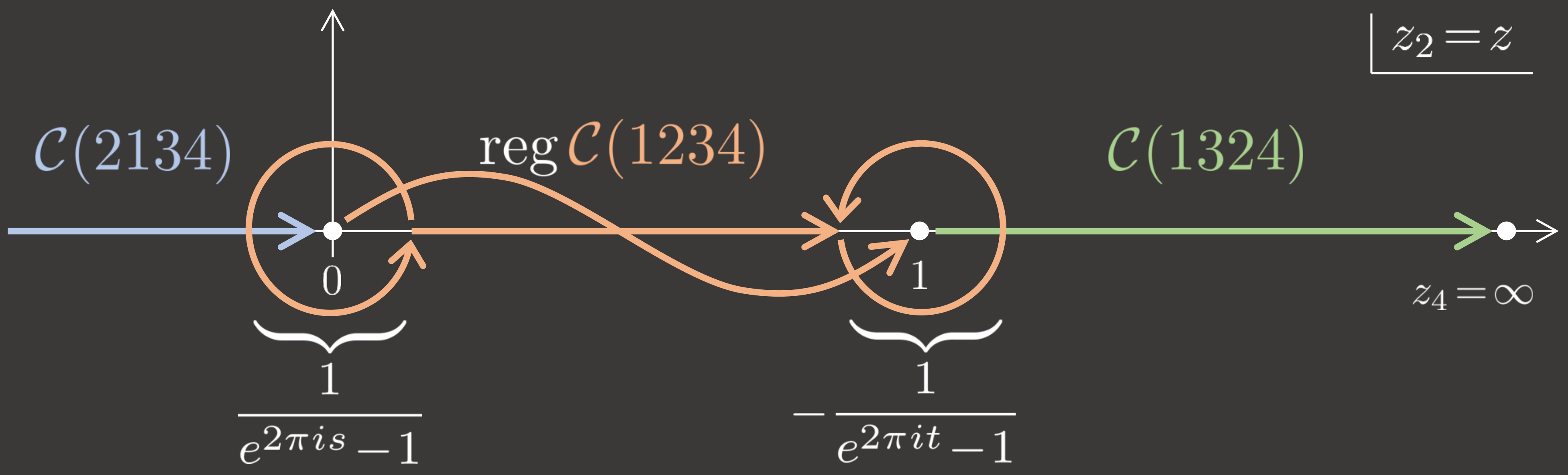


$$\langle \mathrm{reg}\,\mathcal{C}(1234), \mathcal{C}(2134)\rangle = \frac{+e^{\pi i s}}{e^{2\pi i s}-1}$$

$$\langle \mathrm{reg}\,\mathcal{C}(1234), \mathcal{C}(1324)\rangle = -\frac{-e^{\pi i t}}{e^{2\pi i t}-1}$$

$$\langle \mathrm{reg}\,\mathcal{C}(1234), \mathcal{C}(1234)\rangle =$$

We can now compute intersection numbers:

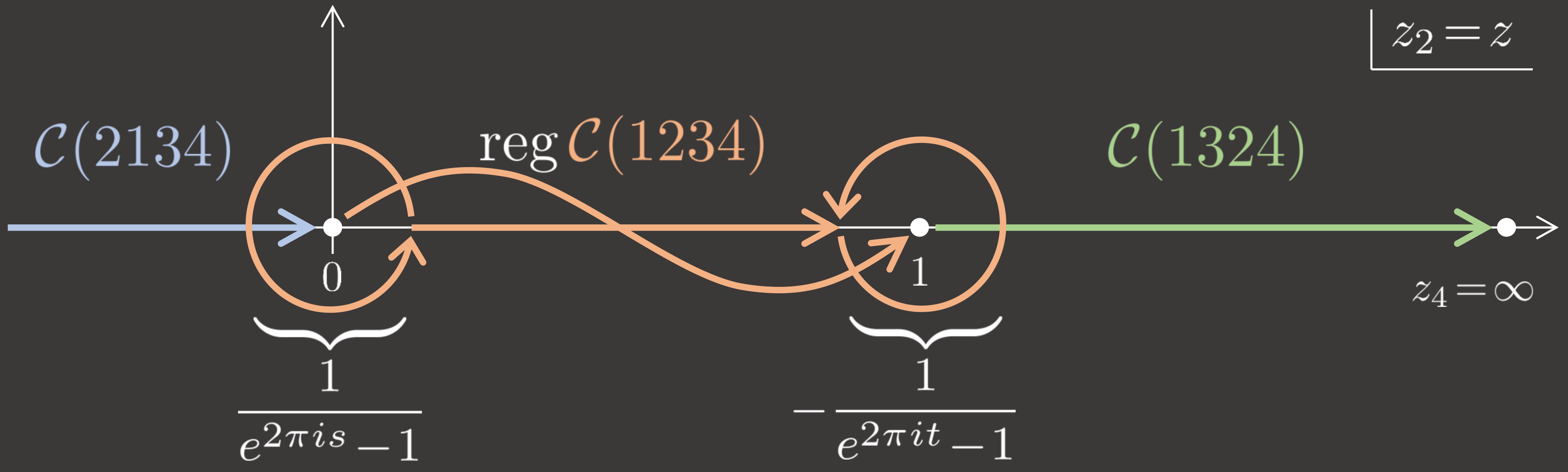


$$\langle \mathrm{reg}\,\mathcal{C}(1234), \mathcal{C}(2134)\rangle = \frac{+e^{\pi i s}}{e^{2\pi i s}-1}$$

$$\langle \mathrm{reg}\,\mathcal{C}(1234), \mathcal{C}(1324)\rangle = -\frac{-e^{\pi i t}}{e^{2\pi i t}-1}$$

$$\langle \mathrm{reg}\,\mathcal{C}(1234), \mathcal{C}(1234)\rangle = \frac{-1}{e^{2\pi i s}-1} - 1 - \frac{+1}{e^{2\pi i t}-1}$$

We can now compute intersection numbers:

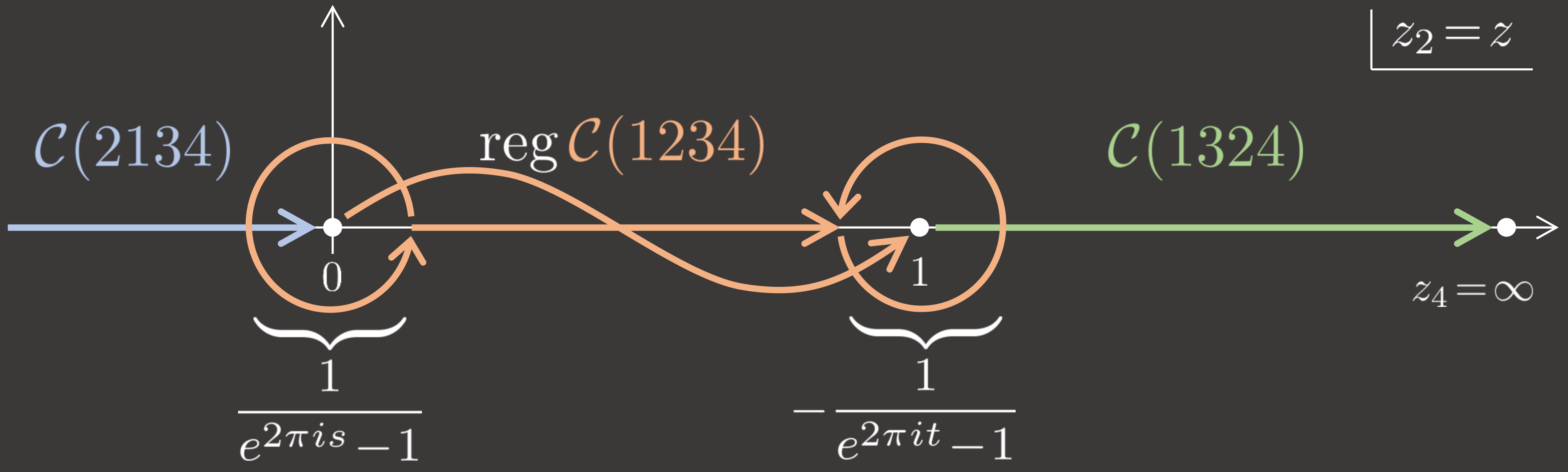


$$\langle \operatorname{reg} \mathcal{C}(1234), \mathcal{C}(2134) \rangle = \frac{+e^{\pi i s}}{e^{2\pi i s} - 1} \xrightarrow{\text{field-theory}} \frac{i}{2\pi}\left(-\frac{1}{s}\right)$$

$$\langle \operatorname{reg} \mathcal{C}(1234), \mathcal{C}(1324) \rangle = -\frac{-e^{\pi i t}}{e^{2\pi i t} - 1}$$

$$\langle \operatorname{reg} \mathcal{C}(1234), \mathcal{C}(1234) \rangle = \frac{-1}{e^{2\pi i s} - 1} - 1 - \frac{+1}{e^{2\pi i t} - 1}$$

We can now compute intersection numbers:

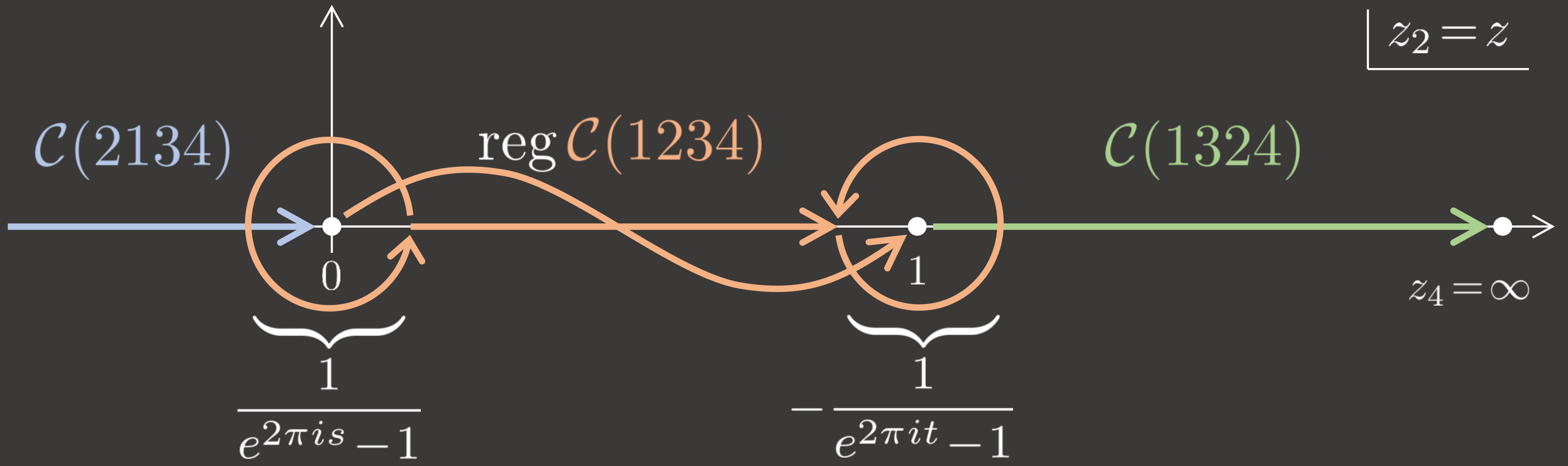


$$\langle \operatorname{reg}\mathcal{C}(1234), \mathcal{C}(2134)\rangle = \frac{+e^{\pi i s}}{e^{2\pi i s}-1} \xrightarrow{\text{field-theory}} \frac{i}{2\pi}\left(-\frac{1}{s}\right)$$

$$\langle \operatorname{reg}\mathcal{C}(1234), \mathcal{C}(1324)\rangle = -\frac{-e^{\pi i t}}{e^{2\pi i t}-1} \xrightarrow{\text{field-theory}} \frac{i}{2\pi}\left(-\frac{1}{t}\right)$$

$$\langle \operatorname{reg}\mathcal{C}(1234), \mathcal{C}(1234)\rangle = \frac{-1}{e^{2\pi i s}-1} - 1 - \frac{+1}{e^{2\pi i t}-1}$$

We can now compute intersection numbers:



$$\langle \mathrm{reg}\,\mathcal{C}(1234), \mathcal{C}(2134)\rangle = \frac{+e^{\pi i s}}{e^{2\pi i s}-1} \xrightarrow{\text{field-theory}} \frac{i}{2\pi}\left(-\frac{1}{s}\right)$$

$$\langle \mathrm{reg}\,\mathcal{C}(1234), \mathcal{C}(1324)\rangle = -\frac{-e^{\pi i t}}{e^{2\pi i t}-1} \xrightarrow{\text{field-theory}} \frac{i}{2\pi}\left(-\frac{1}{t}\right)$$

$$\langle \mathrm{reg}\,\mathcal{C}(1234), \mathcal{C}(1234)\rangle = \frac{-1}{e^{2\pi i s}-1} - 1 - \frac{+1}{e^{2\pi i t}-1} \xrightarrow{\text{field-theory}} \frac{i}{2\pi}\left(\frac{1}{s}+\frac{1}{t}\right)$$

Twisted period relations: $\langle \varphi, \widetilde{\varphi} \rangle = \langle \varphi, \mathcal{C} \rangle \, \langle \mathcal{C}, \widetilde{\mathcal{C}} \rangle^{-1} \langle \widetilde{\mathcal{C}}, \widetilde{\varphi} \rangle$

Twisted period relations: $\langle \varphi, \widetilde{\varphi} \rangle = \langle \varphi, \mathcal{C} \rangle \, \langle \mathcal{C}, \widetilde{\mathcal{C}} \rangle^{-1} \langle \widetilde{\mathcal{C}}, \widetilde{\varphi} \rangle$

Choosing $\mathcal{C} = \mathcal{C}(1234)$ & $\widetilde{\mathcal{C}} = \mathcal{C}(2134)$ gives:

Twisted period relations: $\langle \varphi, \widetilde{\varphi} \rangle = \langle \varphi, \mathcal{C} \rangle \, \langle \mathcal{C}, \widetilde{\mathcal{C}} \rangle^{-1} \langle \widetilde{\mathcal{C}}, \widetilde{\varphi} \rangle$

Choosing $\mathcal{C} = \mathcal{C}(1234)$ & $\widetilde{\mathcal{C}} = \mathcal{C}(2134)$ gives:

$$\mathcal{A}_4^{\text{closed}} = \mathcal{A}^{\text{open}}(1234) \left(\frac{e^{\pi i s}}{e^{2\pi i s} - 1} \right)^{-1} \mathcal{A}^{\text{open}}(2134)$$

Twisted period relations: $\langle \varphi, \widetilde{\varphi} \rangle = \langle \varphi, \mathcal{C} \rangle \, \langle \mathcal{C}, \widetilde{\mathcal{C}} \rangle^{-1} \langle \widetilde{\mathcal{C}}, \widetilde{\varphi} \rangle$

Choosing $\mathcal{C} = \mathcal{C}(1234)$ & $\widetilde{\mathcal{C}} = \mathcal{C}(2134)$ gives:

$$\mathcal{A}_4^{\text{closed}} = \mathcal{A}^{\text{open}}(1234) \underbrace{\left(\frac{e^{\pi i s}}{e^{2\pi i s} - 1} \right)^{-1}}_{2i \, \sin \pi s} \mathcal{A}^{\text{open}}(2134)$$

Twisted period relations: $\langle \varphi, \widetilde{\varphi} \rangle = \langle \varphi, \mathcal{C} \rangle \, \langle \mathcal{C}, \widetilde{\mathcal{C}} \rangle^{-1} \langle \widetilde{\mathcal{C}}, \widetilde{\varphi} \rangle$

Choosing $\mathcal{C} = \mathcal{C}(1234)$ & $\widetilde{\mathcal{C}} = \mathcal{C}(2134)$ gives:

$$\mathcal{A}_4^{\text{closed}} = \mathcal{A}^{\text{open}}(1234) \underbrace{\left(\frac{e^{\pi i s}}{e^{2\pi i s} - 1} \right)^{-1}}_{2i \, \sin \pi s} \mathcal{A}^{\text{open}}(2134)$$

These are the Kawai–Lewellen–Tye relations at 4-pt! [KLT '85]

Everything fully generalizes to higher-pt amplitudes:

Everything fully generalizes to higher-pt amplitudes:

$$\mathcal{A}_n^{\text{closed}} = \sum_{\alpha,\beta} \mathcal{A}^{\text{open}}(\alpha) \, \langle \mathcal{C}(\alpha), \mathcal{C}(\beta) \rangle^{-1} \, \mathcal{A}^{\text{open}}(\beta)$$

Everything fully generalizes to higher-pt amplitudes:

$$\mathcal{A}_n^{\text{closed}} = \sum_{\alpha,\beta} \mathcal{A}^{\text{open}}(\alpha)\, \underbrace{\langle \mathcal{C}(\alpha), \mathcal{C}(\beta) \rangle}_{\text{matrix of intersection numbers}}{}^{-1}\, \mathcal{A}^{\text{open}}(\beta)$$

Everything fully generalizes to higher-pt amplitudes:

$$\mathcal{A}_n^{\text{closed}} = \sum_{\alpha,\beta} \mathcal{A}^{\text{open}}(\alpha) \underbrace{\langle \mathcal{C}(\alpha), \mathcal{C}(\beta) \rangle^{-1}}_{\substack{\text{matrix of intersection numbers} \\ = \text{ inverse of the KLT kernel}}} \mathcal{A}^{\text{open}}(\beta)$$

Everything fully generalizes to higher-pt amplitudes:

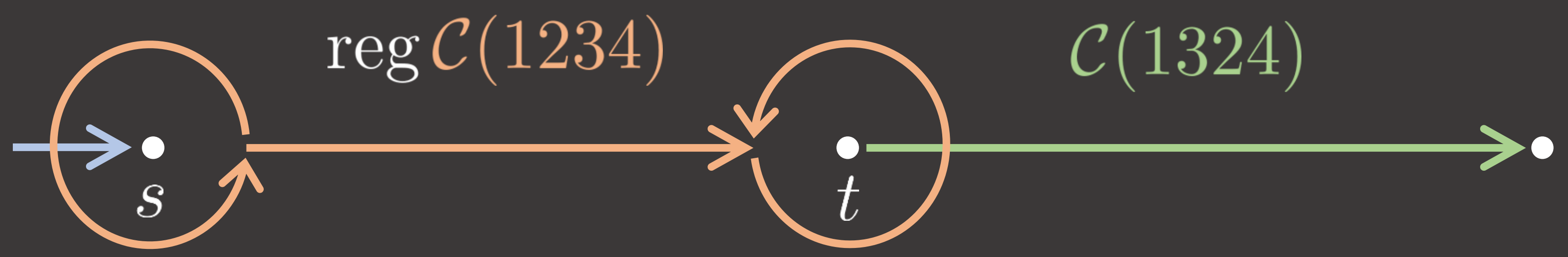
$$\mathcal{A}_n^{\text{closed}} = \sum_{\alpha,\beta} \mathcal{A}^{\text{open}}(\alpha)\, \underbrace{\langle \mathcal{C}(\alpha), \mathcal{C}(\beta) \rangle^{-1}}_{\substack{\text{matrix of intersection numbers} \\ = \text{inverse of the KLT kernel}}} \mathcal{A}^{\text{open}}(\beta)$$

$\mathcal{M}_{0,n}(\mathbb{R})$ tiled by *associahedra*:

Everything fully generalizes to higher-pt amplitudes:

$$\mathcal{A}_n^{\text{closed}} = \sum_{\alpha,\beta} \mathcal{A}^{\text{open}}(\alpha)\, \underbrace{\langle \mathcal{C}(\alpha), \mathcal{C}(\beta) \rangle^{-1}}_{\substack{\text{matrix of intersection numbers} \\ = \text{ inverse of the KLT kernel}}} \mathcal{A}^{\text{open}}(\beta)$$

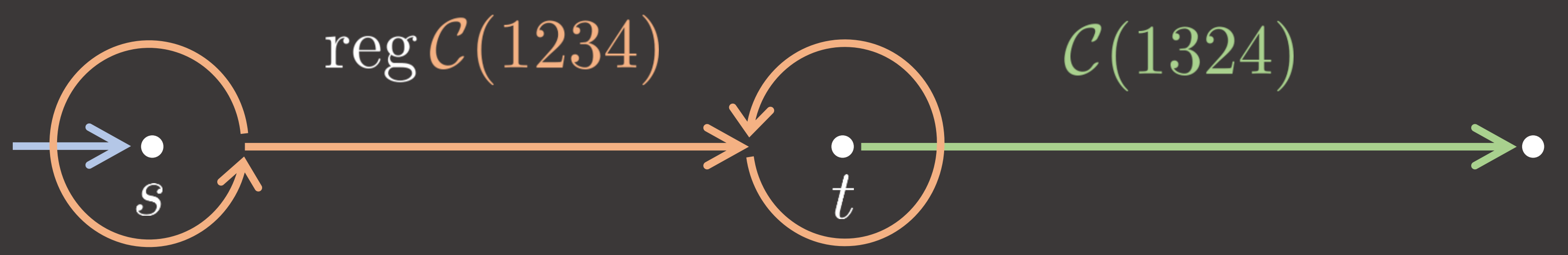
$\mathcal{M}_{0,n}(\mathbb{R})$ tiled by *associahedra*:

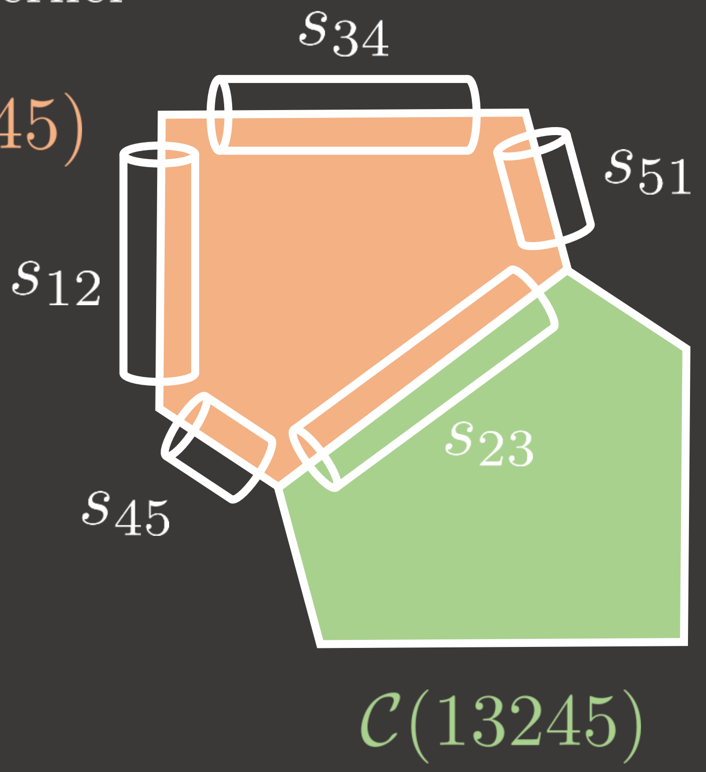


Everything fully generalizes to higher-pt amplitudes:

$$\mathcal{A}_n^{\text{closed}} = \sum_{\alpha,\beta} \mathcal{A}^{\text{open}}(\alpha)\, \underbrace{\langle \mathcal{C}(\alpha), \mathcal{C}(\beta)\rangle^{-1}}_{\substack{\text{matrix of intersection numbers} \\ = \text{ inverse of the KLT kernel}}} \mathcal{A}^{\text{open}}(\beta)$$

$\mathcal{M}_{0,n}(\mathbb{R})$ tiled by *associahedra*:



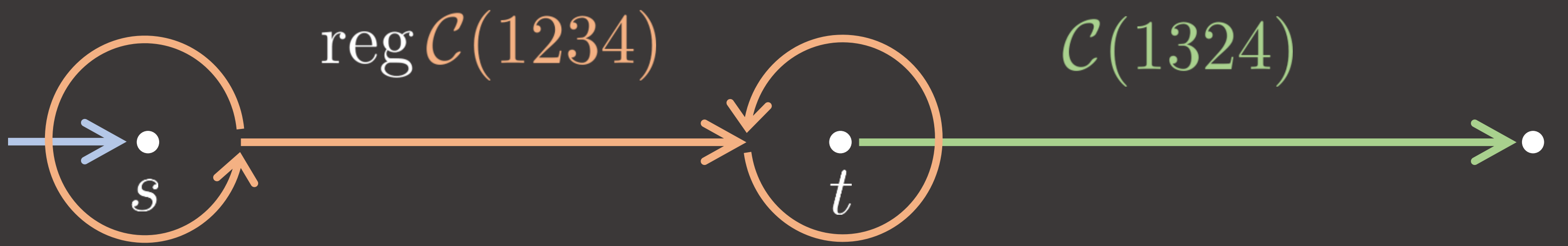


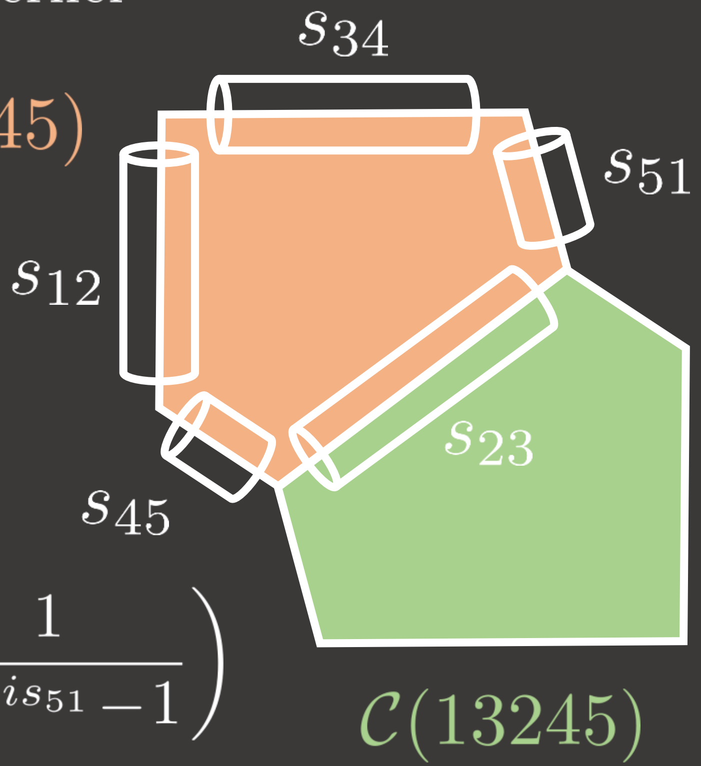
Everything fully generalizes to higher-pt amplitudes:

$$\mathcal{A}_n^{\text{closed}} = \sum_{\alpha,\beta} \mathcal{A}^{\text{open}}(\alpha)\, \underbrace{\langle \mathcal{C}(\alpha), \mathcal{C}(\beta) \rangle^{-1}}_{\substack{\text{matrix of intersection numbers} \\ = \text{ inverse of the KLT kernel}}}\, \mathcal{A}^{\text{open}}(\beta)$$

$\mathcal{M}_{0,n}(\mathbb{R})$ tiled by *associahedra*:

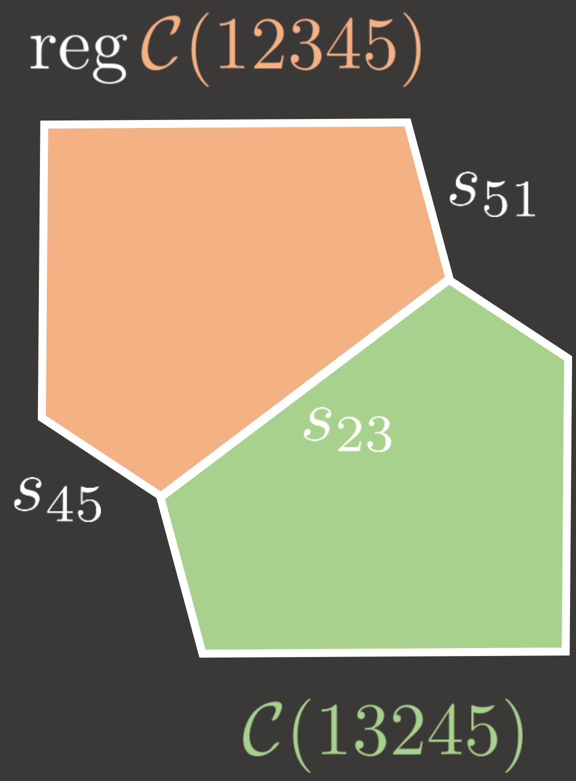
reg $\mathcal{C}(12345)$





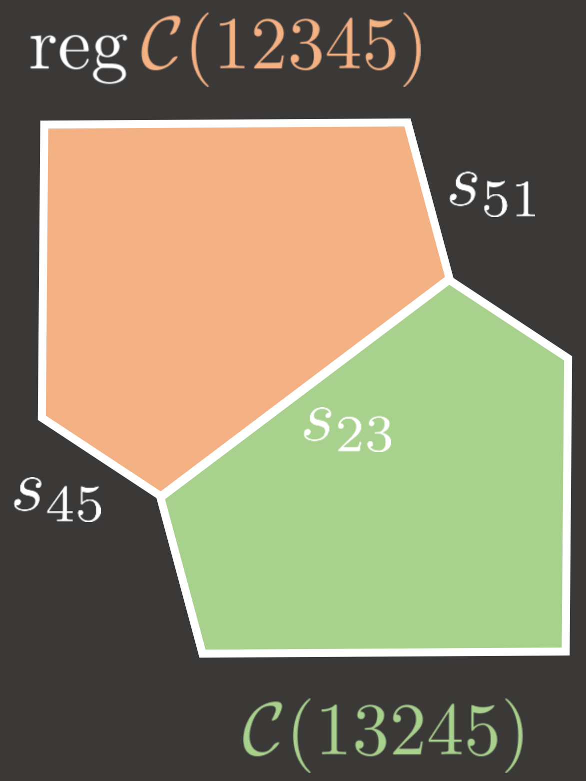
$$\langle \text{reg}\, \mathcal{C}(12345), \mathcal{C}(13245) \rangle = \frac{e^{\pi i s_{23}}}{e^{2\pi i s_{23}} - 1} \left(\frac{-1}{e^{2\pi i s_{45}} - 1} - 1 - \frac{1}{e^{2\pi i s_{51}} - 1} \right)$$

In the field-theory limit intersections localize on vertices of the associahedra:



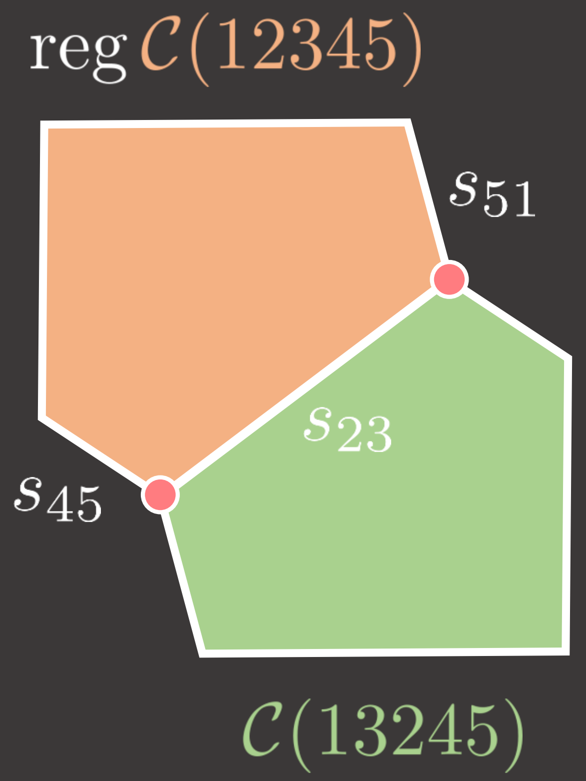
In the field-theory limit intersections localize on vertices of the associahedra:

$$\langle \mathrm{reg}\, \mathcal{C}(12345), \mathcal{C}(13245) \rangle \xrightarrow{\text{field-theory}} -\left(\frac{i}{2\pi}\right)^2 \frac{1}{s_{23}} \left(\frac{1}{s_{45}} + \frac{1}{s_{51}} \right)$$



In the field-theory limit intersections localize on vertices of the associahedra:

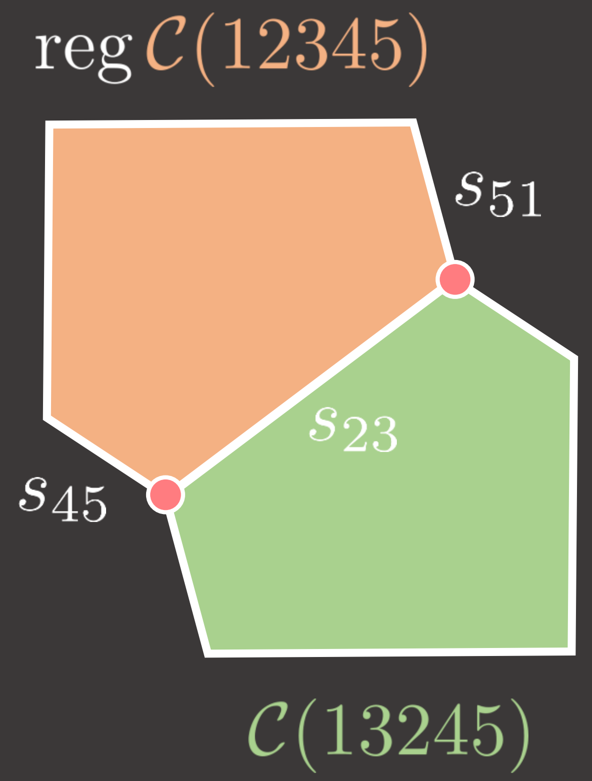
$$\langle \operatorname{reg}\mathcal{C}(12345), \mathcal{C}(13245)\rangle \xrightarrow{\text{field-theory}} -\left(\frac{i}{2\pi}\right)^2 \frac{1}{s_{23}}\left(\frac{1}{s_{45}} + \frac{1}{s_{51}}\right)$$



In the field-theory limit intersections localize on vertices of the associahedra:

$$\langle \mathrm{reg}\,\mathcal{C}(12345), \mathcal{C}(13245)\rangle \xrightarrow{\text{field-theory}} -\left(\frac{i}{2\pi}\right)^2 \frac{1}{s_{23}}\left(\frac{1}{s_{45}} + \frac{1}{s_{51}}\right)$$

$$\langle \mathcal{C}(\alpha), \mathcal{C}(\beta)\rangle \xrightarrow{\text{field-theory}} \sum \left(\begin{array}{c} \text{trivalent diagrams planar} \\ \text{w.r.t. permutations } \alpha \;\&\; \beta \end{array} \right)$$

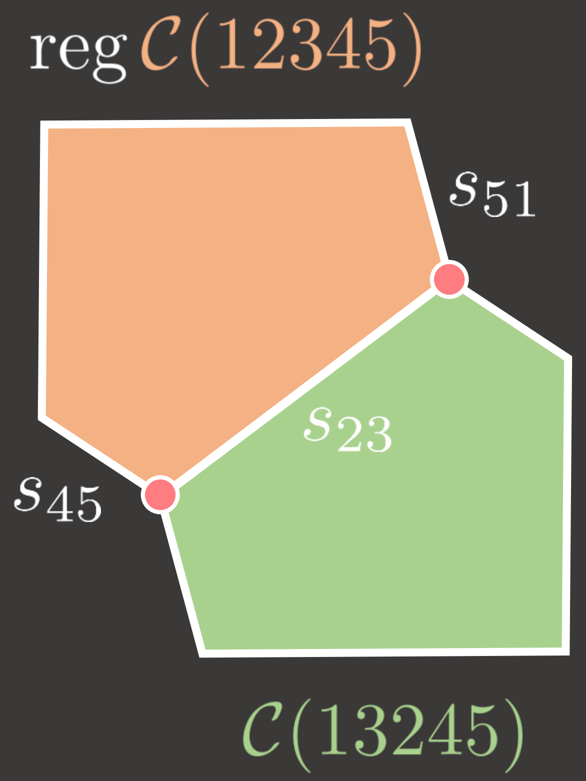


In the field-theory limit intersections localize on vertices of the associahedra:

$$\langle \operatorname{reg} \mathcal{C}(12345), \mathcal{C}(13245) \rangle \xrightarrow{\text{field-theory}} -\left(\frac{i}{2\pi}\right)^2 \frac{1}{s_{23}} \left(\frac{1}{s_{45}} + \frac{1}{s_{51}} \right)$$

$$\langle \mathcal{C}(\alpha), \mathcal{C}(\beta) \rangle \xrightarrow{\text{field-theory}} \sum \left(\begin{array}{c} \text{trivalent diagrams planar} \\ \text{w.r.t. permutations } \alpha \;\&\; \beta \end{array} \right)$$

$$= \text{bi-adjoint scalar amplitude } m(\alpha|\beta)$$

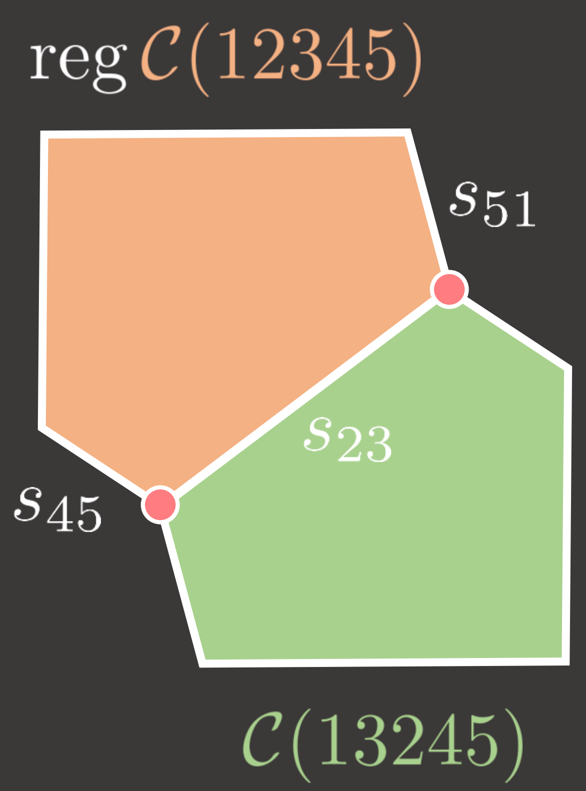


In the field-theory limit intersections localize on vertices of the associahedra:

$$\langle \mathrm{reg}\, \mathcal{C}(12345), \mathcal{C}(13245) \rangle \xrightarrow{\text{field-theory}} -\left(\frac{i}{2\pi}\right)^2 \frac{1}{s_{23}} \left(\frac{1}{s_{45}} + \frac{1}{s_{51}} \right)$$

$$\langle \mathcal{C}(\alpha), \mathcal{C}(\beta) \rangle \xrightarrow{\text{field-theory}} \sum \left(\begin{array}{c} \text{trivalent diagrams planar} \\ \text{w.r.t. permutations } \alpha \;\&\; \beta \end{array} \right)$$

$$= \text{bi-adjoint scalar amplitude } m(\alpha|\beta)$$



For details see:

- *"Combinatorics and Topology of Kawai–Lewellen–Tye Relations"*, SM, [hep-th/1706.08527]
- *"Inverse of the String Theory KLT Kernel"*, SM, [hep-th/1610.04230]

Recap so far:

Inverse of the KLT kernel describes how different associahedra intersect one another in the moduli space

Recap so far:

Inverse of the KLT kernel describes how different associahedra intersect one another in the moduli space

This is the reason why bi-adjoint scalar amplitudes appear in the KLT relations

Can we apply these powerful tools to study other problems?

Can we apply these powerful tools to study other problems?

$$\mathcal{A}^{\text{closed}} = \int \prod_{i<j} |z_i - z_j|^{\alpha' s_{ij}} \; \varphi_L \, \varphi_R$$

Can we apply these powerful tools to study other problems?

$$\mathcal{A}^{\text{closed}} = \int \prod_{i<j} |z_i - z_j|^{\alpha' s_{ij}} \, \varphi_L \, \varphi_R$$

$$\mathcal{A}^{\text{CHY}} = \int \prod_i \delta\left(\sum_{j \neq i} \frac{s_{ij}}{z_i - z_j} \right) \varphi_L \, \varphi_R$$

Can we apply these powerful tools to study other problems?

$$\mathcal{A}^{\text{closed}} = \int \underbrace{\prod_{i<j} |z_i - z_j|^{\alpha' s_{ij}}}_{\text{Koba–Nielsen factor}} \varphi_L \, \varphi_R$$

$$\mathcal{A}^{\text{CHY}} = \int \prod_i \delta\left(\sum_{j \neq i} \frac{s_{ij}}{z_i - z_j} \right) \varphi_L \, \varphi_R$$

Can we apply these powerful tools to study other problems?

$$\mathcal{A}^{\text{closed}} = \int \underbrace{\prod_{i<j} |z_i - z_j|^{\alpha' s_{ij}}}_{\text{Koba–Nielsen factor}} \varphi_L \, \varphi_R$$

$$\mathcal{A}^{\text{CHY}} = \int \underbrace{\prod_i \delta\Big(\sum_{j \neq i} \frac{s_{ij}}{z_i - z_j} \Big)}_{\text{saddle points of KN factor}} \varphi_L \, \varphi_R$$

Can we apply these powerful tools to study other problems?

$$\mathcal{A}^{\text{closed}} = \int \underbrace{\prod_{i<j} |z_i - z_j|^{\alpha' s_{ij}}}_{\text{Koba–Nielsen factor}} \varphi_L \, \varphi_R = \frac{1}{(\alpha')^{n-3}} \mathcal{A}^{\text{ft}} + \dots$$

$$\mathcal{A}^{\text{CHY}} = \int \underbrace{\prod_i \delta\Bigg(\sum_{j \neq i} \frac{s_{ij}}{z_i - z_j} \Bigg)}_{\text{saddle points of KN factor}} \varphi_L \, \varphi_R$$

Can we apply these powerful tools to study other problems?

$$\mathcal{A}^{\text{closed}} = \int \underbrace{\prod_{i<j} |z_i - z_j|^{\alpha' s_{ij}}}_{\text{Koba–Nielsen factor}} \varphi_L \, \varphi_R = \frac{1}{(\alpha')^{n-3}} \mathcal{A}^{\text{ft}} + \ldots$$

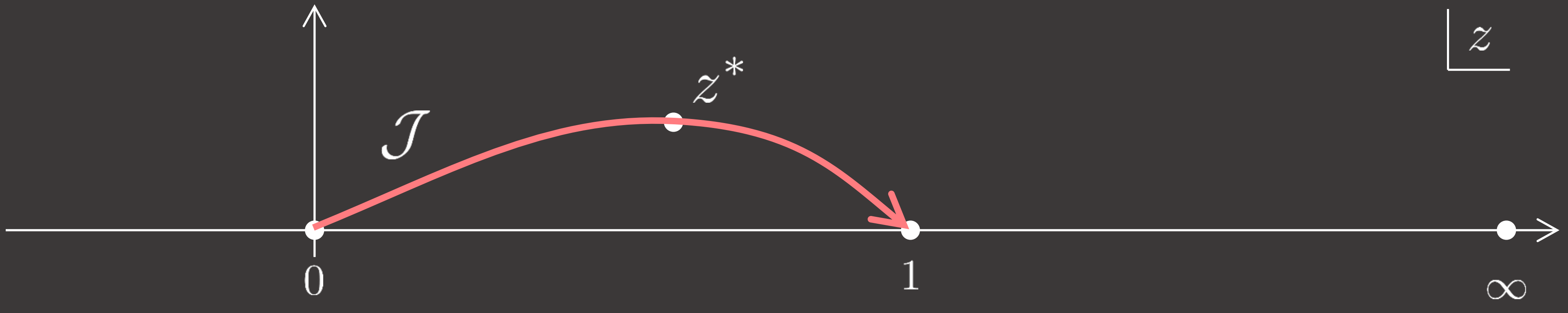
$$\mathcal{A}^{\text{CHY}} = \int \underbrace{\prod_i \delta\left(\sum_{j \neq i} \frac{s_{ij}}{z_i - z_j} \right)}_{\text{saddle points of KN factor}} \varphi_L \, \varphi_R$$

Here we illustrate what intersection theory has to say about this connection at the example of 4-pt massless amplitude

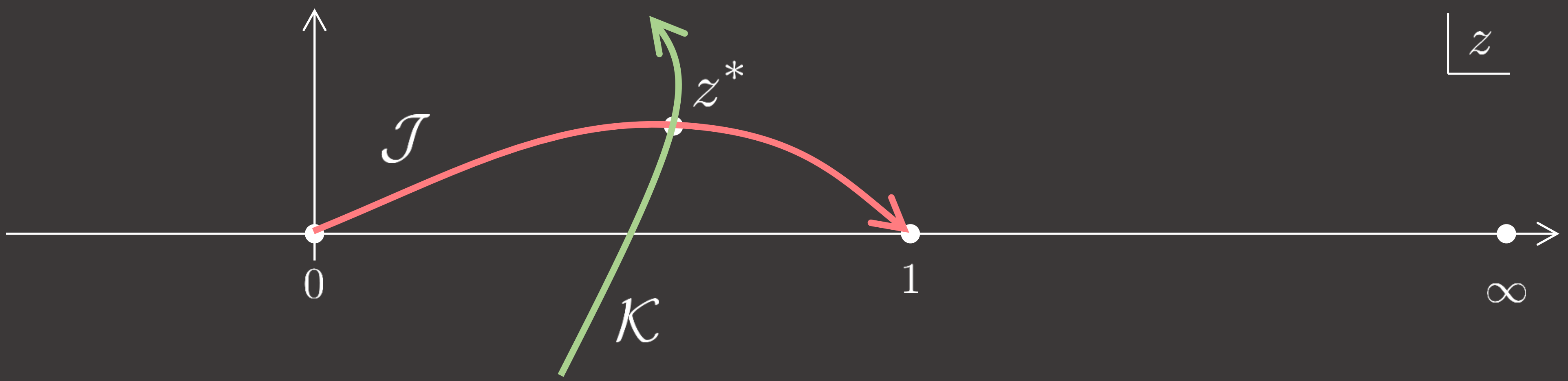
We use paths of *steepest descent* $\mathcal{J}$ and *steepest ascent* $\mathcal{K}$ intersecting at the saddle point of $z^{\alpha' s}(1-z)^{\alpha' t}$ denoted by z^*:

We use paths of *steepest descent* $\mathcal{J}$ and *steepest ascent* $\mathcal{K}$ intersecting at the saddle point of $z^{\alpha' s}(1-z)^{\alpha' t}$ denoted by z^*:

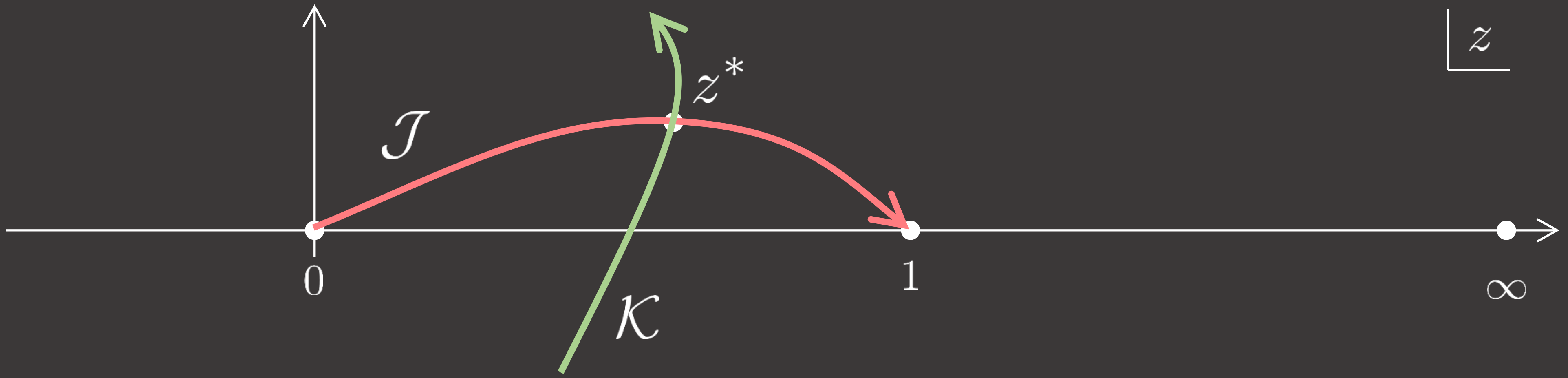
We use paths of *steepest descent* $\mathcal{J}$ and *steepest ascent* $\mathcal{K}$ intersecting at the saddle point of $z^{\alpha' s}(1-z)^{\alpha' t}$ denoted by z^*:



We use paths of *steepest descent* $\mathcal{J}$ and *steepest ascent* $\mathcal{K}$ intersecting at the saddle point of $z^{\alpha' s}(1-z)^{\alpha' t}$ denoted by z^*:

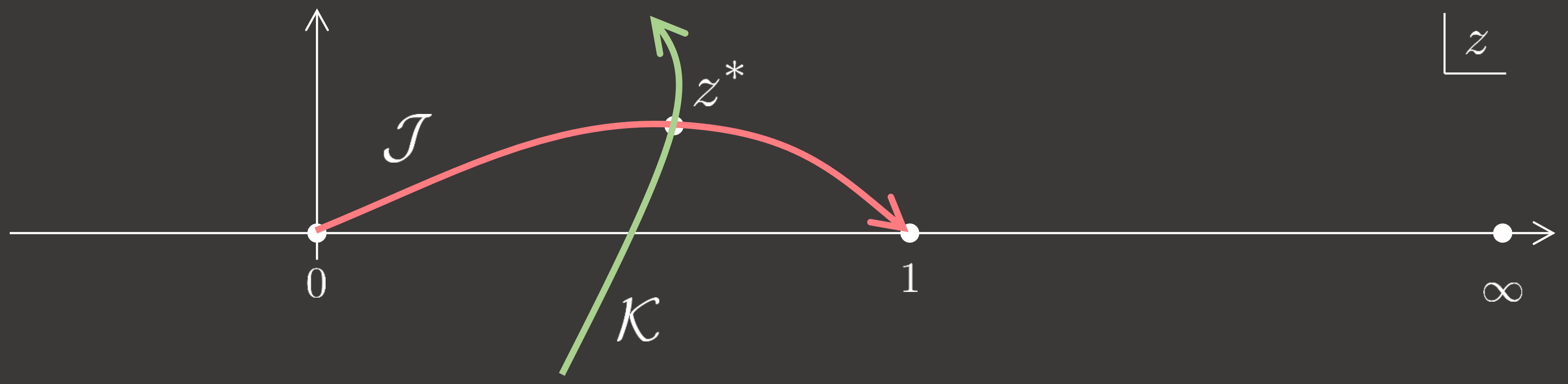


We use paths of *steepest descent* $\mathcal{J}$ and *steepest ascent* $\mathcal{K}$ intersecting at the saddle point of $z^{\alpha' s}(1-z)^{\alpha' t}$ denoted by z^*:



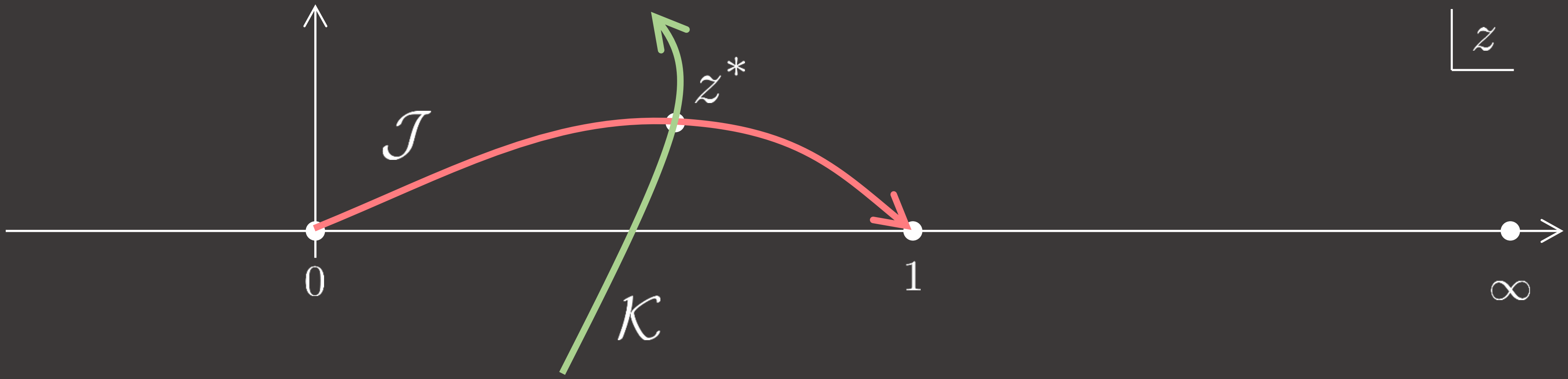
$$\mathcal{A}_4^{\text{closed}} = \left(\int_{\mathcal{J}} z^{\alpha' s}(1-z)^{\alpha' t} \varphi_L \right) \langle \mathcal{J}, \mathcal{K} \rangle^{-1} \left(\int_{\mathcal{K}} z^{\alpha' s}(1-z)^{\alpha' t} \varphi_R \right)$$

We use paths of *steepest descent* $\mathcal{J}$ and *steepest ascent* $\mathcal{K}$ intersecting at the saddle point of $z^{\alpha' s}(1-z)^{\alpha' t}$ denoted by z^*:



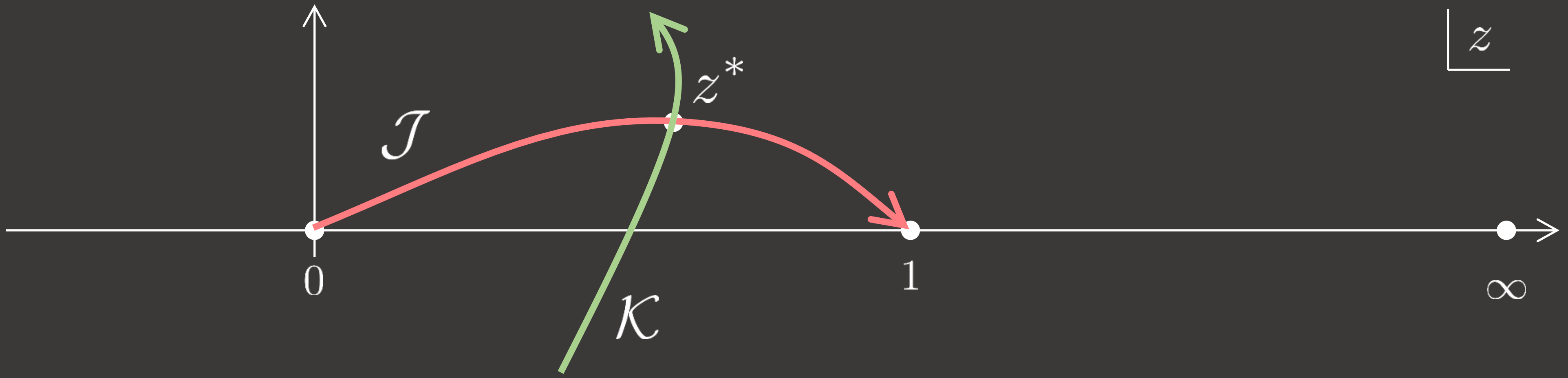
$$\mathcal{A}_4^{\text{closed}} = \left(\int_{\mathcal{J}} z^{\alpha' s}(1-z)^{\alpha' t}\, \varphi_L \right) \underbrace{\langle \mathcal{J}, \mathcal{K} \rangle^{-1}}_{+1} \left(\int_{\mathcal{K}} z^{\alpha' s}(1-z)^{\alpha' t}\, \varphi_R \right)$$

We use paths of *steepest descent* $\mathcal{J}$ and *steepest ascent* $\mathcal{K}$ intersecting at the saddle point of $z^{\alpha' s}(1-z)^{\alpha' t}$ denoted by z^*:



$$\mathcal{A}_4^{\text{closed}} = \underbrace{\left(\int_{\mathcal{J}} z^{\alpha' s}(1-z)^{\alpha' t}\, \varphi_L \right)}_{\text{most convergent}} \underbrace{\langle \mathcal{J}, \mathcal{K} \rangle^{-1}}_{+1} \left(\int_{\mathcal{K}} z^{\alpha' s}(1-z)^{\alpha' t}\, \varphi_R \right)$$

We use paths of *steepest descent* $\mathcal{J}$ and *steepest ascent* $\mathcal{K}$ intersecting at the saddle point of $z^{\alpha' s}(1-z)^{\alpha' t}$ denoted by z^*:



$$\mathcal{A}_4^{\text{closed}} = \underbrace{\left(\int_{\mathcal{J}} z^{\alpha' s}(1-z)^{\alpha' t}\, \varphi_L \right)}_{\text{most convergent}} \underbrace{\langle \mathcal{J}, \mathcal{K} \rangle^{-1}}_{+1} \underbrace{\left(\int_{\mathcal{K}} z^{\alpha' s}(1-z)^{\alpha' t}\, \varphi_R \right)}_{\text{most divergent}}$$

Since we're only interested in the leading term, we can invert one KN factor: [Aomoto, Deligne, Kita, ...; Siegel, Huang, Yuan, Casali, Tourkine, Azevedo, Jusinskas, ...]

Since we're only interested in the leading term, we can invert one KN factor: [Aomoto, Deligne, Kita, ...; Siegel, Huang, Yuan, Casali, Tourkine, Azevedo, Jusinskas, ...]

$$\frac{1}{\alpha'}\mathcal{A}_4^{\mathrm{ft}} + \ldots = \left(\int_{\mathcal{J}} z^{\alpha' s}(1-z)^{\alpha' t}\, \varphi_L \right) \langle \mathcal{J}, \mathcal{K} \rangle^{-1} \left(\int_{\mathcal{K}} z^{+\alpha' s}(1-z)^{+\alpha' t}\, \varphi_R \right)$$

Since we're only interested in the leading term, we can invert one KN factor: [Aomoto, Deligne, Kita, ...; Siegel, Huang, Yuan, Casali, Tourkine, Azevedo, Jusinskas, ...]

$$-\frac{1}{\alpha'}\mathcal{A}_4^{\mathrm{ft}} + \ldots = \left(\int_{\mathcal{J}} z^{\alpha' s}(1-z)^{\alpha' t}\, \varphi_L \right) \langle \mathcal{J}, \mathcal{K} \rangle^{-1} \left(\int_{\mathcal{K}} z^{-\alpha' s}(1-z)^{-\alpha' t}\, \varphi_R \right)$$

Since we're only interested in the leading term, we can invert one KN factor: [Aomoto, Deligne, Kita, ...; Siegel, Huang, Yuan, Casali, Tourkine, Azevedo, Jusinskas, ...]

$$-\frac{1}{\alpha'}\mathcal{A}_4^{\mathrm{ft}} + \ldots = \underbrace{\left(\int_{\mathcal{J}} z^{\alpha' s}(1-z)^{\alpha' t}\,\varphi_L\right)}_{\text{most convergent}} \underbrace{\langle \mathcal{J}, \mathcal{K}\rangle^{-1}}_{+1} \underbrace{\left(\int_{\mathcal{K}} z^{-\alpha' s}(1-z)^{-\alpha' t}\,\varphi_R\right)}_{\text{most convergent}}$$

Since we're only interested in the leading term, we can invert one KN factor: [Aomoto, Deligne, Kita, ...; Siegel, Huang, Yuan, Casali, Tourkine, Azevedo, Jusinskas, ...]

$$-\frac{1}{\alpha'}\mathcal{A}_4^{\mathrm{ft}} + \ldots = \underbrace{\left(\int_{\mathcal{J}} z^{\alpha' s}(1-z)^{\alpha' t}\,\varphi_L\right)}_{\text{most convergent}} \underbrace{\langle \mathcal{J}, \mathcal{K}\rangle^{-1}}_{+1} \underbrace{\left(\int_{\mathcal{K}} z^{-\alpha' s}(1-z)^{-\alpha' t}\,\varphi_R\right)}_{\text{most convergent}}$$

$$\xrightarrow{\alpha' \to \infty} -\frac{2\pi i}{\alpha'} \frac{\varphi_L\,\varphi_R}{\frac{\partial}{\partial z}\left(\frac{s}{z} + \frac{t}{z-1}\right)}\Bigg|_{z=z^*}$$

Since we're only interested in the leading term, we can invert one KN factor: [Aomoto, Deligne, Kita, ...; Siegel, Huang, Yuan, Casali, Tourkine, Azevedo, Jusinskas, ...]

$$-\frac{1}{\alpha'}\mathcal{A}_4^{\text{ft}} + \ldots = \underbrace{\left(\int_{\mathcal{J}} z^{\alpha' s}(1-z)^{\alpha' t}\,\varphi_L\right)}_{\text{most convergent}} \underbrace{\langle \mathcal{J}, \mathcal{K}\rangle^{-1}}_{+1} \underbrace{\left(\int_{\mathcal{K}} z^{-\alpha' s}(1-z)^{-\alpha' t}\,\varphi_R\right)}_{\text{most convergent}}$$

$$\xrightarrow{\alpha'\to\infty} -\frac{2\pi i}{\alpha'}\frac{\varphi_L\,\varphi_R}{\frac{\partial}{\partial z}\left(\frac{s}{z}+\frac{t}{z-1}\right)}\Bigg|_{z=z^*} = -\frac{2\pi i}{\alpha'}\underbrace{\int \delta\left(\frac{s}{z}+\frac{t}{z-1}\right)\varphi_L\,\varphi_R}_{\mathcal{A}_4^{\text{CHY}}}$$

Since we're only interested in the leading term, we can invert one KN factor: [Aomoto, Deligne, Kita, ...; Siegel, Huang, Yuan, Casali, Tourkine, Azevedo, Jusinskas, ...]

$$-\frac{1}{\alpha'}\mathcal{A}_4^{\text{ft}} + \cancel{\ldots} = \underbrace{\left(\int_{\mathcal{J}} z^{\alpha' s}(1-z)^{\alpha' t}\,\varphi_L\right)}_{\text{most convergent}} \underbrace{\langle \mathcal{J}, \mathcal{K}\rangle^{-1}}_{+1} \underbrace{\left(\int_{\mathcal{K}} z^{-\alpha' s}(1-z)^{-\alpha' t}\,\varphi_R\right)}_{\text{most convergent}}$$

$$\xrightarrow{\alpha'\to\infty} -\frac{2\pi i}{\alpha'} \frac{\varphi_L\,\varphi_R}{\frac{\partial}{\partial z}\left(\frac{s}{z}+\frac{t}{z-1}\right)}\Bigg|_{z=z^*} = -\frac{2\pi i}{\alpha'} \underbrace{\int \delta\left(\frac{s}{z}+\frac{t}{z-1}\right)\varphi_L\,\varphi_R}_{\mathcal{A}_4^{\text{CHY}}}$$

The result is exact in α', so $\lim_{\alpha'\to 0}\mathcal{A}_4^{\text{closed}} = \frac{2\pi i}{\alpha'}\mathcal{A}_4^{\text{CHY}} + \ldots$

Fully generalizes to higher-pt amplitudes:

Fully generalizes to higher-pt amplitudes:

$$\lim_{\alpha' \to 0} \mathcal{A}_n^{\text{closed}} = \left(\frac{2\pi i}{\alpha'}\right)^{n-3} \mathcal{A}_n^{\text{CHY}} + \ldots$$

Fully generalizes to higher-pt amplitudes:

$$\lim_{\alpha' \to 0} \mathcal{A}_n^{\text{closed}} = \left(\frac{2\pi i}{\alpha'} \right)^{n-3} \mathcal{A}_n^{\text{CHY}} + \ldots$$

(recall that the integrands are logarithmic forms & the kinematics is massless)

Fully generalizes to higher-pt amplitudes:

$$\lim_{\alpha' \to 0} \mathcal{A}_n^{\text{closed}} = \left(\frac{2\pi i}{\alpha'}\right)^{n-3} \mathcal{A}_n^{\text{CHY}} + \ldots$$

(recall that the integrands are logarithmic forms & the kinematics is massless)

In fact, this is the first sign that CHY formalism is a part of a more general structure, which mathematicians call *intersection numbers of twisted* *cocycles*. This is a topic on its own

Fully generalizes to higher-pt amplitudes:

$$\lim_{\alpha' \to 0} \mathcal{A}_n^{\text{closed}} = \left(\frac{2\pi i}{\alpha'} \right)^{n-3} \mathcal{A}_n^{\text{CHY}} + \ldots$$

(recall that the integrands are logarithmic forms & the kinematics is massless)

In fact, this is the first sign that CHY formalism is a part of a more general structure, which mathematicians call *intersection numbers of twisted* *cocycles*. This is a topic on its own

For details see:

- *"Scattering Amplitudes from Intersection Theory"*, SM, [hep-th/1711.00469]

To summarize, we've seen that intersection theory plays an important role in understanding of KLT relations, both in string and field-theory

To summarize, we've seen that intersection theory plays an important role in understanding of KLT relations, both in string and field-theory

In this language amplitudes come into three classes:

To summarize, we've seen that intersection theory plays an important role in understanding of KLT relations, both in string and field-theory

In this language amplitudes come into three classes:

$\langle \text{cycle}, \text{cycle} \rangle$ inverse of the KLT kernel

To summarize, we've seen that intersection theory plays an important role in understanding of KLT relations, both in string and field-theory

In this language amplitudes come into three classes:

$\langle \text{cycle}, \text{cycle} \rangle$ inverse of the KLT kernel

$\langle \text{cycle}, \text{cocycle} \rangle$ open string

To summarize, we've seen that intersection theory plays an important role in understanding of KLT relations, both in string and field-theory

In this language amplitudes come into three classes:

$$\begin{aligned} \langle \text{cycle}, \text{cycle} \rangle \quad & \text{inverse of the KLT kernel} \\ \langle \text{cycle}, \text{cocycle} \rangle \quad & \text{open string} \\ \langle \text{cocycle}, \text{cocycle} \rangle \quad & \text{closed string,} \end{aligned}$$

To summarize, we've seen that intersection theory plays an important role in understanding of KLT relations, both in string and field-theory

In this language amplitudes come into three classes:

$\langle \text{cycle}, \text{cycle} \rangle$	inverse of the KLT kernel
$\langle \text{cycle}, \text{cocycle} \rangle$	open string
$\langle \text{cocycle}, \text{cocycle} \rangle$	closed string, CHY

To summarize, we've seen that intersection theory plays an important role in understanding of KLT relations, both in string and field-theory

In this language amplitudes come into three classes:

$$\langle \text{cycle}, \text{cycle} \rangle \quad \text{inverse of the KLT kernel}$$

$$\langle \text{cycle}, \text{cocycle} \rangle \quad \text{open string}$$

$$\langle \text{cocycle}, \text{cocycle} \rangle \quad \text{closed string, CHY}$$

We've also seen evidence that intersection theory is a useful tool for the study of the connections between string theory amplitudes and CHY formulae

Thank you!