Twofold Invariances of Pure Gravity

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Building the Gravity Action
Motivation

- **Double copy** (KLT, BCJ, ...): $A_{GR} \sim \sum A_{YM} \bar{A}_{YM}$
- Implies existence of **extra symmetry** in gravity amplitudes: $SO(D - 1, 1) \times \overline{SO}(D - 1, 1)$
  - Symmetry is hidden in the canonical perturbation theory $(g_{ab} = \eta_{ab} + h_{ab})$ of the Einstein-Hilbert action:
    $$S = \int d^D x \sqrt{-g} \left( \frac{R}{16\pi G} + \mathcal{L}_{GF} \right)$$
  - Symmetry should be manifested as **factorization of indices** of graviton $h_{a\bar{b}}$ to all orders
  - Action invariant under double Lorentz transformations of $k_a$ and $\bar{k}_{\bar{a}}$ and double gauge transformations of $\epsilon_a$ and $\bar{\epsilon}_{\bar{a}}$, where $h_{a\bar{b}} \sim \epsilon_a \bar{\epsilon}_{\bar{b}}$
  - Find a way to make this double Lorentz symmetry manifest
  - First step towards understanding the double copy of gravity and YM **from the action itself** (cf. Z. Bern and A. K. Grant, hep-th/9904026)
Define \( h_{ab}^n = h_{a_1}^a h_{a_2}^a \cdots h_{a_{n-1}}^a \) and \([h^n] = h_{ab}^n \eta^{ba}\).  

Graviton cycles:

\[ [h^{2n}] = \text{even cycle}, \quad [h^{2n+1}] = \text{odd cycle} \]

- Even cycles \( \implies \) compatible with index factorization
- Odd cycles \( \implies \) incompatible with index factorization

- Canonical perturbation theory introduces many odd cycles (e.g., through \( \sqrt{-g} \)) \( \implies \) obstruction to seeing double copy symmetry
Field Redefinition and Gauge Fixing

How can we eliminate odd graviton cycles from the action?

- **Nonlinear field redefinition**
- **Nonlinear, local gauge fixing**
- Leaves amplitudes unchanged (Haag’s theorem)

Field redefinition ($\alpha_1 = 1$)

\[ h_{ab} \rightarrow \alpha_1 h_{ab} + \alpha_2 \eta_{ab}[h] \]
\[ + \alpha_3 h_{ab}^2 + \alpha_4 h_{ab}[h] + \alpha_5 \eta_{ab}[h^2] + \alpha_6 \eta_{ab}[h]^2 + \alpha_7 \eta_{ab}[h^3] + \alpha_8 \eta_{ab}[h^2][h] + \alpha_9 \eta_{ab}[h]^2 + \alpha_{10} h_{ab}[h]^2 + \alpha_{11} \eta_{ab}[h^3] + \alpha_{12} \eta_{ab}[h^2][h] + \alpha_{13} \eta_{ab}[h]^3 + \cdots \]

Gauge fixing: \( L_{GF} = -\eta^{ab} F_a F_b \),

\[ F_a = \partial^b h^{cd}(\beta_1 \eta_{ab} \eta_{cd} + \beta_2 \eta_{ac} \eta_{bd} + \beta_3 h_{ab} \eta_{cd} + \beta_4 h_{ac} \eta_{bd} + \beta_5 \eta_{ab} h_{cd} + \beta_6 \eta_{ac} h_{bd} + \beta_7 \eta_{ab} \eta_{cd}[h] + \beta_8 \eta_{ac} \eta_{bd}[h] + \cdots) \]

Choose \( \alpha_i \) and \( \beta_i \) to cancel all odd cycles in the action
Index-Factorizing the Action
Defining our fields

- Definitions:
- Field basis:

\[ g_{ab} = (e^{\pi})_{ab} = \eta_{ab} + \pi_{ab} + \frac{1}{2!} \pi_{ab}^2 + \cdots \quad \text{and} \quad g^{ab} = (e^{-\pi})^{ab}, \]

where

\[ \pi_{ab} = h_{ab} - \frac{1}{D - 2} \eta_{ab} [h] \]

- Gauge-fixing term

\[ \mathcal{L}_{GF} = -\frac{D - 2}{64\pi G} g^{ab} \omega_a \omega_b, \]

where

\[ \omega_a = \partial_a \log \sqrt{-g} = -\frac{1}{D - 2} \partial_a [h] \]

- Classical gauge condition: \( \omega_a = 0 \), equivalently, \( \nabla_b \nabla_a x^b = 0 \)
- Exponential field

\[ \sigma_{ab} = \eta_{ab} + h_{ab} + \frac{1}{2!} h_{ab}^2 + \cdots = (e^h)_{ab} \quad \text{and} \quad \sigma^{ab} = (e^{-h})^{ab} \]
The Action

- All-orders action for the graviton, expanded around flat spacetime:

\[
S = \frac{1}{16\pi G} \int d^D x \partial_a \sigma_{ce} \partial_b \sigma^{de} \left( \frac{1}{4} \sigma^{ab} \delta^c_d - \frac{1}{2} \sigma^{cb} \delta^a_d \right)
\]

- Equivalent to original EH action
  - Every order in $h$ is index-factorizable
  - Exponential field basis is suggestive (cf. NLSM)
  - Perturbation theory is simple compared to canonical expansion: $\sim \frac{3}{4} n^2$ terms compared to $\propto 2^n$ terms

- How to *automate* the process of assigning barred/unbarred indices?
Auxiliary Dimensions

- Extend the spacetime from $D \to 2D$ dimensions: $x^A = (x^a, x^{\bar{a}})$, $\partial_A = (\partial_a, \partial_{\bar{a}})$
- Original spacetime on the “diagonal”: $x = \bar{x}$
- Flat metric tensors:
  \[
  \eta_{AB} = \begin{bmatrix}
  \eta_{ab} & 0 \\
  0 & \eta_{\bar{a}\bar{b}}
  \end{bmatrix}
  \]
- New graviton field
  \[
  H_{AB} = \begin{bmatrix}
  0 & h_{a\bar{b}} \\
  h_{\bar{a}b} & 0
  \end{bmatrix},
  \]
  where $h_{a\bar{b}}$ and $h_{\bar{a}b}$ are transposes and are arbitrary $D$-by-$D$ matrices (symmetric part = graviton, antisymmetric part = two-form)
**Auxiliary Dimensions**

- Exponential field

\[
\Sigma_{AB} = (e^H)_{AB} = \begin{bmatrix}
(cosh \ h)_{ab} & (sinh \ h)_{a\bar{b}} \\
(sinh \ h)_{\bar{a}b} & (cosh \ h)_{\bar{a}\bar{b}}
\end{bmatrix}
\]

\[
(cosh \ h)_{ab} = \eta_{ab} + \frac{1}{2!} h_{ab}^2 + \cdots
\]

\[
(sinh \ h)_{a\bar{b}} = h_{a\bar{b}} + \frac{1}{3!} h_{a\bar{b}}^3 + \cdots
\]

- Consistent index factorization is automatic:

\[
h_{ab}^{2n} = h_a^{\bar{a}_1} h_{\bar{a}_1}^{a_2} \cdots h_{a_{2n-2}}^{\bar{a}_{2n-1}} h_{\bar{a}_{2n-1}}^{a_{2n-1}} \bar{b}
\]

\[
h_{a\bar{b}}^{2n+1} = h_a^{\bar{a}_1} h_{\bar{a}_1}^{a_2} \cdots h_{a_{2n}}^{\bar{a}_{2n-1}} h_{\bar{a}_{2n-1}}^{a_{2n-1}} \bar{b}
\]

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**Index-Factorized Action**

\[
S = \frac{1}{16\pi G} \int d^D x \ d^D \bar{x} \ \delta^D (x - \bar{x}) \partial_A \Sigma_{CE} \partial_B \Sigma^{DE} \left( \frac{1}{16} \Sigma^{AB} \delta^C_D - \frac{1}{4} \Sigma^{CB} \delta^A_D \right)
\]
The Index-Factorized Action is given by:

\[
S = \frac{1}{16\pi G} \int d^D x \ d^D \bar{x} \delta^D (x - \bar{x}) \partial_A \Sigma_{CE} \partial_B \Sigma^{DE} \left( \frac{1}{16} \Sigma^{AB} \delta^C_D - \frac{1}{4} \Sigma^{CB} \delta^A_D \right)
\]

- **Doubly Lorentz invariant**, manifest \(SO(D - 1, 1) \times \overline{SO}(D - 1, 1)\)

- **\(\mathbb{Z}_2\) parity** swapping spacetimes:
  \[
  x^a \leftrightarrow x^\bar{a} \\
  h_{\bar{a} \bar{b}} \leftrightarrow h_{\bar{a} \bar{b}} \\
  H \leftrightarrow \tau H \tau \quad \tau = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
  \eta \leftrightarrow \tau \eta \tau = \eta
  \]

- **\(\mathbb{Z}_2\) parity** \(\Rightarrow\) two-form appears in pairs in action \(\Rightarrow\) decouples from tree-level graviton amplitudes
\[ O_2 = -\frac{1}{4} h_{a\bar{b}} h_{c\bar{d}} K^{\bar{a}\bar{b}c\bar{d}}, \text{ where} \]
\[ K^{\bar{a}\bar{b}c\bar{d}} = p^2 \eta^{ac} \eta^{\bar{b}\bar{d}} - \eta^{ac} p^\bar{b} p^\bar{d} - p^a p^c \eta^{\bar{b}\bar{d}} \]

- Antisymmetric part of \( h_{a\bar{b}} \) has noninvertible kinetic term
  - But can introduce additional gauge-fixing for two-form:
  \[ K^\xi_{\bar{a}\bar{b}c\bar{d}} = p^2 \eta^{ac} \eta^{\bar{b}\bar{d}} - \left(1 + \frac{1}{\xi}\right) \eta^{ac} p^\bar{b} p^\bar{d} - \left(1 - \frac{1}{\xi}\right) p^a p^c \eta^{\bar{b}\bar{d}} \]
  - Invert via \( K^\xi_{\bar{a}\bar{b}c\bar{d}} \Delta_{c\bar{d}e\bar{f}} = i\delta^a_e \delta^\bar{b}_\bar{f} \) to get:

\[ \Delta_{\bar{a}\bar{b}c\bar{d}} = \frac{i}{p^2} \left( \eta^{ac} \eta^{\bar{b}\bar{d}} - (1 + \xi) \eta^{ac} \frac{p^\bar{b} p^\bar{d}}{p^2} - (1 - \xi) \frac{p^a p^c}{p^2} \eta^{\bar{b}\bar{d}} \right) \]

- Amplitudes are \( \xi \)-independent and match those computed in canonical field basis
Alternative Representations
A Family of Index-Factorized Actions

- Can take our index-factorized action and send
  
  $$h_{a\bar{b}} \rightarrow \sum_{m \text{ even}} \sum_{n \text{ odd}} [h^m] h^{n}_{a\bar{b}}$$

  while preserving index factorization

- For example, send
  
  $$h_{a\bar{b}} \rightarrow (\sinh^{-1} h)_{a\bar{b}}, \quad \Sigma_{AB} \rightarrow \begin{bmatrix} (\sqrt{1 + h^2})_{ab} & \frac{h_{a\bar{b}}}{h_{a\bar{b}}} \\ \frac{h_{a\bar{b}}}{h_{a\bar{b}}} & (\sqrt{1 + h^2})_{a\bar{b}} \end{bmatrix}$$

- Use this to simplify action further by reducing number of terms in perturbation expansion
For example, inspired by the Cayley basis for the NLSM, we can redefine

\[ h_{\bar{a}b} \rightarrow \log \left( \frac{1 + \frac{1}{2} h}{1 - \frac{1}{2} h} \right)_{\bar{a}b} = h_{\bar{a}b} + \frac{1}{12} h^3_{\bar{a}b} + \frac{1}{80} h^5_{\bar{a}b} + \cdots \]

Action just as before, but with \( \sigma_{ab} \rightarrow \left( \frac{1 + \frac{1}{2} h}{1 - \frac{1}{2} h} \right)_{ab} \)

**New duality** under small and large perturbations, \( h_{ab} \) and \( h_{ab}^{-1} \):

- \( h_{ab}/2 \rightarrow (h_{ab}/2)^{-1} \) sends \( \sigma_{ab} \rightarrow -\sigma_{ab} \) and \( S \rightarrow -S \) (i.e., flips sign of \( \hbar \))

Far simpler at \( \mathcal{O}(h^n) \) compared to canonical perturbation theory: \( \sim \frac{3}{16} n^2 \) Lorentz-invariant terms
Generalizing to Curved Spacetime
Definitions

- **Arbitrary background spacetime** metric $\bar{g}_{ab}$
- Field basis

\[
g_{ab} = \bar{g}_{ab} + \pi_{ab} + \frac{1}{2!} \pi_{ab}^2 + \cdots
\]

\[
\pi_{ab}^n = \pi_{ab_1} \bar{g}_{b_1 a_1} \pi_{a_1 b_2} \bar{g}_{b_2 a_2} \cdots \bar{g}_{b_{n-1} a_{n-1}} \pi_{a_{n-1} b}
\]

\[
\pi_{ab} = h_{ab} - \frac{1}{D-2} \bar{g}_{ab} [h]
\]

\[
[h^n] = h_{ab} \bar{g}^{ba}
\]

- **Gauge-fixing**

\[
\mathcal{L}_{GF} = -\frac{D-2}{64\pi G} g^{ab} \Omega_a \Omega_b
\]

\[
\Omega_a = \omega_a - \bar{\omega}_a = -\frac{1}{D-2} \partial_a [h]
\]

\[
\omega_a = \partial_a \log \sqrt{-g} \quad \text{and} \quad \bar{\omega}_a = \partial_a \log \sqrt{-\bar{g}}
\]

- **Classical gauge condition:** $\Omega_a = 0$, equivalently $\nabla_b \nabla_a x^b = \bar{\nabla}_b \bar{\nabla}_a \bar{x}^b$.

- **Exponential field**

\[
\sigma_{ab} = \bar{g}_{ab} + h_{ab} + \frac{1}{2!} h_{ab}^2 + \cdots = (e^h)_{ab}, \quad \sigma^{ab} = (e^{-h})^{ab}
\]
A Index-Factorized Gravity Action in Curved Spacetime

Action

\[ S = \frac{1}{16\pi G} \int \! d^D x \sqrt{-\bar{g}} \left[ \bar{\nabla}_a \sigma_{ce} \bar{\nabla}_b \sigma^{de} \left( \frac{1}{4} \sigma^{ab} \delta^c_d - \frac{1}{2} \sigma^{cb} \delta^a_d \right) + \sigma^{ab} \bar{R}_{ab} \right] \]

- Tadpole from \( \sigma^{ab} \bar{R}_{ab} \) and matter action vanishes on background Einstein equation.
- If \( \bar{R}_{ab} = 0 \), i.e., for arbitrary vacuum background (e.g., black hole), action index factorizes:

Index-Factorized Action

\[ S = \frac{1}{16\pi G} \int \! d^D x \, d^D \bar{x} \, \delta^D (x - \bar{x}) \sqrt{-\bar{g}} \times \]

\[ \times \bar{\nabla}_A \Sigma_{CE} \bar{\nabla}_B \Sigma^{DE} \left( \frac{1}{16} \Sigma^{AB} \delta^C_D - \frac{1}{4} \Sigma^{CB} \delta^A_D \right) \]
Equations of motion

- \( E_{ab} = 16\pi G \frac{\delta S}{\delta \sigma_{ab}} \)

- **Jacobian:**

\[
J^{cd}_{ab} = \sqrt{-g} \frac{\delta g^{ab}}{-\bar{g} \delta \sigma^{cd}} = \frac{1}{2} \left( \delta^a_c \delta^b_d + \delta^a_d \delta^b_c \right) - \frac{1}{D - 2} g^{ab} g_{cd}
\]

- **Einstein’s equations:**

\[
E_{ab} = J^{cd}_{ab} \left( R_{cd} - \frac{1}{2} R g_{cd} \right) = R_{ab}
\]

\[
= \frac{1}{2} \bar{\nabla}_c \left( \sigma^{cd} \bar{\nabla}_a \sigma_{bd} + \sigma^{cd} \bar{\nabla}_b \sigma_{ad} - \sigma^{cd} \bar{\nabla}_d \sigma_{ab} \right)
+ \frac{1}{2} \left( \sigma^{ce} \sigma_{df} - \sigma^{cf} \sigma_{de} \right) \bar{\nabla}_d \sigma_{ac} \bar{\nabla}_f \sigma_{be}
+ \frac{1}{4} \bar{\nabla}_a \sigma_{cd} \bar{\nabla}_b \sigma^{cd} + \bar{R}_{ab}
\]

\[
= 8\pi G J^{cd}_{ab} T_{cd} = 8\pi G \left( T_{ab} - \frac{1}{D - 2} T g_{ab} \right)
\]
Equations of motion

- \( E_{ab} = 16\pi G \frac{\delta S}{\delta \sigma^{ab}} \)

- **Jacobian:**

  \[
  J_{ab}^{\ cd} = \sqrt{\frac{-g}{-\bar{g}}} \frac{\delta g^{ab}}{\delta \sigma^{cd}} = \frac{1}{2} \left( \delta_c^a \delta_d^b + \delta_d^a \delta_c^b \right) - \frac{1}{D-2} g^{ab} g_{cd}
  \]

- **Index-factorized form of Einstein’s equations:**

  \[
  [E_{AB}]_{x=\bar{x}} = \left[ \frac{1}{4} \nabla_C \left( \Sigma^{CD} \bar{\nabla}_B \Sigma_{AD} + \Sigma^{DC} \bar{\nabla}_A \Sigma_{DB} - \frac{1}{2} \Sigma^{CD} \bar{\nabla}_D \Sigma_{AB} \right) + \frac{1}{8} \left( \Sigma^{EC} \Sigma^{FD} - \Sigma^{FC} \Sigma^{ED} \right) \bar{\nabla}_D \Sigma_{AC} \bar{\nabla}_F \Sigma_{EB} + \frac{1}{8} \bar{\nabla}_A \Sigma_{CD} \bar{\nabla}_B \Sigma^{CD} \right]_{x=\bar{x}} = 0
  \]
Conclusions

- Field redefinition and gauge-fixing for index factorization of $h_{a\bar{b}}$ in the action for general relativity
  - All Feynman diagrams automatically index-factorize
  - Manifest the hidden $SO(D-1,1) \times \tilde{SO}(D-1,1) \times \mathbb{Z}_2$ symmetry of gravity
- Auxiliary doubling of spacetime dimensions automates the process
- Amplitudes consistent with canonical perturbation theory
  - Two-form decouples at tree level
- Action is very simple, with far fewer terms than canonical perturbation theory
  - Cayley-like basis has extra small/large perturbation duality
- Index-factorized action generalizes nicely to arbitrary curved vacuum spacetime
- This is the initial step in a program of finding an action-level understanding of double copy