Twofold Invariances of Pure Gravity

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Building the Gravity Action

Motivation

- **Double copy** (KLT, BCJ, ...): $A_{\mathrm{GR}} \sim \sum A_{\mathrm{YM}} \bar{A}_{\mathrm{YM}}$
- Implies existence of **extra symmetry** in gravity amplitudes: $SO(D-1,1) \times \overline{SO}(D-1,1)$
  - Symmetry is hidden in the canonical perturbation theory ($g_{ab} = \eta_{ab} + h_{ab}$) of the Einstein-Hilbert action:

$$S = \int \mathrm{d}^D x \sqrt{-g} \left(\frac{R}{16\pi G} + \mathcal{L}_{\mathrm{GF}} \right)$$

  - Symmetry should be manifested as **factorization of indices** of graviton $h_{a\bar{b}}$ to all orders
  - Action invariant under double Lorentz transformations of k_a and $\bar{k}_{\bar{a}}$ and double gauge transformations of ϵ_a and $\bar{\epsilon}_{\bar{a}}$, where $h_{a\bar{b}} \sim \epsilon_a \bar{\epsilon}_{\bar{b}}$
- Find a way to make this double Lorentz symmetry manifest
- First step towards understanding the double copy of gravity and YM *from the action itself* (cf. Z. Bern and A. K. Grant, hep-th/9904026)

Index Factorization

- Define $h^n_{ab} = h_a{}^{a_1} h_{a_1}{}^{a_2} \cdots h_{a_{n-1}b}$ and $[h^n] = h^n_{ab}\eta^{ba}$.
- Graviton cycles:

$$[h^{2n}] = \text{even cycle}, \qquad [h^{2n+1}] = \text{odd cycle}$$

  - Even cycles $\implies$ compatible with index factorization
  - Odd cycles $\implies$ incompatible with index factorization
- Canonical perturbation theory introduces many odd cycles (e.g., through $\sqrt{-g}$) $\implies$ obstruction to seeing double copy symmetry

Field Redefinition and Gauge Fixing

- How can we eliminate odd graviton cycles from the action?
  - **Nonlinear field redefinition**
  - **Nonlinear, local gauge fixing**
  - Leaves amplitudes unchanged (Haag's theorem)
- Field redefinition ($\alpha_1 = 1$)

$$\begin{aligned} h_{ab} \;\to\; & \alpha_1 h_{ab} + \alpha_2 \eta_{ab}[h] \\ & + \alpha_3 h_{ab}^2 + \alpha_4 h_{ab}[h] + \alpha_5 \eta_{ab}[h^2] + \alpha_6 \eta_{ab}[h]^2 \\ & + \alpha_7 h_{ab}^3 + \alpha_8 h_{ab}^2[h] + \alpha_9 h_{ab}[h^2] + \alpha_{10} h_{ab}[h]^2 \\ & + \alpha_{11} \eta_{ab}[h^3] + \alpha_{12} \eta_{ab}[h]^2[h] + \alpha_{13} \eta_{ab}[h]^3 + \cdots \end{aligned}$$

- Gauge fixing: $\mathcal{L}_{\mathrm{GF}} = -\eta^{ab} F_a F_b$,

$$\begin{aligned} F_a = \partial^b h^{cd} (\beta_1 \; & \eta_{ab}\eta_{cd} + \beta_2 \; \eta_{ac}\eta_{bd} + \\ & \beta_3 \; h_{ab}\eta_{cd} + \beta_4 \; h_{ac}\eta_{bd} + \beta_5 \; \eta_{ab} h_{cd} + \beta_6 \; \eta_{ac} h_{bd} \\ & + \beta_7 \; \eta_{ab}\eta_{cd}[h] + \beta_8 \; \eta_{ac}\eta_{bd}[h] + \cdots) \end{aligned}$$

- Choose α_i and β_i to cancel all odd cycles in the action

Index-Factorizing the Action

Defining our fields

- Definitions:
- Field basis:

$$g_{ab} = (e^{\pi})_{ab} = \eta_{ab} + \pi_{ab} + \frac{1}{2!}\pi^2_{ab} + \cdots \qquad \text{and} \qquad g^{ab} = (e^{-\pi})^{ab},$$

where

$$\pi_{ab} = h_{ab} - \frac{1}{D-2}\eta_{ab}[h]$$

- Gauge-fixing term

$$\mathcal{L}_{\mathrm{GF}} = -\frac{D-2}{64\pi G} g^{ab}\omega_a\omega_b,$$

where

$$\omega_a = \partial_a \log\sqrt{-g} = -\frac{1}{D-2}\partial_a[h]$$

  - Classical gauge condition: $\omega_a = 0$, equivalently, $\nabla_b\nabla_a x^b = 0$
- Exponential field

$$\sigma_{ab} = \eta_{ab} + h_{ab} + \frac{1}{2!}h^2_{ab} + \cdots = (e^h)_{ab} \qquad \text{and} \qquad \sigma^{ab} = (e^{-h})^{ab}$$

The Action

- All-orders action for the graviton, expanded around flat spacetime:

$$S = \frac{1}{16\pi G} \int \mathrm{d}^D x \, \partial_a \sigma_{ce} \partial_b \sigma^{de} \left(\frac{1}{4} \sigma^{ab} \delta^c_d - \frac{1}{2} \sigma^{cb} \delta^a_d \right)$$

- Equivalent to original EH action
  - Every order in h is index-factorizable
  - Exponential field basis is suggestive (cf. NLSM)
  - Perturbation theory is simple compared to canonical expansion: $\sim \frac{3}{4} n^2$ terms compared to $\propto 2^n$ terms
- How to *automate* the process of assigning barred/unbarred indices?

Auxiliary Dimensions

- Extend the spacetime from $D \to 2D$ dimensions: $x^A = (x^a, x^{\bar{a}})$, $\partial_A = (\partial_a, \partial_{\bar{a}})$
- Original spacetime on the "diagonal": $x = \bar{x}$
- Flat metric tensors:
$$\eta_{AB} = \begin{bmatrix} \eta_{ab} & 0 \\ 0 & \eta_{\bar{a}\bar{b}} \end{bmatrix}$$
- New graviton field
$$H_{AB} = \begin{bmatrix} 0 & h_{a\bar{b}} \\ h_{\bar{a}b} & 0 \end{bmatrix},$$
where $h_{a\bar{b}}$ and $h_{\bar{a}b}$ are transposes and are arbitrary D-by-D matrices (symmetric part = graviton, antisymmetric part = two-form)

Auxiliary Dimensions

- Exponential field

$$\Sigma_{AB} = (e^H)_{AB} = \begin{bmatrix} (\cosh h)_{ab} & (\sinh h)_{a\bar{b}} \\ (\sinh h)_{\bar{a}b} & (\cosh h)_{\bar{a}\bar{b}} \end{bmatrix}$$

$$(\cosh h)_{ab} = \eta_{ab} + \frac{1}{2!} h^2_{ab} + \cdots$$

$$(\sinh h)_{a\bar{b}} = h_{a\bar{b}} + \frac{1}{3!} h^3_{a\bar{b}} + \cdots$$

- Consistent index factorization is automatic:

$$h^{2n}_{ab} = h_a{}^{\bar{a}_1} h_{\bar{a}_1}{}^{a_2} \cdots h_{a_{2n-2}}{}^{\bar{a}_{2n-1}} h_{\bar{a}_{2n-1} b}$$

$$h^{2n+1}_{a\bar{b}} = h_a{}^{\bar{a}_1} h_{\bar{a}_1}{}^{a_2} \cdots h_{\bar{a}_{2n-1}}{}^{a_{2n}} h_{a_{2n}\bar{b}}$$

Index-Factorized Action

$$S = \frac{1}{16\pi G} \int \mathrm{d}^D x \, \mathrm{d}^D \bar{x} \, \delta^D(x - \bar{x}) \partial_A \Sigma_{CE} \partial_B \Sigma^{DE} \left(\frac{1}{16} \Sigma^{AB} \delta^C_D - \frac{1}{4} \Sigma^{CB} \delta^A_D \right)$$

Auxiliary Dimensions

Index-Factorized Action

$$S = \frac{1}{16\pi G} \int \mathrm{d}^D x \, \mathrm{d}^D \bar{x} \, \delta^D(x - \bar{x}) \partial_A \Sigma_{CE} \partial_B \Sigma^{DE} \left(\frac{1}{16} \Sigma^{AB} \delta^C_D - \frac{1}{4} \Sigma^{CB} \delta^A_D \right)$$

- **Doubly Lorentz invariant**, manifest $SO(D-1,1) \times \overline{SO}(D-1,1)$
- $\mathbb{Z}_2$ **parity** swapping spacetimes:

$$x^a \leftrightarrow x^{\bar{a}}$$

$$h_{a\bar{b}} \leftrightarrow h_{\bar{a}b}$$

$$H \leftrightarrow \tau H \tau \qquad \tau = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}$$

$$\eta \leftrightarrow \tau \eta \tau = \eta$$

  - $\mathbb{Z}_2$ parity $\implies$ two-form appears in pairs in action $\implies$ decouples from tree-level graviton amplitudes

Propagator

- $\mathcal{O}_2 = -\frac{1}{4} h_{a\bar{b}} h_{c\bar{d}} K^{a\bar{b}c\bar{d}}$, where

$$K^{a\bar{b}c\bar{d}} = p^2 \eta^{ac} \eta^{\bar{b}\bar{d}} - \eta^{ac} p^{\bar{b}} p^{\bar{d}} - p^a p^c \eta^{\bar{b}\bar{d}}$$

- Antisymmetric part of $h_{a\bar{b}}$ has noninvertible kinetic term
  - But can introduce additional gauge-fixing for two-form:

$$K_\xi^{a\bar{b}c\bar{d}} = p^2 \eta^{ac} \eta^{\bar{b}\bar{d}} - \left(1 + \frac{1}{\xi}\right) \eta^{ac} p^{\bar{b}} p^{\bar{d}} - \left(1 - \frac{1}{\xi}\right) p^a p^c \eta^{\bar{b}\bar{d}}$$

  - Invert via $K_\xi^{a\bar{b}c\bar{d}} \Delta_{c\bar{d}e\bar{f}} = i \delta^a_e \delta^{\bar{b}}_{\bar{f}}$ to get:

Propagator

$$\Delta_{a\bar{b}c\bar{d}} = \frac{i}{p^2} \left(\eta_{ac} \eta_{\bar{b}\bar{d}} - (1 + \xi) \eta_{ac} \frac{p_{\bar{b}} p_{\bar{d}}}{p^2} - (1 - \xi) \frac{p_a p_c}{p^2} \eta_{\bar{b}\bar{d}} \right)$$

  - Amplitudes are ξ-independent and match those computed in canonical field basis

Alternative Representations

A Family of Index-Factorized Actions

- Can take our index-factorized action and send

$$h_{a\bar{b}} \to \sum_{m \text{ even}} \sum_{n \text{ odd}} [h^m] h^n_{a\bar{b}}$$

  while preserving index factorization
- For example, send

$$h_{a\bar{b}} \to (\sinh^{-1} h)_{a\bar{b}}, \quad \Sigma_{AB} \to \begin{bmatrix} \left(\sqrt{1+h^2}\right)_{ab} & h_{a\bar{b}} \\ h_{\bar{a}b} & \left(\sqrt{1+h^2}\right)_{\bar{a}\bar{b}} \end{bmatrix}$$

- Use this to simplify action further by reducing number of terms in perturbation expansion

A Cayley-like Basis

- For example, inspired by the Cayley basis for the NLSM, we can redefine

$$h_{a\bar{b}} \to \log\left(\frac{1+\frac{1}{2}h}{1-\frac{1}{2}h}\right)_{a\bar{b}} = h_{a\bar{b}} + \frac{1}{12}h^3_{a\bar{b}} + \frac{1}{80}h^5_{a\bar{b}} + \cdots$$

- Action just as before, but with $\sigma_{ab} \to \left(\frac{1+\frac{1}{2}h}{1-\frac{1}{2}h}\right)_{ab}$
- **New duality** under small and large perturbations, h_{ab} and h^{-1}_{ab}:
  - $h_{ab}/2 \to (h_{ab}/2)^{-1}$ sends $\sigma_{ab} \to -\sigma_{ab}$ and $S \to -S$ (i.e., flips sign of $\hbar$)
- Far simpler at $\mathcal{O}(h^n)$ compared to canonical perturbation theory: $\sim \frac{3}{16}n^2$ Lorentz-invariant terms

Generalizing to Curved Spacetime

Definitions

- **Arbitrary background spacetime** metric $\bar{g}_{ab}$
- Field basis

$$g_{ab} = \bar{g}_{ab} + \pi_{ab} + \frac{1}{2!}\pi^2_{ab} + \cdots \qquad \pi_{ab} = h_{ab} - \frac{1}{D-2}\bar{g}_{ab}[h]$$

$$\pi^n_{ab} = \pi_{ab_1}\bar{g}^{b_1 a_1}\pi_{a_1 b_2}\bar{g}^{b_2 a_2}\cdots\bar{g}^{b_{n-1}a_{n-1}}\pi_{a_{n-1}b} \qquad [h^n] = h^n_{ab}\bar{g}^{ba}$$

- Gauge-fixing

$$\mathcal{L}_{\mathrm{GF}} = -\frac{D-2}{64\pi G}g^{ab}\Omega_a\Omega_b$$

$$\Omega_a = \omega_a - \bar{\omega}_a = -\frac{1}{D-2}\partial_a[h]$$

$$\omega_a = \partial_a \log\sqrt{-g} \qquad \text{and} \qquad \bar{\omega}_a = \partial_a \log\sqrt{-\bar{g}}$$

  - Classical gauge condition: $\Omega_a = 0$, equivalently $\nabla_b\nabla_a x^b = \bar{\nabla}_b\bar{\nabla}_a\bar{x}^b$.
- Exponential field

$$\sigma_{ab} = \bar{g}_{ab} + h_{ab} + \frac{1}{2!}h^2_{ab} + \cdots = (e^h)_{ab}, \quad \sigma^{ab} = (e^{-h})^{ab}$$

A Index-Factorized Gravity Action in Curved Spacetime

Action

$$S = \frac{1}{16\pi G} \int \mathrm{d}^D x \sqrt{-\bar{g}} \left[\bar{\nabla}_a \sigma_{ce} \bar{\nabla}_b \sigma^{de} \left(\frac{1}{4} \sigma^{ab} \delta^c_d - \frac{1}{2} \sigma^{cb} \delta^a_d \right) + \sigma^{ab} \bar{R}_{ab} \right]$$

- Tadpole from $\sigma^{ab} \bar{R}_{ab}$ and matter action vanishes on background Einstein equation
- If $\bar{R}_{ab} = 0$, i.e., for *arbitrary vacuum background* (e.g., black hole), action index factorizes:

Index-Factorized Action

$$S = \frac{1}{16\pi G} \int \mathrm{d}^D x \, \mathrm{d}^D \bar{x} \, \delta^D(x - \bar{x}) \sqrt{-\bar{g}} \times$$
$$\times \bar{\nabla}_A \Sigma_{CE} \bar{\nabla}_B \Sigma^{DE} \left(\frac{1}{16} \Sigma^{AB} \delta^C_D - \frac{1}{4} \Sigma^{CB} \delta^A_D \right)$$

Equations of motion

- $E_{ab} = 16\pi G \frac{\delta S}{\delta \sigma^{ab}}$
- **Jacobian**:

$$J^{ab}{}_{cd} = \sqrt{\frac{-g}{-\bar{g}}} \frac{\delta g^{ab}}{\delta \sigma^{cd}} = \frac{1}{2}\left(\delta^a_c \delta^b_d + \delta^a_d \delta^b_c\right) - \frac{1}{D-2} g^{ab} g_{cd}$$

- Einstein's equations:

$$\begin{aligned} E_{ab} &= J^{cd}{}_{ab}\left(R_{cd} - \frac{1}{2} R\, g_{cd}\right) = R_{ab} \\ &= \frac{1}{2}\bar{\nabla}_c\left(\sigma^{cd}\bar{\nabla}_a\sigma_{bd} + \sigma^{cd}\bar{\nabla}_b\sigma_{ad} - \sigma^{cd}\bar{\nabla}_d\sigma_{ab}\right) \\ &\quad + \frac{1}{2}\left(\sigma^{ce}\sigma^{df} - \sigma^{cf}\sigma^{de}\right)\bar{\nabla}_d\sigma_{ac}\bar{\nabla}_f\sigma_{be} \\ &\quad + \frac{1}{4}\bar{\nabla}_a\sigma_{cd}\bar{\nabla}_b\sigma^{cd} + \bar{R}_{ab} \\ &= 8\pi G\, J^{cd}{}_{ab} T_{cd} = 8\pi G\left(T_{ab} - \frac{1}{D-2} T g_{ab}\right) \end{aligned}$$

Equations of motion

- $E_{ab} = 16\pi G \frac{\delta S}{\delta \sigma^{ab}}$
- **Jacobian**:

$$J^{ab}{}_{cd} = \sqrt{\frac{-g}{-\bar{g}}} \frac{\delta g^{ab}}{\delta \sigma^{cd}} = \frac{1}{2}\left(\delta^a_c \delta^b_d + \delta^a_d \delta^b_c\right) - \frac{1}{D-2} g^{ab} g_{cd}$$

- **Index-factorized form of Einstein's equations**:

$$\begin{aligned}
[E_{AB}]_{x=\bar{x}} = \Bigg[& \frac{1}{4}\bar{\nabla}_C \left(\Sigma^{CD}\bar{\nabla}_B \Sigma_{AD} + \Sigma^{DC}\bar{\nabla}_A \Sigma_{DB} - \frac{1}{2}\Sigma^{CD}\bar{\nabla}_D \Sigma_{AB}\right) \\
& + \frac{1}{8}\left(\Sigma^{EC}\Sigma^{FD} - \Sigma^{FC}\Sigma^{ED}\right)\bar{\nabla}_D \Sigma_{AC} \bar{\nabla}_F \Sigma_{EB} \\
& + \frac{1}{8}\bar{\nabla}_A \Sigma_{CD} \bar{\nabla}_B \Sigma^{CD} \Bigg]_{x=\bar{x}} \\
= 0 &
\end{aligned}$$

Conclusions

- Field redefinition and gauge-fixing for index factorization of $h_{a\bar{b}}$ in the action for general relativity
  - All Feynman diagrams automatically index-factorize
  - Manifest the hidden $SO(D-1,1) \times \overline{SO}(D-1,1) \times \mathbb{Z}_2$ symmetry of gravity
- Auxiliary doubling of spacetime dimensions automates the process
- Amplitudes consistent with canonical perturbation theory
  - Two-form decouples at tree level
- Action is very simple, with far fewer terms than canonical perturbation theory
  - Cayley-like basis has extra small/large perturbation duality
- Index-factorized action generalizes nicely to arbitrary curved vacuum spacetime
- This is the initial step in a program of finding an action-level understanding of double copy