New integration-by-parts techniques for gravity amplitudes

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Talk at Gravity Meets QCD, in collaboration with

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Outline

1) Background
2) Improving integration by parts
3) On-shell IBP & Pure gravity @ 2 loops
4) Logarithmic vacuum IBP & SUGRA @ 4- and 5-loops
Challenges in gravity calculations

- Many intrigues about gravity in the UV, warranting explicit calculations.

- **2-loop pure gravity**: quantum equivalence & evanescent operators. High tensor powers
  [Bern, Cheung, Chi, Davies, Dixon, Nohle ’15]

- **5-loop \( \mathcal{N} = 8 \) Supergravity**: enhanced UV cancellation? High loop order
  [Gang Yang ’16, Wei-Ming Chen’s talk]
Problem 1: Two-loop pure gravity

Conventional dim. reg. used
$1/\epsilon$ coefficient changes,
$\ln \mu$ dependence does not!

[Bern, Cheung, Chi, Davies, Dixon, Nohle ’15]

[Duff, Van Nieuwenhuizen, ’80, Siegel ’81, Fradkin, Tseytlin ’85, Grisaru, Neilsen, Siegel, Zanon 84, Sezgin, Van Nieuwenhuizen ’84]
Problem 1: Two-loop pure gravity

- Previously (++++) amplitude studied. Is the finite part of amplitude for general helicities also duality-invariant?  
  [4d approach: Dixon, Chi]

- Loop-level BCJ: pure gravity = pure YM ⊗ pure YM 
  Rank-8 tensors involved.  [Bern, Davies, Nohle ’15]
Problem 2: SUGRA UV cancellations

- Which integral-level properties behind enhanced cancellations?
  - e.g. $\mathcal{N} = 4$, 3 loops: Bern, Davies, Dennen, Huang '12
  - $\mathcal{N} = 5$, 4 loops: Bern, Davies, Dennen '14

- **Critical dimension** of 5-loop $\mathcal{N} = 8$ SUGRA?
  - $d = 24/5$ (symmetry prediction), or
  - $d = 26/5$ (enhanced cancellation)

- Need efficient method for extracting 5-loop UV divergences. Conventional IBP seems difficult.
Integration by parts

Total derivatives integrate to zero in dim. reg.

Vacuum bubble reduction example:

\[ F(a) = \int d^d l \frac{1}{(l^2 - m^2)^a}, \quad \int d^d l \frac{\partial}{\partial l^\mu} \left[ \frac{l^\mu}{(l^2 - m^2)^a} \right] = 0 \]

\[ \Rightarrow (d - 2a) F(a) - 2am^2 F(a + 1) = 0 \quad [\text{Smirnov '12 book}] \]

\[ F(1) = -i\pi^{d/2} \Gamma(1 - d/2)(m^2)^{d/2-1} \quad \text{the master integral} \]
Integration by parts – improved basis

- For full amplitude, only interested in tensor integrals *w/o doubled propagators*.  
  - Related by *on-shell IBP reduction*

  Syzygy approach - Gluza, Kajda, Kosower ’10, Schabinger ’11, Chen, Liu, Xie, Zhang, Zhou ’15,
  Algebraic curves - Zhang ’14, Sogaard, Zhang ’14, Georgoudis, Zhang ’15,
  Full IBP reduction: Larsen, Zhang ’15, Ita ’15

- For UV divergence, only interested in *log-divergent vacuum graphs*.
  - Propose *Logarithmic vacuum reduction*
On-shell IBP reduction

- IBP relation

\[ \int d^d l \frac{\partial}{\partial l^\mu} \left( \frac{v_\mu(l, p)}{\prod_i \rho_i} \right) = 0, \quad \rho_i = (l - q_i)^2 - m_i^2 \]

- “Good” IBP vector satisfies

\[ v_\mu \frac{\partial \rho_i}{\partial l^\mu} = \rho_i \cdot f_i(l, p) \]

IBP relation w/o doubled propagators!

[Gudja, Kluza, Kosower ’10]

Syzygy equations, a problem in computational algebraic geometry
On-shell IBP: Larsen-Zhang method

- **Change coordinates** \((l^0, l^1, \ldots)\) to inverse propagators & irreducible numerators \((\rho_i, \rho_j^{irred})\)

- Merging results from a spanning set of cuts

[Larsen, Zhang ’15]
On-shell IBP reduction – method 2

- $v^\mu$ is a **tangent vector** to the maximal-cut surface!

$$v^\mu \frac{\partial \rho_i}{\partial l^\mu} = \rho_i \cdot f_i(l,p)$$

- **One-loop cut surfaces** are spheres. [H. Ita ’15]
  
  Tangent vectors: rotation generators.
Finding good IBP relations – method 2

- $v^\mu$ is a **tangent vector** to the maximal-cut surface!

\[ v^\mu \frac{\partial \rho_i}{\partial l^\mu} = \rho_i \cdot f_i(l, p) \]

- **Zero radius case**: conical surface
  
  Tangent vectors: scalings from singularity
Finding good IBP relations – method 2

- Planar 2-loop cut = (1 loop cut) + (1 loop cut)’ + (central rung cut). Combine 1-loop vectors.

- Works for 4 dims [Ita ’15] and d dimensions [Ita, Jacquier, MZ, in progress]
2-loop pure gravity results

Two planar + One non-planar + Self energy + daughter graphs
2-loop pure gravity (rank 8) results

- **Double box**: (++++) in ~ 80 mins, one CPU, ~ 100 × faster than FIRE5 [Smirnov '15].

- General polarizations: used fake tensor projector (YM projector $P_3$ squared [Kosower]). Takes 1-2 days. (Alternative: direct calculation w/o projection)

- **Non-planar**: Tested at rank-4 (pure YM, $\mathcal{N} = 4$ SUGRA).

- **Internal self energy**: Extension of LZ method
  \[ \int d^d l \frac{\partial}{\partial l^\mu} \left( \frac{\nu^\mu(l, p)}{\rho^1 \rho^2 \cdots \rho^2_k \cdots \rho_n} \right) = 0 \]
Logarithmic vacuum reduction

Toy example: 2-loop $\mathcal{N} = 4$ SUGRA in $d = 5$. Need integral-level properties to explain? Yes.

Color structure explanation in [Bern, Dennen, Huang '12]

Taylor expansion

Lorentz invariance

See [Mastrolia, Peraro, Primo '16]

$= 0$?
IBP for vacuum graphs

\[ \wedge a \wedge b \wedge c = I_{a,b,c} \quad \text{invariant under } S_3 \text{ permutation of } (a, b, c). \]

\[ = \int d^{5-2\epsilon} l_1 \, d^{5-2\epsilon} l_2 \frac{1}{(l_1^2 - m^2)^a (l_2^2 - m^2)^b [(l_1 + l_2)^2 - m^2]^c} \]

Need **relations between UV poles** of different \( I_{a,b,c} \).
IBP for vacuum graphs

\[ 0 = \int d^{5-2\epsilon} l_1 \, d^{5-2\epsilon} l_2 \, \frac{\partial}{\partial l_1^\mu} \left[ \frac{l_B^\mu}{(l_1^2 - m^2)^a (l_2^2 - m^2)^b [(l_1 + l_2)^2 - m^2]^c} \right], \]

where \( A, B = 1, 2 \), \( a + b + c = 5 \). RHS_{\text{A=B=1}} - RHS_{\text{A=B=2}}

\[ \Rightarrow 0 = \text{Res}_{\epsilon=0} \left[ (b - a) \cdot I_{a,b,c} - c \cdot I_{a-1,b,c+1} + c \cdot I_{a,b-1,c+1} \right]_{m=0} + O \left( \frac{m^2}{l^{2.6}} \right) \]

UV finite from naïve power counting
IBP for vacuum graphs

\[ 0 = \text{Res}_{\epsilon=0} \left[ (b - a) \cdot I_{a,b,c} - c \cdot I_{a-1,b,c+1} + c \cdot I_{a,b-1,c+1} \right]_{m=0} \]

With \( a = 1, b = 2, c = 2, \)

\[ 0 = \text{Res}_{\epsilon=0} \left[ I_{1,2,2} + 2I_{1,1,3} - 2I_{0,2,3} \right] \]

So UV div. = 0. Clean cancellation using one IBP relation.

UV finite b/c no sub-divergence
General prescription for vacuum IBP

Compute, setting $m = 0, \epsilon = 0$,

(Applicable if no sub-divergence)

$$0 = \int d^d l_1 d^d l_2 \frac{\partial}{\partial l_1^\mu} \frac{l_2^\mu}{\prod \rho_i^{a_i}}, \quad 1 \leq A, B \leq L, \quad dL - 2 \sum a_i = 0$$

Involving only log-divergent integrals, and all such integrals.
Vacuum graphs @ 4-loops, $d = 11/2$

- 4-loop critical dimension of $\mathcal{N} = 8$ SUGRA.

\[\begin{align*}
  &\text{“Prism graph"}, \\
  &|S_3 \times Z_2| = 12 \text{ fold symmetry}
\end{align*}\]

\[\begin{align*}
  &\text{“Utility graph"}, \\
  &|S_3 \times S_3 \times Z_2| = 72 \text{ fold symmetry}
\end{align*}\]
4-loop results

- ~ 1400 integrals, with $\leq 1$ numerators & $\leq 1$ cancelled propagators, reduced to 3 master integrals in ~ 6 mins by Mathematica code

- Agree with [Bern, Carrasco, Dixon, Johansson, Roiban ’12]
Vacuum graphs @ 5-loops, \( d = \frac{22}{5} \)

- 5-loop \( \mathcal{N} = 4 \) SYM form factor integrand obtained in BCJ form [Gang Yang, '16].
- UV behavior of double copy?
- Five cubic vacuum diagrams.

[Wikipedia]
5-loop cube topology, $d = 22/5$

- 48-fold symmetry,
- 15 dot products $l_A \cdot l_B, 1 \leq A, B \leq 5$
- 12 propagators, need 3 irreducible numerators

$$N_1 = \left( \sum_{\text{clockwise}} l_{\text{top}} \right) \cdot \left( \sum_{\text{clockwise}} l_{\text{bottom}} \right)$$

3 pairs of opposite faces, defining irred. numerators $N_1, N_2, N_3$
5-loop cube topology - results

- ~ 15000 integrals, ≤ 4 numerators, ≤ 4 canceled propagators, reduced to 8 master integrals.

- Sufficient for reducing cube & daughter vacuum diagrams [Roiban] from BCJ double-copy form factor.

- Used fast linear solver based on finite-field methods, Finred [v. Manteuffel ’15]
Conclusion

- Recent improvements of integration-by-parts achieved by careful **trimming of the basis**.

- **On-shell IBP** without doubled propagators:
  First real application in **2-loop pure gravity**

- **Vacuum IBP** with only log-divergent integrals:
  efficient extraction of (e.g. 5-loop) **UV divergences** & isolate integral properties behind **enhanced cancellations in SUGRA**
Thank you!