New integration-by-parts techniques for gravity amplitudes

Mao Zeng, Bhaumik Institute for Theoretical Physics, UC Los Angeles

Talk at *Gravity Meets QCD*, in collaboration with

- **Zvi Bern, Michael Enciso, Julio Parra Martinez**
- **Harald Ita, Mattheiu Jacquier**

Outline

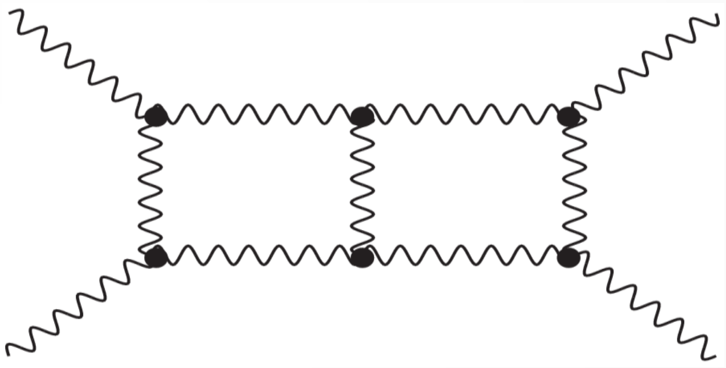
1) Background
2) Improving integration by parts
3) On-shell IBP & Pure gravity @ 2 loops
4) Logarithmic vacuum IBP & SUGRA @ 4- and 5-loops

Challenges in gravity calculations

- Many intrigues about gravity in the UV, warranting explicit calculations.
- **2-loop pure gravity**: quantum equivalence & evanescent operators. High tensor powers
  [Bern, Cheung, Chi, Davies, Dixon, Nohle '15]
- **5-loop $\mathcal{N} = 8$ Supergravity**: enhanced UV cancellation? High loop order
  [Gang Yang '16, Wei-Ming Chen's talk]

Problem 1: Two-loop pure gravity

Conventional dim. reg. used
$1/\epsilon$ coefficient changes,
$\ln\mu$ dependence does not!
[Bern, Cheung, Chi, Davies, Dixon, Nohle '15]



Cosmological constant ⟷ **3-form matter**

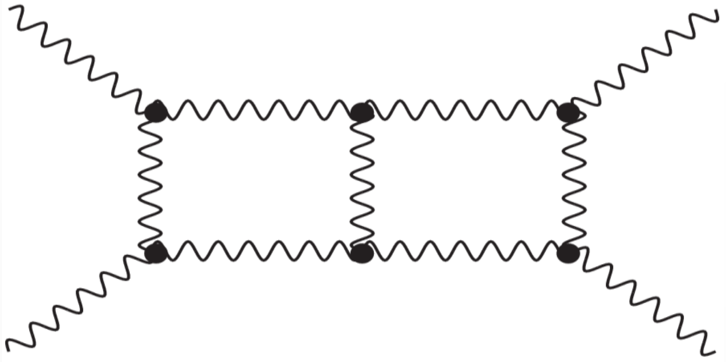
Anti-symm. tensor ⟷ **2-form matter**

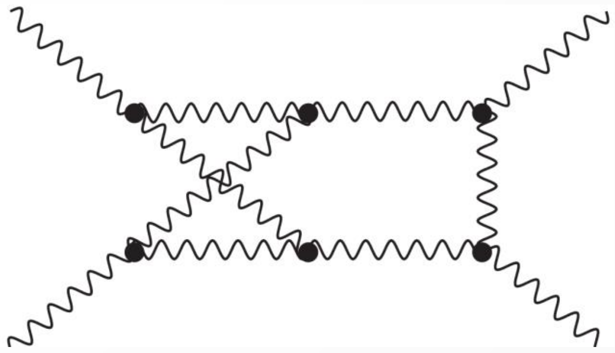
[Duff, Van Nieuwenhuizen, '80, Siegel '81, Fradkin, Tseytlin '85, Grisaru, Neilsen, Siegel, Zanon 84, Sezgin, Van Nieuwenhuizen '84]

Problem 1: Two-loop pure gravity

- Previously (++++) amplitude studied. Is the **finite part** of amplitude for **general helicities** also duality-invariant?
  [4d approach: Dixon, Chi]

- Loop-level BCJ: pure gravity = pure YM ⊗ pure YM
  Rank-8 tensors involved. [Bern, Davies, Nohle ’15]





Problem 2: SUGRA UV cancellations

- Which integral-level properties behind *enhanced cancellations*?

  e.g. $\mathcal{N}=4$, 3 loops: Bern, Davies, Dennen, Huang '12
  $\mathcal{N}=5$, 4 loops: Bern, Davies, Dennen '14

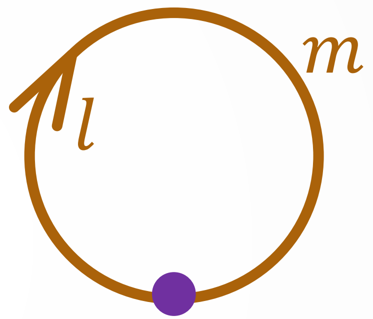
- **Critical dimension** of 5-loop $\mathcal{N}=8$ SUGRA ?
  $d=24/5$ (symmetry prediction), or
  $d=26/5$(enhanced cancellation) ?

- Need efficient method for extracting **5-loop UV divergences**. Conventional IBP seems difficult.

Integration by parts

Total derivatives integrate to zero in dim. reg.

Vacuum bubble reduction example:



$$F(a) = \int d^d l \, \frac{1}{(l^2 - m^2)^a}, \qquad \int d^d l \, \frac{\partial}{\partial l^\mu} \left[\frac{l^\mu}{(l^2 - m^2)^a} \right] = 0$$

$$\Longrightarrow \ (d - 2a) F(a) - 2am^2 F(a+1) = 0$$ [Smirnov '12 book]

$$F(1) = -i\pi^{d/2} \Gamma(1 - d/2) (m^2)^{d/2-1}$$ the **master integral**

Integration by parts – improved basis

- For full amplitude, only interested in tensor integrals **w/o doubled propagators**.
  - Related by **on-shell IBP reduction**

Syzygy approach - Gluza, Kajda, Kosower '10, Schabinger '11,
Chen, Liu, Xie, Zhang, Zhou '15,
Algebraic curves - Zhang '14, Sogaard, Zhang '14, Georgoudis, Zhang '15,
Full IBP reduction: Larsen, Zhang '15, Ita '15

- For UV divergence, only interested in **log-divergent vacuum graphs**.
  - Propose **Logarithmic vacuum reduction**

On-shell IBP reduction

[Gudja, Kluza, Kosower '10]

- IBP relation

$$\int d^d l \, \frac{\partial}{\partial l^\mu} \left(\frac{\overbrace{v^\mu(l,p)}^{\text{IBP vector}}}{\prod_i \rho_i} \right) = 0, \quad \rho_i = (l - q_i)^2 - m_i^2$$

- "Good" IBP vector satisfies

$$v^\mu \frac{\partial \rho_i}{\partial l^\mu} = \rho_i \cdot f_i(l,p)$$

Syzygy equations, a problem in computational algebraic geometry

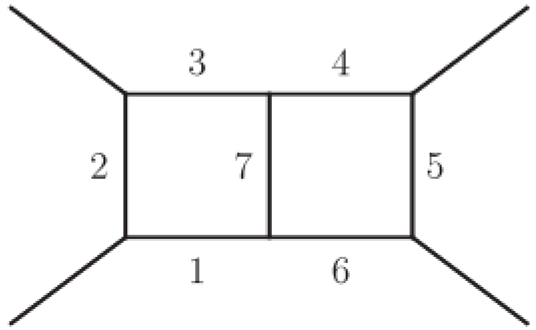
IBP relation w/o doubled propagators!

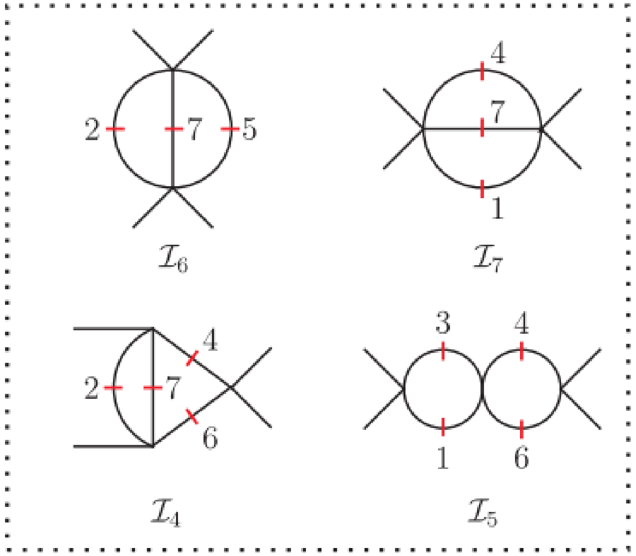
On-shell IBP: Larsen-Zhang method

[Larsen, Zhang ’15]

- **Change coordinates** $(l^0, l^1, \ldots)$ to inverse propagators & irred. numerators (ρ_i, ρ_j^{irred})

- Merging results from **a spanning set of cuts**



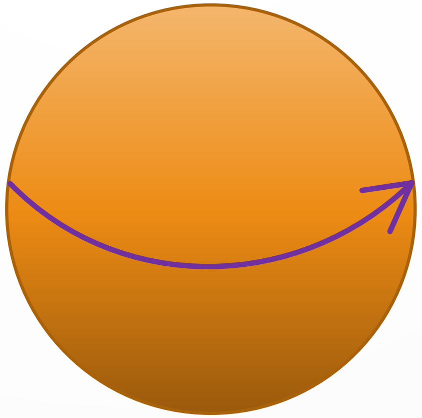


On-shell IBP reduction – method 2

- v^μ is a **tangent vector** to the maximal-cut surface!

$$v^\mu \frac{\partial \rho_i}{\partial l^\mu} = \rho_i \cdot f_i(l,p)$$

- **One-loop cut surfaces** are spheres. [H. Ita '15]
  Tangent vectors: rotation generators.

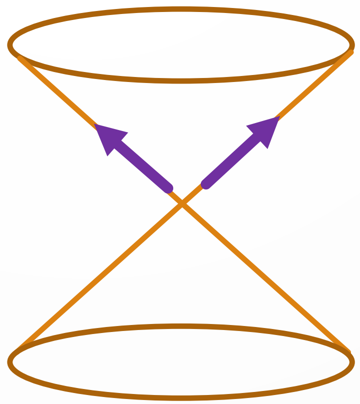


Finding good IBP relations – method 2

- v^μ is a **tangent vector** to the maximal-cut surface!

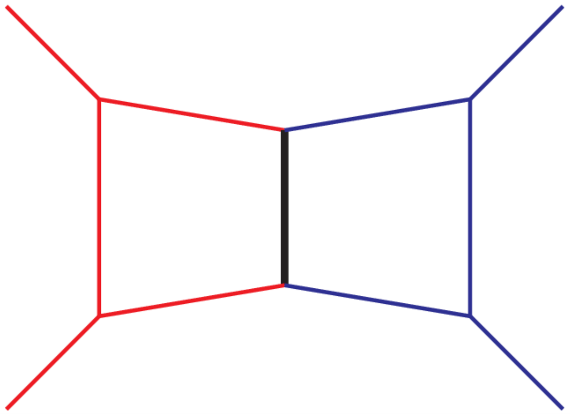
$$v^\mu \frac{\partial \rho_i}{\partial l^\mu} = \rho_i \cdot f_i(l,p)$$

- **Zero radius case**: conical surface
  Tangent vectors: scalings from singularity



Finding good IBP relations – method 2

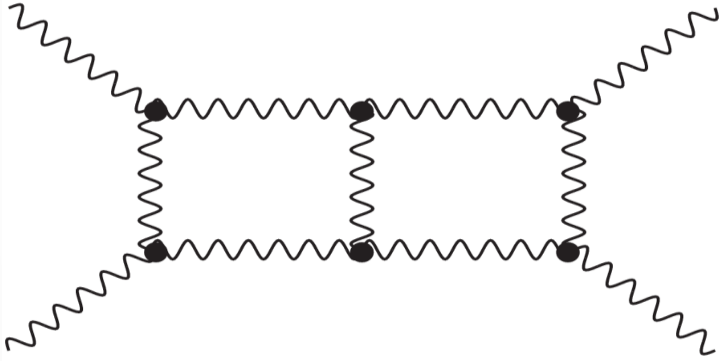
- Planar 2-loop cut = (1 loop cut) + (1 loop cut)' + (central rung cut). Combine 1-loop vectors.

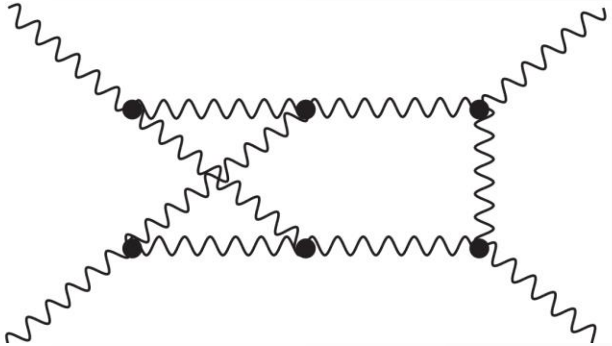


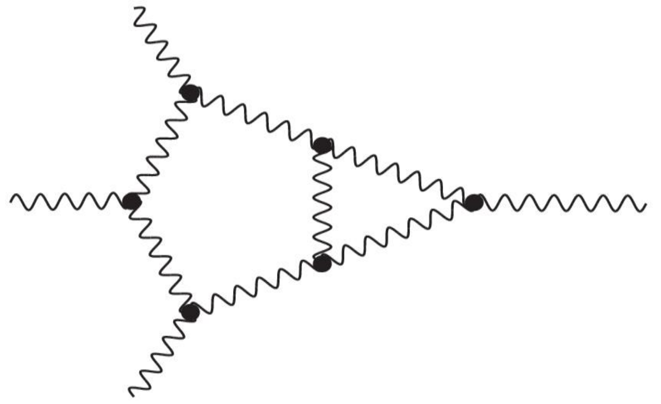
- Works for 4 dims [Ita '15] and d dimensions [Ita, Jacquier, MZ, in progress]

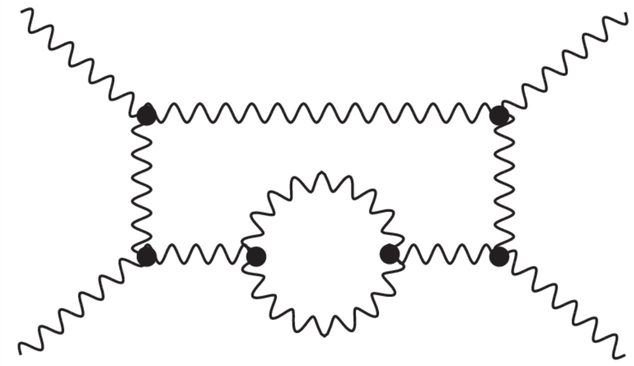
2-loop pure gravity results

Two planar + One non-planar + Self energy + daughter graphs

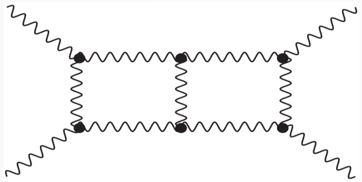




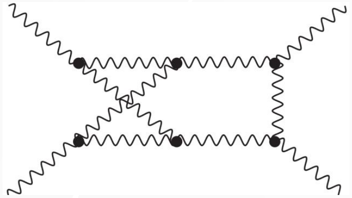




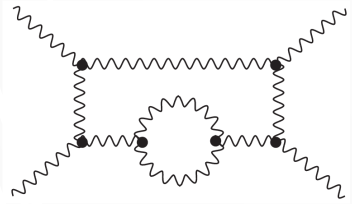
2-loop pure gravity (rank 8) results



- **Double box**: (++++) in ~ 80 mins, one CPU, $\sim 100\times$ faster than FIRE5 [Smirnov '15].
- General polarizations: used fake tensor projector (YM projector P_3 squared [Kosower]). Takes 1-2 days. (Alternative: direct calculation w/o projection)



- **Non-planar**: Tested at rank-4 (pure YM, $\mathcal{N}=4$ SUGRA).



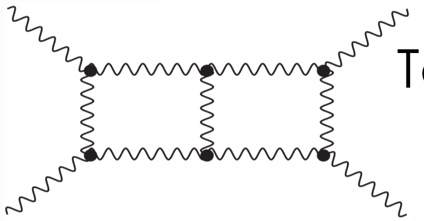
- **Internal self energy**: Extension of LZ method

$$\int d^d l \, \frac{\partial}{\partial l^\mu} \left(\frac{v^\mu(l,p)}{\rho^1 \rho^2 \dots \rho_k^2 \dots \rho_n} \right) = 0$$

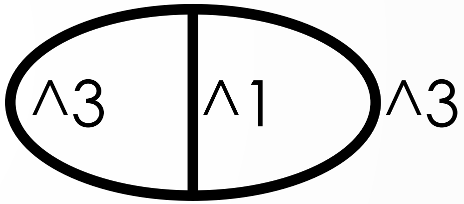
Logarithmic vacuum reduction

- Toy example: 2-loop $\mathcal{N} = 4$ SUGRA in $d = 5$. Need integral-level properties to explain? **Yes**.

  Color structure explanation in [Bern, Dennen, Huang ’12]

- 

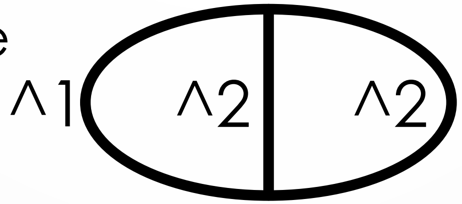
  Taylor expansion $\longrightarrow$

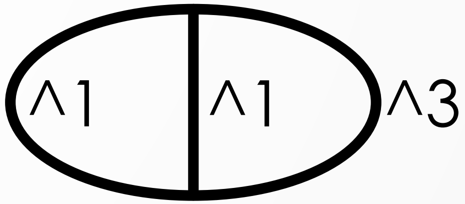
  $\cdot\, N(p_i, l_i)$

Lorentz invariance $\longrightarrow$

See [Mastrolia, Peraro, Primo ’16]

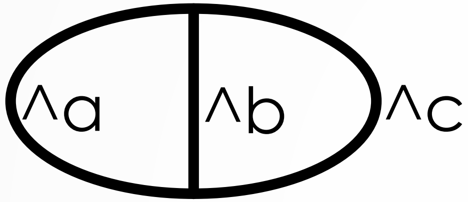


$+2\,\cdot$



$= 0\,?$

IBP for vacuum graphs



$= I_{a,b,c}$

invariant under S_3 permutation of (a, b, c).

$$= \int d^{5-2\epsilon} l_1 \, d^{5-2\epsilon} l_2 \frac{1}{(l_1^2 - m^2)^a (l_2^2 - m^2)^b [(l_1 + l_2)^2 - m^2]^c}$$

Need **relations between UV poles** of different $I_{a,b,c}$.

IBP for vacuum graphs

$$0 = \int d^{5-2\epsilon} l_1 \, d^{5-2\epsilon} l_2 \, \frac{\partial}{\partial l_A^\mu} \left[\frac{l_B^\mu}{(l_1^2 - m^2)^a (l_2^2 - m^2)^b [(l_1 + l_2)^2 - m^2]^c} \right],$$

where $A, B = 1, 2, \; a + b + c = 5.$ $\text{RHS}_{A=B=1} - \text{RHS}_{A=B=2}$

$$\Rightarrow 0 = \text{Res}_{\epsilon=0} \Big[(b-a) \cdot I_{a,b,c} - c \cdot I_{a-1,b,c+1} + c \cdot I_{a,b-1,c+1} \Big]_{m\,=\,0}$$

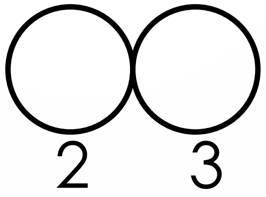
$$\cancel{+O\underbrace{\left(\frac{m^2}{l^{2\cdot 6}}\right)}_{\text{UV finite from naïve power counting}}}$$

IBP for vacuum graphs

$$0 = \mathrm{Res}_{\epsilon=0}\left[(b-a)\cdot I_{a,b,c} - c\cdot\ I_{a-1,b,c+1} + c\cdot\ I_{a,b-1,c+1}\right]_{m=0}$$

With $a = 1, b = 2, c = 2,$

$$0 = \mathrm{Res}_{\epsilon=0}\left[\ I_{1,2,2} + 2\ I_{1,1,3} - \underbrace{\cancel{2\ I_{0,2,3}}}\right]$$



UV finite b/c no sub-divergence

So UV div. $=0$. Clean cancellation using one IBP relation.

General prescription for vacuum IBP

Compute, setting $m = 0, \epsilon = 0$,

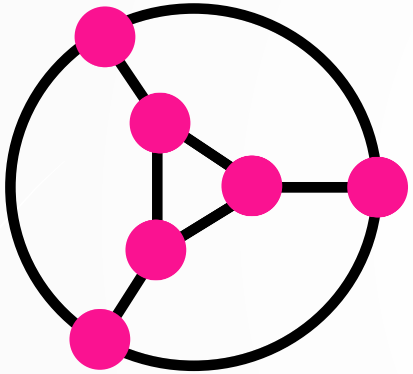
(Applicable if no sub-divergence)

$$0 = \int d^d l_1 d^d l_2 \frac{\partial}{\partial l_A^\mu} \frac{l_B^\mu}{\prod \rho_i^{a_i}}, \quad 1 \le A, B \le L, \quad d\,L - 2\sum_i a_i = 0$$

Involving only **log-divergent** integrals, and all such integrals.

Vacuum graphs @ 4-loops, $d = 11/2$

- 4-loop critical dimension of $\mathcal{N} = 8$ SUGRA.



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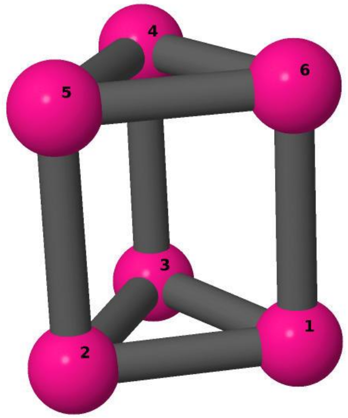
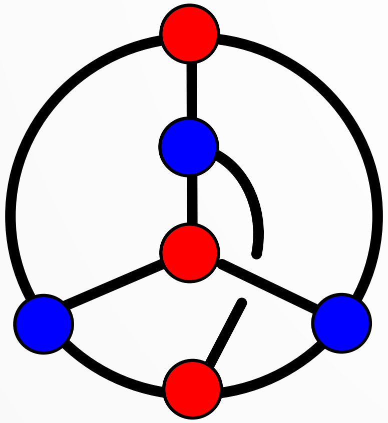


"Prism graph",
$|S_3 \times Z_2| = 12$ fold
symmetry



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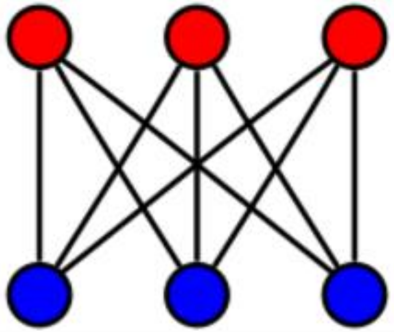
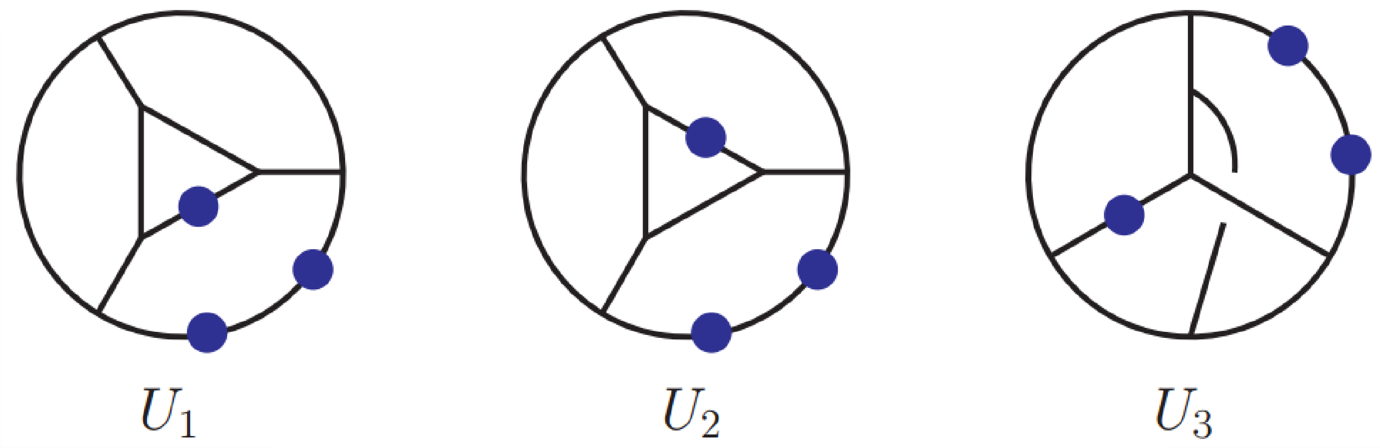


[Wikipedia]

"Utility graph",
$|S_3 \times S_3 \times Z_2| = 72$
fold symmetry

4-loop results

- ~ 1400 integrals, with ≤ 1 numerators & ≤ 1 cancelled propagators, reduced to **3 master integrals** in ~ 6 mins by *Mathematica* code
- Agree with [Bern, Carrasco, Dixon, Johansson, Roiban '12]



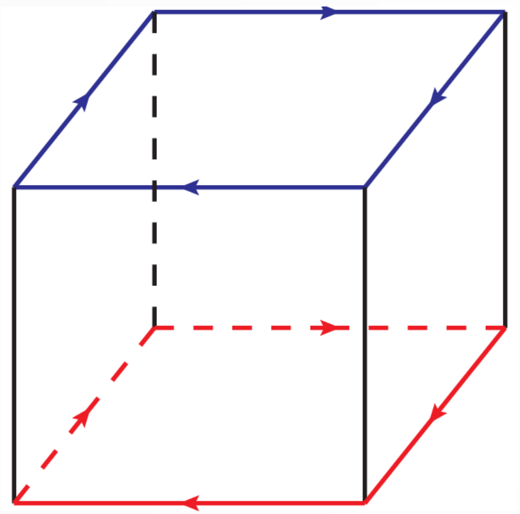
Vacuum graphs @ 5-loops, $d = 22/5$

- 5-loop $\mathcal{N} = 4$ SYM form factor integrand obtained in BCJ form [Gang Yang, '16].
  UV behavior of double copy?
- Five cubic vacuum diagrams.

[Wikipedia]

5-loop cube topology, $d = 22/5$

- 48-fold symmetry,
  15 dot products $l_A \cdot l_B, 1 \le A, B \le 5$
  12 propagators, need **3 irreducible numerators**



$$N_1 = \left(\sum_{\text{clockwise}} l_{\text{top}} \right) \cdot \left(\sum_{\text{clockwise}} l_{\text{bottom}} \right)$$

3 pairs of **opposite faces**, defining irred. numerators N_1, N_2, N_3

5-loop cube topology - results

- ~ 15000 integrals, ≤ 4 numerators, ≤ 4 canceled propagators, reduced to **8 master integrals**.
- Sufficient for reducing cube & daughter vacuum diagrams [Roiban] from BCJ double-copy form factor.
- Used **fast linear solver based on finite-field methods**, ***Finred*** [v. Manteuffel '15]

Conclusion

- Recent improvements of integration-by-parts achieved by careful **trimming of the basis**.
- **On-shell IBP** without doubled propagators: First real application in **2-loop pure gravity**
- **Vacuum IBP** with only log-divergent integrals: efficient extraction of (e.g. 5-loop) **UV divergences** & isolate integral properties behind **enhanced cancellations in SUGRA**

Thank you!