An overview of lattice QCD+QED progress for g-2

Christoph Lehner (BNL)

December 4, 2018 – Schwinger Fest 2018, UCLA
There is a tension of $3.7\sigma$ for the muon $a_\mu = (g_\mu - 2)/2$:

$$a_\mu^{\text{EXP}} - a_\mu^{\text{SM}} = 27.4 \left\{ \begin{array}{l} 2.7 \end{array} \right\} \left\{ \begin{array}{l} 2.6 \end{array} \right\} \left\{ \begin{array}{l} 0.1 \end{array} \right\} \left\{ \begin{array}{l} 6.3 \end{array} \right\} \times 10^{-10}$$

HVP  HLbL  other  EXP

2019: $\delta a_\mu^{\text{EXP}} \to 4.5 \times 10^{-10}$ (avg. of BNL/estimate of 2019 Fermilab result)

Targeted final uncertainty of Fermilab E989: $\delta a_\mu^{\text{EXP}} \to 1.6 \times 10^{-10}$

$\Rightarrow$ by 2019 consolidate HVP/HLbL, over the next years uncertainties to $O(1 \times 10^{-10})$
There is also a tension of $-2.4\sigma$ for the muon $a_e = (g_e - 2)/2$:

$$a_e^{\text{EXP}} - a_e^{\text{SM}} = -87 \times 10^{-14},$$

SM uncertainty far from dominant, however, check of five-loop QED calculation by Aoyama/Kinoshita/Nio is desirable (and a six-loop approximate answer?)

Possible future progress by lattice methods:

- Numerical Stochastic Perturbation Theory Burgio et al. 1998
  
  "The final goal of this project is . . . to push one loop further the computation of electron's $g$-2"

- Diagrammatic Monte-Carlo Prokof'ev & B.V.Svistunov 1998
The HVP contribution to the muon g-2

Talks by Mattia Bruno (Mo/4:30), Hartmut Wittig (Tue/3:00), Vera Guelpers (Tue/4:00), Kotaroh Miura (Tue/4:30), Christine Davies (Tue/5:00), Davide Giusti (Tue/5:30), Marina Marinkovic (Wed/10:45), Aaron Meyer (Wed/11:30)

4 hours of lattice talks on HVP
Status of HVP determinations

Green: LQCD, Orange: LQCD+Dispersive, Purple: Dispersive
Knowledge of isospin-breaking corrections and separation of vector and axial-vector components needed to use $\tau$ decay data. This connection can be provided by lattice QCD+QED. (Talk M. Bruno)
Starting from the vector current $J_\mu(x) = i \sum_f Q_f \overline{\Psi}_f(x) \gamma_\mu \Psi_f(x)$ we may write

$$a_{\mu}^{\text{HVP LO}} = \sum_{t=0}^{\infty} w_t C(t)$$

with

$$C(t) = \frac{1}{3} \sum_{\vec{x}} \sum_{j=0,1,2} \langle J_j(\vec{x}, t) J_j(0) \rangle$$

and $w_t$ capturing the photon and muon part of the HVP diagrams (Bernecker-Meyer 2011).

The correlator $C(t)$ is computed in lattice QCD+QED at physical pion mass with non-degenerate up and down quark masses including up, down, strange, and charm quark contributions. The missing bottom quark contributions are computed in pQCD.
Small interlude - Lattice QCD

- Simulate QFT in terms of fundamental quarks and gluons (QCD) on a supercomputer with discretized four-dimensional space-time lattice
- Hadrons are emergent phenomena of statistical average over background gluon configurations to which quarks are coupled
- In this framework draw diagrams only with respect to quarks, photons, and leptons; gluons and their effects are generated by the statistical average.

Lattice QCD action density, Leinweber, CSSM, Adelaide, 2003
Computing resources (example RBC/UKQCD)

The RBC/UKQCD $g - 2$ project has used on the order of $10^9$ core hours (100k years on a single core) on the Mira supercomputer at Argonne, USQCD clusters at JLab and BNL, the BNL CSI KNL cluster, and the Oakforest and Hokusai supercomputers in Japan.

We have processed on the order of 5 petabytes of QCD data related to this project.
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Next generation of runs on Summit in preparation

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Next generation of runs on Summit in preparation

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Top 10 positions of the 52nd TOP500 in November 2018[16]
Diagrams – Isospin limit

FIG. 1. Quark-connected (left) and quark-disconnected (right) diagram for the calculation of $a_{\mu}^{\text{HVP LO}}$. We do not draw gluons but consider each diagram to represent all orders in QCD.
with $C(t) = 1$.

Perturbations around an isospin-symmetric lattice QCD do not include any time-like region above 2 GeV and find them to be degenerate up and down quark masses. We compute the correlator

$$C(t) = \sum_{j=0,1,2} P_j(0)i.$$  

The appropriate definition of $w_t$, we can therefore write

$$a_{\mu, ud, conn, isospin} \times 10^{10}$$
with \( C(t) = 1 \)

\[
P \sum_{j=0,1,2} h J_j(x, t) J_j(0) i.\]

With appropriate definition of \( w_t \), we can therefore write...

\[
a \mu, s, \text{conn, isospin} \times 10^{10}
\]

**FIG. 3. Strong isospin-breaking correction diagrams.** The crosses denote the insertion of a scalar operator.

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**FIG. 2.** QED-correction diagrams with external pseudo-scalar (left) and vector operators.

**FIG. 7.** Mass-splitting and HVP 1-photon diagrams. In the former the dots are internal photon vertices, and in the latter the dots are external photon vertices. Note that for the HVP some of them (such as \( F \) with no gluons between the two quark loops) are meson operators, in the latter the dots are external photon vertices. Note also that some diagrams are absent for flavor quark loops; we need to make sure not to double-count those, i.e., we need to include the appropriate subtractions! Also note that some diagrams are neglected diagrams are both SU(3) and 1

**Mainz 2018 (prelim)**

**RBC/UKQCD 2018**

**BMW 2017**

**ETMC 2017**

**Mainz 2017**

**HPQCD 2014**

---

The 

\[
\frac{e^2}{4 \pi} a \mu, s, \text{conn, isospin} \times 10^{10}
\]

corrections to the masses \([20]\) with spatial lattice size \( l = 48\) and a heavy quark with mass \( m = 0.00050(1) \), and \( 0.0002(2) \) for the charm \([22]\) and QED corrections. The shifts due to the QED correction is significantly smaller than 0.01 and 0.03, respectively. We keep only the leading corrections explained in Ref. [18]. We use the finite-volume QED prescription [19] and remove the universal 1

The 

\[
 C(4) = 0
\]

corrections to the masses [20] with spatial lattice size \( l = 48\) and three degenerate light quarks and non-degenerate up and down quark masses. We compute the 

\[
P \sum_{j=0,1,2} h J_j(x, t) J_j(0) i.\]

with appropriate definition of \( w_t \), we can therefore write

\[
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P \sum_{j=0,1,2} h J_j(x, t) J_j(0) i.\]

with appropriate definition of \( w_t \), we can therefore write

\[
a \mu, s, \text{conn, isospin} \times 10^{10}
\]
with \( C(t) = \frac{1}{3} \sum P_j \mathcal{J}_j(x, t) \mathcal{J}_j(0) \). With appropriate definition of \( \omega \), we can therefore write

\[ a_\mu, c, \text{conn}, \text{isospin} \times 10^{10} \]

FIG. 3. Strong isospin-breaking correction diagrams. The crosses denote the insertion of a scalar operator.

HPQCD 2014
Mainz 2017
ETMC 2017
BMW 2017
RBC/UKQCD 2018
Mainz 2018 (prelim)
We write $a_{\mu} = \frac{1}{3} \sum_{k=0}^{2} \langle j(X_0, t) J_k(0) \rangle$ with appropriate definition of $\omega_t$, we can therefore write (capture small).

**FIG. 3. Strong isospin-breaking correction diagrams.** The crosses denote the insertion of a scalar operator.

The correlation

$C(t) = \frac{1}{3} \sum_{j=0}^{2} \langle j(\bar{x}, t) J_j(0) \rangle$.

With appropriate definition of $\omega_t$, we can therefore write $a_{\mu}$... small).

**FIG. 2. QED-correction diagrams with external pseudo-scalar and vector operators.**

- **Diagram R:** The sea quark mass shift.
- **Diagram M:** The valence quark mass shift.
- **Diagram O:** The contribution to the HVP.
- **Diagram F:** The disconnected contribution.
- **Diagram S and V:** The QED-connected and QED-disconnected contributions.

For the hadronic vacuum polarization, the contribution of $O$ gives a (10%) correction for isospin splittings [21] for which the disconnected contribution (that likely is very small).

We tune the bare up, down, and strange quark masses to their experimental values. We perform the calculation as a corresponding 30% uncertainty.

The charm sea quark masses in perturbative QCD [13] by integrating missing contributions to the meson spectrum and the hadronic vacuum polarization. The external vertices are pseudo-scalar operators that for the HVP some of them (such as $F$ with no gluons between the two quark loops) are non-diagonal operators. We need to make sure not to double-count those, i.e., we need to include the photon line is possible. For this reason, we subtract the gluons exchanged between the two quark loops contribute a corresponding $10\%$ uncertainty.

Note that only the parts of diagram $F$ with additional gluons exchanged between the two quark loops contribute. We refer to diagrams $S$ and $V$ as the QED-connected and to $F$ as the QED-disconnected contribution. We define the QED and strong isospin-breaking (SIB) correction is computed by inserting scalar operators.

The e depends on the respective experimental measurements [14]. The lattice spacing is determined by setting the charm sea quark masses. We compute the mass of 495.7 MeV [17]. The correlator is expanded in perturbation around an isospin-symmetric lattice QCD.

The correlator $C(t) = \frac{1}{3} \sum_{j=0}^{2} \langle j(\bar{x}, t) J_j(0) \rangle$ is therefore not included separately. With appropriate definition of $\omega_t$, we can therefore write $a_{\mu}$... small).

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We write $a_{\mu} = \frac{1}{3} \sum_{k=0}^{2} \langle j(X_0, t) J_k(0) \rangle$ with appropriate definition of $\omega_t$, we can therefore write (capture small).

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Diagrams – QED corrections

For the finite-volume errors, the two-pion states in $d^0$ are identical to the $I = 1$ contributions of $c$ and can be calculated using the GSL estimate which we use for $c$. For the omega-related finite-volume errors, I will take the fitted $d^0$ and $E^0$ and use this as the full result at finite-volume and compare it to a GS model with omega mass from the fitted $E^0$ and width from the PDG in infinite-volume. I should also compare this to R-ratio results for the $I = 0$ channel.

Do this entire exercise for 24ID and 32ID to estimate discretization errors.

4 QED and SIB diagrams

We will perform a full first-principles calculation of all $O(\alpha)$ and $O(m_u m_d)$ corrections. The corresponding list of diagrams is given in Figs. 1 and 2.

For diagram F we enforce exchange of gluons between the quark loops as otherwise a cut through a single photon line would be possible. This single-photon contribution is counted as part of the HVP NLO and not included for the HVP LO.
For the finite-volume errors, the two-pion states in $d$ are identical to the $I = 1$ contributions of $c$ and can be calculated using the GSL estimate which we use for $c$. For the omega-related finite-volume errors, I will take the fitted $d$ and $E$ and use this as the full result at finite-volume and compare it to a GS model with omega mass from the fitted $E$ and width from the PDG in infinite-volume. I should also compare this to $R$-ratio results for the $I = 0$ channel.

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(a) $V$
(b) $S$
(c) $T$
(d) $T_d$
(e) $D_1$
(f) $D_{1d}$
(g) $D_2$
(h) $D_{2d}$
(i) $F$
(j) $D_3$

Figure 1: QED corrections

(a) $M$
(b) $R$
(c) $R_d$
(d) $O$

Figure 2: SIB corrections

$\alpha$, QED, $a_\mu, QED, \text{conn} \times 10^{10}$
For the finite-volume errors, the two-pion states in $d$ are identical to the $I = 1$ contributions of $c$ and can be calculated using the GSL estimate which we use for $c$. For the omega-related finite-volume errors, I will take the fitted $d^*$ and $E^*$ and use this as the full result at finite-volume and compare it to a GS model with omega mass from the fitted $E^*$ and width from the PDG in infinite-volume. I should also compare this to R-ratio results for the $I = 0$ channel. Do this entire exercise for 24ID and 32ID to estimate discretization errors.

4 QED and SIB diagrams

We will perform a full first-principles calculation of all $O(\alpha)$ and $O(\mu)$ corrections. The corresponding list of diagrams is given in Figs. 1 and 2.

(a) V  (b) S  (c) T  (d) $T_d$  (e) D1  (f) $D_{1d}$

(g) D2  (h) $D_{2d}$  (i) F  (j) D3

Figure 1: QED corrections

(a) M  (b) R  (c) $R_d$

Figure 2: SIB corrections

$RBC/UKQCD$ 2018

\[ a_\mu, \text{QED, disc} \times 10^{10} \]
Diagrams – Strong isospin breaking

For the HVP R is negligible since $\Delta m_u \approx -\Delta m_d$ and O is SU(3) and $1/N_c$ suppressed.
For the finite-volume errors, the two-pion states in $d!$ are identical to the $I = 1$ contributions of $c$ and can be calculated using the GSL estimate which we use for $c$. For the omega-related finite-volume errors, I will take the fitted $d!$ and $E!$ and use this as the full result at finite-volume and compare it to a GS model with omega mass from the fitted $E!$ and width from the PDG in infinite-volume. I should also compare this to R-ratio results for the $I = 0$ channel. Do this entire exercise for 24ID and 32ID to estimate discretization errors.

**4 QED and SIB diagrams**

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(a) V  
(b) S  
(c) T  
(d) T  
(e) D1  
(f) D1  
(g) D2  
(h) D2  
(i) F  
(j) D3

**Figure 1: QED corrections**

(a) M  
(b) R  
(c) R$_d$  
(d) O

**Figure 2: SIB corrections**
Lattice & Dispersive Analysis
Regions of precision (R-ratio data here is from Fred Jegerlehner 2017)

FIG. 4. Comparison of $w_tC(t)$ obtained using R-ratio data [1] and lattice data on our 64I ensemble.

The precision of lattice data deteriorates exponentially as we go to large $t$, however, is precise at intermediate distances. The R-ratio is very precise at long distances.

Note: in this plot a direct comparison of R-ratio and lattice data is not appropriate. Continuum limit, infinite-volume corrections, charm contributions, and IB corrections are missing from lattice data shown here.
We therefore also consider a window method. Following Meyer-Bernecker 2011 and smearing over $t$ to define the continuum limit we write

$$a_\mu = a^\text{SD}_\mu + a^W_\mu + a^\text{LD}_\mu$$

with

$$a^\text{SD}_\mu = \sum_t C(t) w_t [1 - \Theta(t, t_0, \Delta)] ,$$

$$a^W_\mu = \sum_t C(t) w_t [\Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta)] ,$$

$$a^\text{LD}_\mu = \sum_t C(t) w_t \Theta(t, t_1, \Delta) ,$$

$$\Theta(t, t', \Delta) = \left[ 1 + \tanh \left( \frac{(t - t')}{\Delta} \right) \right] / 2 .$$

In this version of the calculation, we use

$$C(t) = \frac{1}{12\pi^2} \int_0^\infty d(\sqrt{s}) R(s) s e^{-\sqrt{s}t}$$

with $R(s) = \frac{3s}{4\pi\alpha^2} \sigma(s, e^+ e^- \rightarrow \text{had})$ to compute $a^\text{SD}_\mu$ and $a^\text{LD}_\mu$. 

Window method (implemented in RBC/UKQCD 2018)
How does this translate to the time-like region?

Most of $\pi\pi$ peak is captured by window from $t_0 = 0.4$ fm to $t_1 = 1.5$ fm, so replacing this region with lattice data reduces the dependence on BaBar versus KLOE data sets.
Error budget from RBC/UKQCD 2018 (Fred’s alphaQED17 results used for window result)

\[
\begin{align*}
715.4(16.3)_S(3.0)_C(7.8)_V(1.9)_A(0.4)_Z(1.7)_E(2.3)_L \\
(1.5)_{E_{48}}(0.1)_{E_{64}}(0.3)_b(0.2)_c(1.1)_{\bar{S}}(0.3)_{\bar{Q}}(0.0)_{M} \\
692.5(1.4)_S(0.2)_C(0.2)_V(0.3)_A(0.2)_Z(0.0)_E(0.0)_{E_{48}} \\
(0.0)_b(0.1)_c(0.0)_{\bar{S}}(0.0)_{\bar{Q}}(0.0)_{M}(0.7)_{RST}(2.1)_{RSY}
\end{align*}
\]

\( a_\mu \)

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<tr>
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TABLE I. Individual and summed contributions to \( a_\mu \) multiplied by \( 10^{10} \). The left column lists results for the window method with \( t_0 = 0.4 \) fm and \( t_1 = 1 \) fm. The right column shows results for the pure first-principles lattice calculation. The respective uncertainties are defined in the main text.

For the pure lattice number the dominant errors are (S) statistics, (V) finite-volume errors, and (C) the continuum limit extrapolation uncertainty.

For the window method there are additional R-ratio systematic (RSY) and R-ratio statistical (RST) errors.
HVP - Thoughts

- This is a vibrant field with a lot of progress over the last 1-2 years!

- Lattice efforts by many groups, results at physical pion mass, QED, SIB corrections available. New methods to reduce statistical and systematic errors.

- Intermediate target: consolidate error at $O(3 \times 10^{-10})$ from first principles

- In early 2018, the lattice uncertainty is still $O(15 \times 10^{-10})$

- A target of $O(6 \times 10^{-10})$ seems realistic for early 2019 but requires a focused effort (estimate based on RBC/UKQCD unpublished progress)

- In a few years, new spacelike measurements from MUonE experiment (t-channel scattering) may be available (see M.Marinkovic's talk)
The HLbL contribution to the muon g-2

Talks by Antoine Gérardin (Tue/2:00), Luchang Jin (Tue/2:30)
Two new avenues for a model-independent value for the HLbL

Dispersive analysis + Experimental/lattice input

Direct lattice calculation

Truncation of cuts and states; Talk by P.Stoffer

7 quark-level topologies
7 quark-level topologies of direct lattice calculation

Hierarchy imposed by QED charges of dominant up- and down-quark contribution

\[ Q_u^4 + Q_d^4 = \frac{17}{81} \]

\[ (Q_u^2 + Q_d^2)^2 = \frac{25}{81} \]

\[ (Q_u^3 + Q_d^3)(Q_u + Q_d) = \frac{9}{81} \]

\[ (Q_u^2 + Q_d^2)(Q_u + Q_d)^2 = \frac{5}{81} \]

\[ (Q_u + Q_d)^4 = \frac{1}{81} \]

Further insight for magnitude of individual topologies can be gained by studying long-distance behavior of QCD correlation functions (Bijnens, RBC, \ldots)
7 quark-level topologies of direct lattice calculation

Hierarchy imposed by QED charges of dominant up- and down-quark contribution

\[
Q_u^4 + Q_d^4 = \frac{17}{81} \quad \quad \quad (Q_u^2 + Q_d^2)^2 = \frac{25}{81}
\]

**Dominant diagrams in top row:** connected and leading disconnected diagram

\[
(Q_u^3 + Q_d^3)(Q_u + Q_d) = \frac{9}{81}
\]

\[
(Q_u^2 + Q_d^2)(Q_u + Q_d)^2 = \frac{5}{81}
\]

\[
(Q_u + Q_d)^4 = \frac{1}{81}
\]

Further insight for magnitude of individual topologies can be gained by studying long-distance behavior of QCD correlation functions (Bijnens, RBC, …)
Finite-volume and infinite-volume formulations

- $a_{\mu}^{HLbL}$ in finite-volume QCD and QED:
  - PRD93(2016)014503 (RBC/UKQCD): Connected diagram with $m_\pi = 171$ MeV; $a_{\mu}^{HLbL} = 13.21(68) \times 10^{-10}$
  - PRL118(2017)022005 (RBC/UKQCD): Connected and leading disconnected diagram with $m_\pi = 139$ MeV; $a_{\mu}^{HLbL} = 5.35(1.35) \times 10^{-10}$ (potentially large finite-volume systematics)

Strategy: extrapolate away $1/L^n$ ($n \geq 2$) errors

- $a_{\mu}^{HLbL}$ in finite-volume QCD and infinite-volume QED:
  - Method proposed and successfully tested against the lepton-loop analytic result: arXiv:1510.08384 (Mainz), arXiv:1609.08454 (Mainz)
  - Similar method plus subtraction scheme to reduce systematic errors; successfully tested against lepton-loop analytic result: PRD96(2017)034515 (RBC/UKQCD)

Strategy: FV errors exponentially suppressed but still may be significant, effect on noise?
The finite-volume QED prescription uses the photon propagator

\[ G^{\mu\nu}_L(x) = \frac{\delta^{\mu\nu}}{V} \sum_k \frac{1}{\hat{k}^2} e^{i k x}, \]  

(1)

where \( \hat{k}^2 = \sum_\mu 4 \sin^2(k_\mu/2) \) and \( V = \prod_\mu L_\mu \) with lattice dimensions \( L_\mu \). The sum is over all momenta with components \( k_\mu = 2\pi n_\mu/L_\mu \) with \( n_\mu \in [0, \ldots, L_\mu - 1] \) and the restriction that \( k_0^2 + k_1^2 + k_2^2 \neq 0 \).

For fixed \( x \) and \( y \) can get result for all \( z \) in \( O(V \log V) \) time using convolutions starting at \( t_{\text{src}} \) and \( t_{\text{snk}} \); has statistical advantage for leading disconnected diagram (\( M^2 \) trick)
Table XII. Functions linear in $a^2$ which can be used to extrapolate the data shown in Fig. 10 to $a^2 = 0$. The results from these fits at $a^2 = 0$ are plotted in Fig. 11.

Figure 11. Results for $F_2(0)$ from QED connected light-by-light scattering. These results have been extrapolated to the $a^2 \to 0$ limit using two methods. The upper points use the quadratic fit to all three lattice spacings shown in Fig. 10 while the lower point uses a linear fit to the two left most points in that figure. Here we extrapolate to infinite volume using the linear fits shown to the two, left-most of the three points in each case.

Lepton loop with $m_{\text{lepton}} = m_\mu$
New sampling strategy with $10\times$ reduced noise for same cost (red versus black):

Stochastically evaluate the sum over vertices $x$ and $y$:

- Pick random point $x$ on lattice
- Sample all points $y$ up to a specific distance $r = |x - y|$
- Pick $y$ following a distribution $P(|x - y|)$ that is peaked at short distances
Calculation at physical pion mass with finite-volume QED prescription (QED$_L$) at single lattice cutoff of $a^{-1} = 1.73$ GeV and lattice size $L = 5.5$ fm.

Connected diagram:

\[
a_{\mu}^{cHLbL} = 11.6(0.96) \times 10^{-10}
\]

Leading disconnected diagram:

\[
a_{\mu}^{dHLbL} = -6.25(0.80) \times 10^{-10}
\]

Large cancellation expected from pion-pole-dominance considerations is realized:

\[
a_{\mu}^{HLbL} = a_{\mu}^{cHLbL} + a_{\mu}^{dHLbL} = 5.35(1.35) \times 10^{-10}
\]

Potentially large systematics due to finite-volume QED!
Infinite-volume QED prescription ($\text{QED}_\infty$)

Details:

We define

\[ i^3 G_{\rho,\sigma,\kappa}(x, y, z) = G_{\rho,\sigma,\kappa}(x, y, z) + G_{\sigma,\kappa,\rho}(y, z, x) + \text{other 4 permutations}. \]

and add the Hermitian conjugate with permuted indices (does not alter \( F_2 \) but makes this kernel infrared finite)

\[ G_{\rho,\sigma,\kappa}^{(1)}(x, y, z) = \frac{1}{2} G_{\rho,\sigma,\kappa}(x, y, z) + \frac{1}{2} [G_{\kappa,\sigma,\rho}(z, y, x)]^\dagger \]
For $m_{\text{line}} = 1$ this yields the kernel

$$G^{(1)}_{\sigma, \kappa, \rho}(y, z, x) = \frac{\gamma_0 + 1}{2} i \gamma_\sigma (-\partial_y + \gamma_0 + 1) i \gamma_\kappa (\partial_x + \gamma_0 + 1) i \gamma_\rho \frac{\gamma_0 + 1}{2}$$

$$\times \frac{1}{4\pi^2} \int d^4 \eta \frac{1}{(\eta - z)^2} f(\eta - y) f(x - \eta).$$

Due to current conservation, we can also devise a subtraction scheme that we found suppresses significantly finite-volume and discretization errors (demonstrated in the lepton loop case)

$$G^{(2)}_{\rho, \sigma, \kappa}(x, y, z) = G^{(1)}_{\rho, \sigma, \kappa}(x, y, z) - G^{(1)}_{\rho, \sigma, \kappa}(y, y, z) - G^{(1)}_{\rho, \sigma, \kappa}(x, y, y) + G^{(1)}_{\rho, \sigma, \kappa}(y, y, y)$$
Figure 3. Leptonic light-by-light contribution to the muon non-zero lattice spacing effects are given in Tab. II. On the right by a factor of four, or more, while the finite volume effects are of order $O(1)$. We also study the lattice spacing dependence. This is expected since the non-zero lattice spacing effects become independent of lattice spacing. The finite volume effects are r}

For each volume, we draw a second-order reference between

Without subtraction (left), with subtraction (right)
Lepton loop contribution $a_{\mu}^{LbL}$ in QED

Integrand of lepton loop contribution $a_{\mu}^{LbL}$:

![Graph showing $f(|y|) \times 10^{9}$ as a function of $|y|/\text{fm}$ for different values of $m_l$.]

<table>
<thead>
<tr>
<th>$m_l/m_\mu$</th>
<th>$a_{\mu}^{LbL} \times 10^{11}$ (exact)</th>
<th>$a_{\mu}^{LbL} \times 10^{11}$</th>
<th>Precision</th>
<th>Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>1229.07</td>
<td>1257.5(6.2)(2.4)</td>
<td>0.5%</td>
<td>2.3%</td>
</tr>
<tr>
<td>1</td>
<td>464.97</td>
<td>470.6(2.3)(2.1)</td>
<td>0.7%</td>
<td>1.2%</td>
</tr>
<tr>
<td>2</td>
<td>150.31</td>
<td>150.4(0.7)(1.7)</td>
<td>1.2%</td>
<td>0.06%</td>
</tr>
</tbody>
</table>

1st uncertainty from 3D integration, 2nd uncertainty from extrapolation to small $|y|$. Behavior for small $|y|$ compatible with $f(|y|) \propto m_{\mu} |y| \log^2(m_{\mu}|y|)$. Analytical results for $a_{\mu}^{LbL}$ with $m_l = m_{\mu}, 2m_{\mu}$ reproduced at the percent level. (Laporta + Remiddi '93, numbers courtesy of Massimo Passera)
Preliminary results from Mainz program earlier this year (Nils Asmussen at Mainz g-2 workshop)

Pion Mass Dependence of $a_{\mu}^{cHLbL}$

- the results show an upward trend for decreasing pion mass
- currently collecting more statistics in long distance regime
Preliminary QCD results for infinite-volume extrapolation (RBC/UKQCD 2018); see talk by L.Jin for complete analysis.
Preliminary QCD results for infinite-volume extrapolation (RBC/UKQCD 2018); see talk by L. Jin for complete analysis.

Data used for finite-volume result in PRL118(2016)022005
Roadmap to complete first-principles light-by-light calculation with all errors controlled

- Calculation of connected plus leading disconnected diagram at physical pion mass completed
- Infinite-volume extrapolation done (to be published)
- Discretization errors are now controlled for (four different lattice spacings over two different actions, to be published)
- Calculation of sub-leading disconnected diagrams, starting with 3-1 topology first results
- Crosscheck of dispersive versus lattice (see, e.g., arXiv:1712.00421) desirable
- Progress by two groups (Mainz & RBC/UKQCD), cross checks will be very valuable!
Summary
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- Lattice QCD has indeed come of age
- Significant improvements in methodology and growing computing power go hand-in-hand
- There is a vibrant community with many groups working on the HVP and currently two groups working on the HLbL
- HVP purely from lattice QCD+QED with competitive errors within reach over the next few years
- HLbL first lattice result with all errors controlled soon to be published
Backup
We perform the calculation as a perturbation around an isospin-symmetric lattice QCD computation with two degenerate light quarks with mass $m_{\text{light}}$ and a heavy quark with mass $m_{\text{heavy}}$ tuned to produce a pion mass of 135.0 MeV and a kaon mass of 495.7 MeV.

The correlator is expanded in the fine-structure constant $\alpha$ as well as $\Delta m_{\text{up, down}} = m_{\text{up, down}} - m_{\text{light}}$, and $\Delta m_{\text{strange}} = m_{\text{strange}} - m_{\text{heavy}}$. We write

$$C(t) = C^{(0)}(t) + \alpha C_{\text{QED}}^{(1)}(t) + \sum_f \Delta m_f C_{\Delta m_f}^{(1)}(t) + O(\alpha^2, \alpha \Delta m, \Delta m^2).$$

The correlators of this expansion are computed in lattice QCD with dynamical up, down, and strange quarks. We compute the missing contributions to $a_\mu$ from charm sea quarks in perturbative QCD (RHAD) by integrating the time-like region above 2 GeV and find them to be smaller than $0.3 \times 10^{-10}$. 
We tune the bare up, down, and strange quark masses $m_{\text{up}}$, $m_{\text{down}}$, and $m_{\text{strange}}$ such that the $\pi^0$, $\pi^+$, $K^0$, and $K^+$ meson masses computed in our calculation agree with the respective experimental measurements. The lattice spacing is determined by setting the $\Omega^-$ mass to its experimental value.

We perform the lattice calculations for the light quark contributions using RBC/UKQCD’s 48l and 64l lattice configurations with lattice cutoffs $a^{-1} = 1.730(4)$ GeV and $a^{-1} = 2.359(7)$ GeV and a larger set of ensembles with up to $a^{-1} = 2.774(10)$ GeV for the charm contribution.

From the parameter tuning procedure on the 48l we find $\Delta m_{\text{up}} = -0.00050(1)$, $\Delta m_{\text{down}} = 0.00050(1)$, and $\Delta m_{\text{strange}} = -0.002(2)$.

The shift of the $\Omega^-$ mass due to the QED correction is significantly smaller than the lattice spacing uncertainty and its effect on $C(t)$ is therefore not included separately.
Luscher quantization condition (5.47 fm)
Luscher quantization condition (6.22 fm)
Consolidate continuum limit

Adding a finer lattice
Add $a^{-1} = 2.77$ GeV lattice spacing

- Third lattice spacing for strange data ($a^{-1} = 2.77$ GeV with $m_\pi = 234$ MeV with sea light-quark mass corrected from global fit):

![Graph showing a linear fit in $a^2/\text{fm}^2$.](graph.png)

- For light quark need new ensemble at physical pion mass. Proposed for early science time at Summit Machine at Oak Ridge later this year ($a^{-1} = 2.77$ GeV with $m_\pi = 139$ MeV).
Window method with fixed $t_0 = 0.4 \text{ fm}$

For $t = 1 \text{ fm}$ approximately 50% of uncertainty comes from lattice and 50% of uncertainty comes from the R-ratio. Is there a small slope? More in a few slides!

Can use this to check experimental data sets; see my KEK talk for more details.
Predicts $|F_\pi(s)|^2$:

We can then also predict matrix elements and energies for our other lattices; successfully checked!