Two-pion contributions to the muon $g - 2$

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in collaboration with G. Colangelo and M. Hoferichter

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and with G. Colangelo, M. Hoferichter, and M. Procura


and work in progress

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Mani L. Bhaumik Institute, UCLA
Outline

1. Hadronic contributions to the muon $g - 2$

2. Hadronic vacuum polarisation
   - Dispersion relation for the pion vector form factor
   - Fit strategy
   - Fit results and contribution to the muon $g - 2$

3. Hadronic light-by-light scattering
   - Tensor decomposition and Mandelstam representation
   - Pion pole
   - Pion box
   - $\pi \pi$-rescattering

4. Conclusions and outlook
1 Hadronic contributions to the muon $g - 2$

2 Hadronic vacuum polarisation

3 Hadronic light-by-light scattering

4 Conclusions and outlook
Hadronic vacuum polarisation (HVP)

- problem: QCD is non-perturbative at low energies
- much progress using lattice QCD first-principle calculations
- best current evaluations based on dispersion relations and data (or combinations with lattice)
Hadronic contributions to the muon $g - 2$

Hadronic vacuum polarisation (HVP)

Photon HVP function:

$$\Pi (q^2) = i (q^2 g_{\mu \nu} - q_\mu q_\nu) \Pi (q^2)$$

Unitarity of the $S$-matrix implies the optical theorem:

$$\text{Im} \Pi (s) = \frac{s}{e(s)^2} \sigma (e^+ e^- \rightarrow \text{hadrons})$$
Dispersion relation

Causality implies analyticity:

\[ \text{Cauchy integral formula:} \]
\[
\Pi(s) = \frac{1}{2\pi i} \oint_{\gamma} \frac{\Pi(s')}{{s'} - s} ds'
\]

Deform integration path:
\[
\Pi(s) - \Pi(0) = \frac{s}{\pi} \int_{4M^2}^{\infty} \frac{\text{Im}\Pi(s')}{(s' - s - i\epsilon)s'} ds'
\]
HVP contribution to \((g - 2)_\mu\)

\[
a^\text{HVP}_\mu = \frac{m^2_\mu}{12\pi^3} \int_{s_{\text{thr}}}^\infty ds \frac{\hat{K}(s)}{s} \sigma(e^+e^- \to \text{hadrons})
\]

- basic principles: unitarity and analyticity
- direct relation to experiment: total hadronic cross section \(\sigma(e^+e^- \to \text{hadrons})\)
- can be systematically improved: dedicated \(e^+e^-\) program (BaBar, Belle, BESIII, CMD3, KLOE2, SND)
Hadronic light-by-light (HLbL) scattering

- so far only model calculations
- uncertainty estimate based rather on consensus than on a systematic method
- with recent progress on vacuum polarisation, HLbL starts to dominate the theory uncertainty
- progress with lattice QCD and dispersive approach
### Hadronic contributions to the muon $g - 2$

#### SM contributions to $(g - 2)_\mu$

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<th>$10^{11} \times \Delta a_\mu$</th>
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<td>→ PDG 2016</td>
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<td>QED total</td>
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Hadronic contributions to the muon $g - 2$

**SM contributions to $(g - 2)_\mu$**

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→ PDG 2016

→ Aoyama et al. 2012, 2017

→ Gnendiger et al. 2013

→ Davier et al. 2017

→ Keshavarzï et al. 2018

→ Kurz et al. 2014

→ Nyffeler 2017

→ Colangelo et al. 2014
Overview

1. Hadronic contributions to the muon $g - 2$

2. Hadronic vacuum polarisation
   Dispersion relation for the pion vector form factor
   Fit strategy
   Fit results and contribution to the muon $g - 2$

3. Hadronic light-by-light scattering

4. Conclusions and outlook
Two-pion contribution to HVP

- $\pi\pi$ contribution amounts to more than 70% of HVP contribution
- responsible for a similar fraction of HVP uncertainty
- unitarity relation for $\pi\pi$ contribution to HVP: pion vector form factor (VFF)

$$\sigma(e^+e^- \rightarrow \pi^+\pi^-) \propto |F^V_\pi(s)|^2$$
Two-pion contribution to HVP

- VFF itself fulfils again a unitarity relation:

\[
\begin{align*}
\text{VFF} \quad = \quad & \quad \text{VFF term} \\
& \quad \text{VFF term} \\
& \quad \text{VFF term} \\
& \quad \text{...}
\end{align*}
\]

- use the constraints of analyticity and unitarity to better understand uncertainties in HVP $\pi\pi$ channel


Ananthanarayan et al. 2013, 2016
Dispersive representation of pion VFF

\[ F^V_\pi(s) = \Omega_1^1(s) \times G_\omega(s) \times G^N_{\text{in}}(s) \]

- Omnès function with elastic $\pi\pi$-scattering $P$-wave phase shift $\delta_1^1(s)$ as input:

\[ \Omega_1^1(s) = \exp \left\{ \frac{s}{\pi} \int_{4M^2_\pi}^\infty ds' \frac{\delta_1^1(s')}{s'(s' - s)} \right\} \]
Dispersive representation of pion VFF

\[ F_V^\pi(s) = \Omega_1^1(s) \times G_\omega(s) \times G_{in}^N(s) \]

- isospin-breaking $3\pi$ intermediate state: negligible
  apart from $\omega$ resonance ($\rho - \omega$ interference effect)

\[ G_\omega(s) = 1 + \frac{s}{\pi} \int_{9M^2_\pi}\! ds' \frac{\text{Im}g_\omega(s')}{s'(s' - s)} \left( \frac{1 - \frac{9M^2_\pi}{s'}}{1 - \frac{9M^2_\pi}{M^2_\omega}} \right)^4, \]

\[ g_\omega(s) = 1 + \epsilon_\omega \frac{s}{(M_\omega - \frac{i}{2}\Gamma_\omega)^2 - s} \]
Dispersive representation of pion VFF

\[ F_V^\pi(s) = \Omega_1^1(s) \times G_\omega(s) \times G_{\text{in}}^N(s) \]

- heavier intermediate states: \(4\pi\) (mainly \(\pi^0\omega\), \(\bar{K}K\), \ldots)
- described in terms of a conformal polynomial with cut starting at \(\pi^0\omega\) threshold

\[ G_{\text{in}}^N(s) = 1 + \sum_{k=1}^{N} c_k (z^k(s) - z^k(0)) \]

- correct \(P\)-wave threshold behaviour imposed
Input and systematic uncertainties

- elastic $\pi\pi$-scattering $P$-wave phase shift $\delta_1^1(s)$ from Roy-equation analysis, including uncertainties
  $\rightarrow$ Ananthanarayan et al., 2001; Caprini et al., 2012

- high-energy continuation of phase shift above validity of Roy equations

- $\omega$ width

- systematics in conformal polynomial: order $N$, one mapping parameter
Free fit parameters

- value of the elastic $\pi\pi$-scattering $P$-wave phase shift $\delta^1_1$ at two points (0.8 GeV and 1.15 GeV)
- $\rho - \omega$ mixing parameter $\epsilon_\omega$
- $\omega$ mass
- energy rescaling for the experimental input, which allows for a calibration uncertainty
- $N - 1$ coefficients in the conformal polynomial
VFF fit to the following data

- time-like cross section data from high-statistics $e^+e^-$ experiments SND, CMD-2, BaBar, KLOE
- space-like VFF data from NA7
- Eidelman – Łukaszuk bound on inelastic phase:
  \[ \rightarrow \text{Eidelman, Łukaszuk, 2004} \]
- iterative fit routine including full experimental covariance matrices and avoiding D’Agostini bias
  \[ \rightarrow \text{D’Agostini, 1994; Ball et al. (NNPDF) 2010} \]
VFF fit results

- perfect fits to all experiments possible ($p$-value around 3% to 6%) with a few caveats:
  - either $M_\omega$ or energy recalibration has to be fit (practically identical results)
  - two outliers in KLOE08 set ($>30$ units in $\chi^2$)
  - BESIII covariance matrix cannot be used
- well-known discrepancy between BaBar and KLOE ⇒ fit all data sets and inflate errors by $\sqrt{\chi^2/dof}$
- inelastic effects dominate uncertainty for $(g - 2)_\mu$
Fit results and contribution to $(g - 2)_\mu$

Fit result for the VFF $|F_V^\pi(s)|^2$
Fit results and contribution to $(g - 2)_\mu$
VFF fit result and data without energy rescaling

Fit results and contribution to \((g - 2)_{\mu}\)
Fit results and contribution to $(g - 2)_{\mu}$

Fit result for the VFF $|F^V_{\pi}(s)|^2$
Fit results and contribution to $(g - 2)_\mu$

Relative difference between data sets and fit result

<table>
<thead>
<tr>
<th>Data Set</th>
<th>BaBar</th>
<th>KLOE08</th>
<th>KLOE10</th>
<th>KLOE12</th>
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<tr>
<td>Total Error</td>
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<td></td>
<td></td>
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<td>SND</td>
<td></td>
<td></td>
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<tr>
<td>CMD-2</td>
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</table>

$|\frac{F(s)}{F_{\pi}(s)}|^2_{\text{fit}} - 1$
Contribution to \((g - 2)_\mu\)

- low-energy \(\pi\pi\) contribution:

\[
a_{\mu}^{\text{HVP},\pi\pi} \left|_{\leq 0.63\text{ GeV}} \right. = 132.8(0.4)(1.1) \times 10^{-10}
\]

⇒ compare to 131.1(1.0) → KNT18, 133.3(7) → Ananthanarayan et al., 2016

- \(\pi\pi\) contribution up to 1 GeV:

\[
a_{\mu}^{\text{HVP},\pi\pi} \left|_{\leq 1\text{ GeV}} \right. = 494.8(1.5)(2.1) \times 10^{-10}
\]
Result for $a_{\mu}^{\text{HVP, } \pi\pi}$ below 1 GeV

- SND
- CMD-2
- BaBar
- KLOE''
- Energy scan
- All $e^+e^-$
- All $e^+e^-$, NA7

$10^{10} \times a_{\mu}^{\pi\pi} \leq 1$ GeV
Improved determination of $\delta_1^1(s)$

Fit result for the $\pi\pi$ $P$-wave phase shift $\delta_1^1$

$$
\begin{align*}
\delta_1^1(s_0) &= 110.4(1)(7)^\circ = 110.4(7)^\circ \\
\delta_1^1(s_1) &= 165.7(0.1)(2.4)^\circ = 165.7(2.4)^\circ
\end{align*}
$$
Determination of the pion charge radius

Definition of charge radius:

\[ F_V^\pi(s) = 1 + \frac{1}{6} \langle r^2_\pi \rangle s + \mathcal{O}(s^2) \]

dispersion relation for \( F_V^\pi \) implies sum rule:

\[ \langle r^2_\pi \rangle = \frac{6}{\pi} \int_{4M^2_\pi}^{\infty} ds \frac{\text{Im} F_V^\pi(s)}{s^2} \]

our result:

\[ \langle r^2_\pi \rangle = 0.429(1)(4) \text{ fm}^2 = 0.429(4) \text{ fm}^2 \]

compare to PDG: \( \langle r^2_\pi \rangle = 0.452(11) \text{ fm}^2 \)

(includes potentially model-dependent \( eN \rightarrow e\pi N \))
A puzzle: $\omega$ mass

fit result for $\omega$ mass:

combined fit: $M_\omega = 781.69(9)(3)$ MeV

fits to single experiments: $M_\omega = 781.49 \ldots 782.05$ MeV

compare to PDG value (dominated by $3\pi$ channel):

$M^{\text{PDG}}_\omega = 782.65(12)$ MeV
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   - Pion box
   - $\pi\pi$-rescattering

4. Conclusions and outlook
Dispersive approach

- make use of fundamental principles:
  - gauge invariance, crossing symmetry
  - unitarity, analyticity
- relate HLbL to experimentally accessible quantities
BTT Lorentz decomposition

Lorentz decomposition of the HLbL tensor:

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_i T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2)$$

- Lorentz structures manifestly gauge invariant
- scalar functions $\Pi_i$ free of kinematic singularities
  ⇒ dispersion relation in the Mandelstam variables
Dispersive representation

- write down a double-spectral (Mandelstam) representation for the HLbL tensor
- split the HLbL tensor according to the sum over intermediate (on-shell) states in unitarity relations

\[
\Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{box}} + \Pi_{\mu\nu\lambda\sigma}^{\pi\pi} + \ldots
\]
Dispersive representation

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\[ \Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{box}} + \Pi_{\mu\nu\lambda\sigma}^{\pi\pi} + \ldots \]

one-pion intermediate state
Dispersive representation

- write down a double-spectral (Mandelstam) representation for the HLbL tensor
- split the HLbL tensor according to the sum over intermediate (on-shell) states in unitarity relations

\[ \Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{box}} + \Pi_{\mu\nu\lambda\sigma}^{\pi\pi} + \ldots \]

two-pion intermediate state in both channels
Dispersive representation

- write down a double-spectral (Mandelstam) representation for the HLbL tensor
- split the HLbL tensor according to the sum over intermediate (on-shell) states in unitarity relations

\[ \Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{box}} + \Pi_{\mu\nu\lambda\sigma}^{\pi\pi} + \ldots \]

two-pion intermediate state in first channel
Dispersive representation

- write down a double-spectral (Mandelstam) representation for the HLbL tensor
- split the HLbL tensor according to the sum over intermediate (on-shell) states in unitarity relations

\[ \Pi_{\mu \nu \lambda \sigma} = \Pi^{\pi^0\text{-pole}}_{\mu \nu \lambda \sigma} + \Pi^{\text{box}}_{\mu \nu \lambda \sigma} + \Pi^{\pi \pi}_{\mu \nu \lambda \sigma} + \ldots \]

higher intermediate states
Pion pole

\[ \Pi_{\pi^0} = \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma}(q_3^2, 0)}{q_3^2 - M_{\pi^0}^2} \]

\[ \Pi_{\pi^0} \text{ via crossing symmetry} \]

- input: doubly-virtual and singly-virtual pion transition form factors \( \mathcal{F}_{\gamma^*\gamma^*\pi^0} \) and \( \mathcal{F}_{\gamma^*\gamma\pi^0} \)

- dispersive analysis of transition form factor:

\[ a_{\mu}^{\pi^0} = 62.6^{+3.0}_{-2.5} \times 10^{-11} \]

Box contribution

- simultaneous two-pion cuts in two channels
- Mandelstam representation explicitly constructed
- $q^2$-dependence: pion VFF $F_V^\pi(q_i^2)$ for each off-shell photon factor out
- Wick rotation: integrate over space-like momenta
- dominated by low energies $\leq 1$ GeV
- result: $a_{\mu}^{\pi\text{-box}} = -15.9(2) \times 10^{-11}$
Fit result for the VFF $|F^V_\pi(s)|^2$

- **total error**
- **fit error**
- **NA7**
- **JLab**

$(the\ JLab\ data\ are\ not\ used\ in\ the\ fit)$
Rescattering contribution

- neglect left-hand cut due to multi-particle intermediate states in crossed channel
- two-pion cut in only one channel:

\[
\Pi_{i\pi\pi} = \frac{1}{2} \left( \frac{1}{\pi} \int_{4M_{\pi}^2}^{\infty} dt' \frac{\text{Im}\Pi_{i\pi\pi}(s, t', u')}{t' - t} + \frac{1}{\pi} \int_{4M_{\pi}^2}^{\infty} du' \frac{\text{Im}\Pi_{i\pi\pi}(s, t', u')}{u' - u} \right) + \text{fixed-}t + \text{fixed-}u
\]
Rescattering contribution

- expansion into partial waves
- unitarity gives imaginary parts in terms of helicity amplitudes for $\gamma^* \gamma^{(*)} \rightarrow \pi \pi$:

$$\text{Im}_{\pi \pi} h_{J,\lambda_1 \lambda_2, \lambda_3 \lambda_4}^J (s) \propto \sigma_{\pi}(s) h_{J,\lambda_1 \lambda_2} (s) h_{J,\lambda_3 \lambda_4}^* (s)$$

- framework valid for arbitrary partial waves
- resummation of PW expansion reproduces full result: checked for pion box
$S$-wave rescattering contribution

- pion-pole approximation to left-hand cut
  \[ q^2 \] -dependence given by \( F^V_\pi \)

- phase shifts based on modified inverse-amplitude method (\( f_0(500) \) parameters accurately reproduced)

- result for $S$-waves:

\[
\alpha_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}
\]
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Conclusions and outlook

HVP

• precise dispersive determination of pion VFF
• comprehensive analysis of uncertainties in $\pi\pi$ channel
• valuable to corroborate uncertainties of direct integration methods
• precise prediction for low-energy region, but useful up to 1 GeV:

$$a_{\mu}^{HVP,\pi\pi}|_{\leq 1 \text{ GeV}} = 494.8(1.5)(2.1) \times 10^{-10}$$

• side-products: improved determination of $\pi\pi$ $P$-wave phase shift; pion charge radius
Conclusions and outlook

HLbL

- very precise evaluation of HLbL pion-box contribution:
  \[ a_{\mu}^{\pi-box} = -15.9(2) \times 10^{-11} \]

- precise prediction for $S$-wave $\pi\pi$-rescattering contribution with pion-pole left-hand cut:
  \[ a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11} \]

- $D$-wave contribution work in progress: requires inclusion of higher left-hand cuts

- contributions beyond $\pi\pi$ and pQCD/OPE constraints work in progress
Summary

• our dispersive approach to HVP and HLbL is based on fundamental principles:
  • gauge invariance, crossing symmetry (for HLbL)
  • unitarity, analyticity
• we are focusing on the lightest intermediate states
• relation to experimentally accessible (or again with data dispersively reconstructed) quantities
• precise numerical evaluation of two-pion contributions
• a step towards a model-independent calculation of $a_\mu$