Waveforms from Amplitudes

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[in progress]
+*review of work* with Ben Maybee & Donal O'Connell
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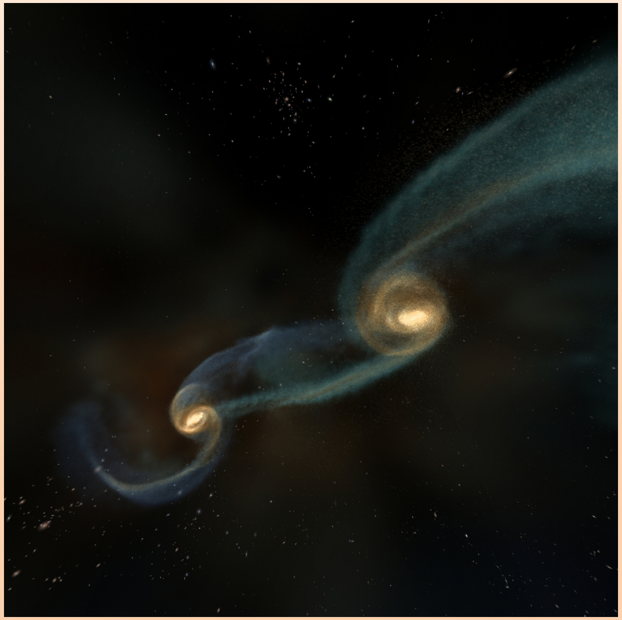
A New Window

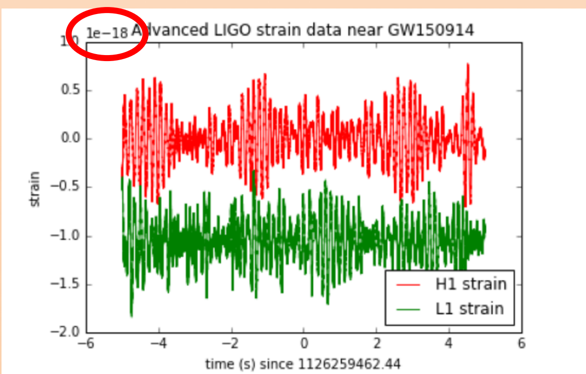
- The universe is a very violent place

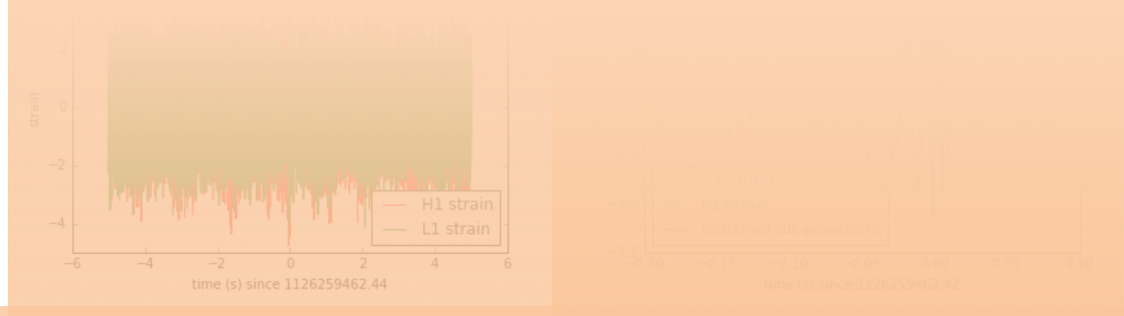


- LIGO/VIRGO observations are letting us see aspects that were heretofore invisible — through gravitational waves

Gravitational Waves

- The waves are anything but violent, though
- LIGO/VIRGO use laser interferometers
  - Incredible isolation and vibration removal
  - Near-perfect mirrors
- Noise is 400 times larger than signal!





- Need theoretical input for waveforms

Gravitational Waves

- General relativists have worked hard for many results
  - Using an Effective One-Body formalism

    Buonanno, Pan, Taracchini, Barausse, Bohé, Cotesta, Shao, Hinderer, Steinhoff, Vines; Damour, Nagar, Bernuzzi, Bini, Balmelli, Messina; Iyer, Sathyaprakash;
  - Numerical Relativity — Pretorius; Campanelli *et al.*; Baker *et al.*
  - ADM — Schäfer, Jaranowski;
  - Direct post-Newtonian — Blanchet
- Joined by Effective Field Theorists

Goldberger, Rothstein; Goldberger, Li, Prabhu, Thompson; Leibovich; Chester; Porto,… Kol; Levi,…

Direct Route to Predictions

- Compute Potential
  - Compute effective-field theory scattering amplitude & match full amplitude to EFT

    Cheung, Rothstein, Solon; Bern, Cheung, Roiban, Shen, Solon, Zeng
  - Extract potential from form of terms in scattering amplitude

    Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove;
    Foffa, Mastrolia, Sturani, Sturm

    Chung, Huang, Kim, Lee

    Guevara, Ochirov, Vines
- Feed potential into Effective-One-Body formalism
- Eikonal Damgaard; Parra-Martinez
- Compute Effective Action

  Plefka, Shi, Steinhoff, Wang
- Other connections to classical scattering

  Goldberger, Ridgway; Goldberger, Ridgway, Li, Prabhu; Shen

Public Service Announcement

- Lower the comprehension potential barrier
- ~~Post-Minkowskian~~, n~~PM~~ $\rightarrow$ order G^n
- ~~Super-classical~~ $\rightarrow$ classically singular

Our Strategy: Take the Scenic Route

DAK, Maybee, O'Connell

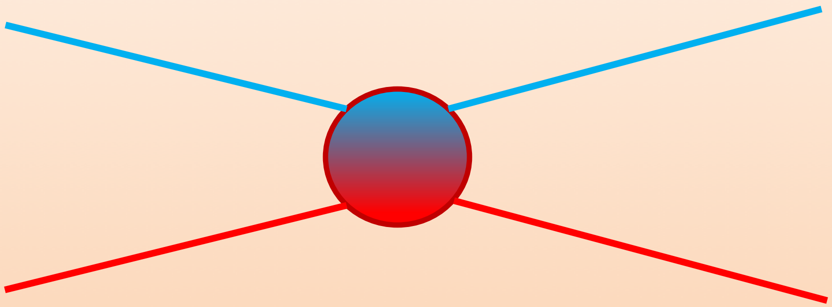
- Pick well-defined observables in the quantum theory
  That are also relevant classically

- Express them in terms of scattering amplitudes in the quantum theory

- Understand how to take the classical limit efficiently

Set-up

- Scatter two massive particles



- Look at two observables:
  - Momentum radiated during the scattering
  - Wave amplitude

We all love scattering amplitudes

But they aren't the final goal or physically meaningful on their own

Wave Packets

- Point particles: localized positions and momenta
- Wavefunction $\phi(p)$
- Initial state: integral over on-shell phase space

$$
\begin{aligned}
|\psi\rangle_{\text{in}} &= \int \hat{d}^4 p_1 \hat{d}^4 p_2\, \hat{\delta}^{(+)}(p_1^2 - m_1^2)\hat{\delta}^{(+)}(p_2^2 - m_2^2)\, \phi(p_1)\phi(p_2) \\
&\quad \times e^{ib\cdot p_1/\hbar}\, |p_1 p_2\rangle_{\text{in}} \\
&= \int d\Phi(p_1) d\Phi(p_2)\, \phi(p_1)\phi(p_2)\;\, e^{ib\cdot p_1/\hbar}\, |p_1 p_2\rangle_{\text{in}}
\end{aligned}
$$

Notation tidies up 2πs

Classical Limit, part 1

- Classical limit requires $\hbar \to 0$: restore $\hbar$
- We're still relativistic field theorists: keep $c = 1$
- Dimensional analysis

$$[M] \neq [L]^{-1}$$
$$[|p\rangle] = [M]^{-1}$$
$$[\mathrm{Ampl}_n] = [M]^{4-n}$$

- $\hbar$ in couplings: $e \to e/\sqrt{\hbar}$; $\kappa \to \kappa/\sqrt{\hbar}$
- Distinguish wavenumber from momentum: $\bar{p} = p/\hbar$
- Net: n-point, L-loop amplitude in scalar QED scales as

$$\hbar^{1-\frac{n}{2}-L}$$

Not the whole story, of course

Radiated Momentum

- Expectation of messenger (photon or graviton) momentum $\mathbf{K}^\mu$

$$R^\mu \equiv \langle k^\mu \rangle = {}_{\text{in}}\langle \psi \big| S^\dagger \mathbf{K}^\mu S \big| \psi \rangle_{\text{in}} = {}_{\text{in}}\langle \psi \big| T^\dagger \mathbf{K}^\mu T \big| \psi \rangle_{\text{in}}$$

Insert complete set of states

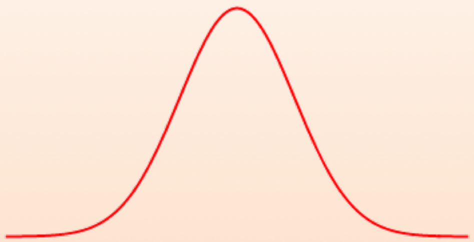
$$\langle k^\mu \rangle = \sum_X \int d\Phi(k) d\Phi(r_1)\, d\Phi(r_2)\; k_X^\mu\; \big| \langle r_1 r_2 k X | T | \psi \rangle \big|^2 \,,$$

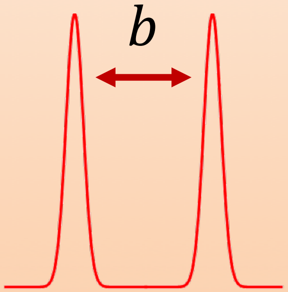
Expressible as scattering amplitude squared or cut of amplitude

Valid to all orders

Classical Limit, part 2

- Three scales
  - ℓ_c: Compton wavelength
  - ℓ_w: wavefunction spread
  - b: impact parameter
- Particles localized: $\ell_c \ll \ell_w$
- Well-separated wave packets: $\ell_w \ll b$





More careful analysis confirms this ‘Goldilocks’ condition

$$\ell_c \ll \ell_w \ll b$$

Classical Limit, part 2

- In-state wavefunctions ϕ and ϕ^* both represent particle
  - Should be sharply peaked
  - Overlap should be $O(1)$
  - Angular-averaged on-shell δ functions transmit constraint to $\sqrt{-q^2}$
- Shrinking on q^2: $\sqrt{-q^2} \ll \hbar/\ell_w$
- Fixed on $\bar{q}^2$: $\sqrt{-\bar{q}^2} \ll 1/\ell_w$

$\Rightarrow$ Natural integration variables for messenger (massless) momenta are wavenumbers, not momenta:

- mismatch $\bar{q}$,
- transfer $\bar{w}$ (from analysis of outgoing states),
- loop $\bar{\ell}$ (from unitarity)

$\Rightarrow$ More factors of $\hbar$ to extract

Classical Limit, *tachlis*

- Introduce notation

$$\left\langle\!\!\left\langle \cdots \right\rangle\!\!\right\rangle$$

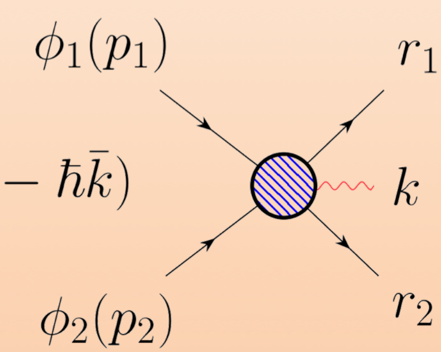
  - Extract $\hbar$s in couplings
  - Extract $\hbar s$ in messenger momenta
  - Take $\hbar \to 0$ limit, Laurent-expanding in $\hbar$ where needed
  - Integrate over massive phase space: $p \to m\, u$

Radiation at LO

- Leading contribution is $O(g^6)$
- Five-point tree amplitude

$$\hbar^3 \sum_X \int d\Phi(\bar{k})d\Phi(r_1)d\Phi(r_2)\; \bar{k}^\mu_X$$

$$\times \left| \int d\Phi(p_1)d\Phi(p_2)\; e^{ib\cdot p_1/\hbar}\, \hat{\delta}^{(4)}(p_1+p_2-r_1-r_2-\hbar\bar{k}) \right|^2$$



$$= g^6 \left\langle\!\!\left\langle \int d\Phi(\bar{k})\, \bar{k}^\mu \left|\mathcal{R}^{(0)}(\bar{k})\right|^2 \right\rangle\!\!\right\rangle$$

Scalar QED Radiation at LO

- Evaluating the five-point tree, one finds for the kernel

$$\mathcal{R}^{(0)}(\bar{k}) = \frac{1}{m_1}\int \hat{d}^4\overline{w}_1 \hat{d}^4\overline{w}_2\ \hat{\delta}(u_1\cdot\overline{w}_1)\hat{\delta}(u_2\cdot\overline{w}_2)\hat{\delta}^{(4)}(\bar{k}-\overline{w}_1-\overline{w}_2)\, e^{i\overline{w}_1\cdot b}$$
$$\times\left\{\frac{Q_1^2 Q_2}{\overline{w}_2^2}\left[-u_2\cdot\varepsilon + \frac{(u_1\cdot u_2)(\overline{w}_2\cdot\varepsilon)}{u_1\cdot\bar{k}} + \frac{(u_2\cdot\bar{k})(u_1\cdot\varepsilon)}{u_1\cdot\bar{k}}\right.\right.$$
$$\left.\left. - \frac{(\bar{k}\cdot\overline{w}_2)(u_1\cdot u_2)(u_1\cdot\varepsilon)}{(u_1\cdot\bar{k})^2}\right] + (1\leftrightarrow 2)\right\}$$

- Matches classical result
- Momentum conservation follows automatically, without addition of the Abraham–Lorentz–Dirac force as in the classical theory

Closer Look at Radiation

- Measure total interated radiation over the whole celestial sphere at r
- Difficult observationally
- Let's instead look at the Mellin moments $\langle (\mathbf{K}^0)^{z-1} \rangle$
- We can use them to get the spectral function: inverse Mellin transform

$$f(\omega) = \frac{1}{2\pi i} \frac{1}{r^2} \int_{c-i\infty}^{c+i\infty} dz\, \omega^{-z} \langle (\mathbb{K}^0)^{z-1} \rangle$$

Closer Look at Radiation

- This would be the spectral function for the total radiation
- Can hope to look at radiation seen at one point on the celestial sphere: one direction
- Want $“\langle(\mathbb{K}^0)^{z-1}\delta^{(2)}(\hat{\mathbb{K}}-\hat{\mathbf{n}})\rangle”$
- The Fourier transform of the spectral function is the waveform

- Spectral function for the power is

$$f_{\text{pow}}(\omega,\hat{\mathbf{n}}) = \frac{g^6}{r^2}\left\langle\!\!\left\langle \int d\Phi(k)\,\left|\mathcal{R}^{(0)}(k)\right|^2 \frac{\delta(\ln k^0 - \ln\omega)}{k^0}\,\delta^{(2)}(\hat{\mathbf{k}}-\hat{\mathbf{n}}) \right\rangle\!\!\right\rangle$$

$$\sim \frac{g^6}{r^2}\omega\left\langle\!\!\left\langle \left|\mathcal{R}^{(0)}(\omega(1,\hat{\mathbf{n}}))\right|^2 \right\rangle\!\!\right\rangle$$

- Spectral function for the amplitude

$$f(\omega,\hat{\mathbf{n}}) \sim \frac{g^3}{r}\sqrt{\left\langle\!\!\left\langle \left|\mathcal{R}^{(0)}(\omega(1,\hat{\mathbf{n}}))\right|^2 \right\rangle\!\!\right\rangle}$$

$$\sim \frac{g^3}{r}\left|\mathcal{R}^{(0)}_{\langle\!\langle\rangle\!\rangle}(\omega(1,\hat{\mathbf{n}}))\right|$$

- Get everything but the phase

Cleanly: Measure the EM Field

- Fix spacetime slice & direction via ζ^μ and n^μ
- Define $\check{F}(x) = F^{\mu\nu}\zeta_\mu n_\nu$

$$\begin{aligned} {}_{\text{out}}\langle\psi|\check{\mathbb{F}}|\psi\rangle_{\text{out}} &= {}_{\text{in}}\langle\psi|S^\dagger \check{\mathbb{F}} S|\psi\rangle_{\text{in}} \\ &= \langle\psi|(1 - iT^\dagger)\check{\mathbb{F}}(1 + iT)|\psi\rangle \\ &= \langle\psi|\check{\mathbb{F}}|\psi\rangle + i\langle\psi|\check{\mathbb{F}}T|\psi\rangle - i\langle\psi|T^\dagger\check{\mathbb{F}}|\psi\rangle + \langle\psi|T^\dagger\check{\mathbb{F}}T|\psi\rangle \end{aligned}$$

0

- Use unitarity $S^\dagger S = 1$ to rewrite

$$\langle\check{\mathbb{F}}\rangle = +i\langle\psi|[\check{\mathbb{F}}, T]|\psi\rangle + \langle\psi|T^\dagger[\check{\mathbb{F}}, T]|\psi\rangle\,,$$

- Similar to expression for Δp

$O(g^3)$ $O(g^5)$

Waveform or Spectral Function Directly

- $\langle \check{F}(t, \boldsymbol{x}) \rangle$ is the waveform at $\boldsymbol{x}$

- Ingredients
  - ∂ is wavenumber
  - $A_\mu(x) = \dfrac{1}{\sqrt{\hbar}} \displaystyle\int d\Phi(k) \big[e^{ik\cdot x/\hbar} \varepsilon_\mu a^\dagger_{\mathbf{k}} + e^{-ik\cdot x/\hbar} \varepsilon^*_\mu a_{\mathbf{k}} \big]$
  - $\mathcal{A}(2 \to 3)$ for lowest-order contribution

- Lowest-order contribution

$$\sim \int_{\text{on-shell}} \hat{d}^4 w \left[\mathcal{A}(2 \to 3) e^{-ik\cdot x/\hbar} - \mathcal{A}(3 \to 2) e^{ik\cdot x/\hbar} \right]$$

Waveform or Spectral Function Directly

- Fourier-transform in t to get spectral function
- For $r \gg b$, difference reconstructs Green function, gives rise to decay

$$\sim \frac{g^3}{r} \left\langle\!\!\left\langle \mathcal{R}^{(0)}(\omega(1, \hat{\mathbf{n}})) \right\rangle\!\!\right\rangle$$

- Spectral function *is* the classical radiation kernel

Scalar QED result

- Radiation kernel in the limit for positive-helicity photon

$$
\begin{aligned}
\mathcal{R}^{(0)}(\bar{k}^{(+)}) = \; & \frac{Q_1^2 Q_2\, e^{ib\cdot\bar{k}}}{2\sqrt{2}\pi m_1\, u_1\cdot\bar{k}\,\sqrt{\gamma^2-1}} \\
& \quad \times \Bigg\{ [\bar{k}|\, u_2\, u_1\, |\bar{k}]\;\, K_0\big(\sqrt{-b^2}\, u_1\cdot\bar{k}/\sqrt{\gamma^2-1}\big) \\
& \qquad + \frac{i\,[\bar{k}|\, b\, u_1\, |\bar{k}]}{\sqrt{\gamma^2-1}\,\sqrt{-b^2}} K_1\big(\sqrt{-b^2}\, u_1\cdot\bar{k}/\sqrt{\gamma^2-1}\big) \Bigg\} \\
& + \frac{Q_1 Q_2^2}{2\sqrt{2}\pi m_1\, u_2\cdot\bar{k}\,\sqrt{\gamma^2-1}} \\
& \quad \times \Bigg\{ [\bar{k}|\, u_1\, u_2\, |\bar{k}]\;\, K_0\big(\sqrt{-b^2}\, u_2\cdot\bar{k}/\sqrt{\gamma^2-1}\big) \\
& \qquad + \frac{i\,[\bar{k}|\, b\, u_2\, |\bar{k}]}{\sqrt{\gamma^2-1}\,\sqrt{-b^2}} K_1\big(\sqrt{-b^2}\, u_2\cdot\bar{k}/\sqrt{\gamma^2-1}\big) \Bigg\} .
\end{aligned}
$$

Summary

- Gravitational-wave astronomy makes new demands on theoretical calculations in GR
- Opportunity for scattering amplitudes: double copy offers simpler avenue to calculations

- Study observables valid in both quantum and classical theories

- Organize formalism to take limit in a simple way
  - Count $\hbar$s
  - Momenta for massive particles, wavenumbers for massless
- Example: radiated momentum
- Direct path to waveform