From scattering amplitudes to classical gravity

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Work together with
A. Cristofoli, P. Damgaard, J. Donoghue, G. Festuccia, H. Gomez, B. Holstein, L. Plante, P. Vanhove
Known for a long time that a particle version of General Relativity can be derived from the Einstein Hilbert Lagrangian.

Expand Einstein-Hilbert Lagrangian:

\[ \mathcal{L}_{EH} = \int d^4x \left[ \sqrt{-g} R \right] \quad g_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa h_{\mu\nu} \]

Derive vertices as in a particle theory - computations using Feynman diagrams!

From scattering amplitudes to classical gravity
Off-shell computation of amplitudes

- Expand Lagrangian, laborious and tedious process....
- Vertices: 3pt, 4pt, 5pt,...n-pt
- Complicated off-shell expressions

\[ V_{\mu\alpha,\nu\beta,\sigma\gamma}^{(3)}(k_1, k_2, k_3) = \kappa \text{sym} \left[ -\frac{1}{2} P_3(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\beta} \eta_{\sigma\gamma}) - \frac{1}{2} P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) \right. \]

\[ + \frac{1}{2} P_3(k_1 \cdot k_2 \eta_{\mu\nu} \eta_{\alpha\beta} \eta_{\sigma\gamma}) + P_6(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\sigma} \eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) \]

\[ - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) \]

\[ + \text{sym} \left[ + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1 \cdot k_2 \eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu}) \right], \]

(DeWitt; Sannan)

Much more complicated than Yang-Mills theory but still many useful applications.

From scattering amplitudes to classical gravity
Gravity as a quantum field theory

- **Viewpoint:** Gravity as a non-abelian gauge field theory with self-interactions

- **Non-renormalisalbe theory!** (‘t Hooft and Veltman)

  Dimensionful coupling:
  \[ G_N = \frac{1}{M_{\text{planck}}^2} \]

- **Traditional belief:** – no known symmetry can remove all UV-divergences

  String theory can by introducing new length scales

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Quantum gravity as an effective field theory

(Weinberg) proposed to view the quantization of general relativity as that of an effective field theory.

\[ \mathcal{L} = \sqrt{-g} \left[ \frac{2R}{\kappa^2} + \mathcal{L}_{\text{matter}} \right] \]

\[ \mathcal{L} = \sqrt{-g} \left\{ \frac{2R}{\kappa^2} + c_1 R^2 + c_2 R^{\mu\nu} R_{\mu\nu} + \ldots \right\} \]
Practical quantum gravity at low energies

- Consistent quantum theory:
  - Quantum gravity at low energies (Donoghue)
  - Direct connection to low energy dynamics of string and super-gravity theories
    - Suggest general relativity augmented by higher derivative operators – the most general modified theory

- A somewhat curious application:
  Classical physics from quantum theory!

**NB:** Contact with General Relativity require some care..!
(Many talks..)
One-loop (off-shell) gravity amplitude computation

Box diagrams:

Triangles:

Bubbles:

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(NEJB, Donoghue, Holstein (2001))
One-loop (off-shell) gravity amplitude computation

From scattering amplitudes to classical gravity

(NEJB, Donoghue, Holstein (2001))
One-loop (off-shell) gravity amplitude computation

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(NEJB, Donoghue, Holstein (2001))
One-loop result for gravity

- Four point amplitude can be deduced to take the form

\[ \mathcal{M} \sim \left( A + Bq^2 + \ldots + \alpha \kappa^4 \frac{1}{q^2} + \beta_1 \kappa^4 \ln(-q^2) + \beta_2 \kappa^4 \frac{m}{\sqrt{-q^2}} + \ldots \right) \]

Focus on deriving these \( \sim \)>

Long-range behavior

(no higher derivative contributions)

Short range behavior

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One-loop and the cut

- It is in fact much **simpler** to capture the long-range behavior from unitarity

\[ C_{i,...,j} = \text{Im} K_{i,...,j} > 0 \text{ M}^{1\text{-loop}} \]

(KLT + on-shell 4D input trees recycled from Yang-Mills
(Badger et al; Forde Kosower)
(e.g. D-dimensions (NEJB, Gomez, Cristofoli, Damgaard))
QCD meets gravity

KLT relationship (Kawai, Lewellen and Tye)

\[ A^M_{\text{closed}} \sim \sum_{\Pi, \Pi} e^{i\pi \Phi(\Pi, \Pi)} A^\text{left open}_M(\Pi) A^\text{right open}_M(\Pi) \]

\[
\left[ (\sim \zeta)^{\mu \mu' \nu \nu' \beta \beta'} \right] = \left[ (\sim \zeta)^{L \mu \nu \beta} \right] \otimes \left[ (\sim \zeta)^{R \mu' \nu' \beta'} \right]
\]

All multiplicity S-kernel

(NEJB, Damgaard, Feng, Søndergaard, Vanhove)

(Bern, Dixon, Dunbar, Perelstein, Rozowsky)

(many talks)

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Massive scalar-scalar scattering

- Will consider scalar-scalar scattering amplitudes mediated through graviton field theory interaction

\[ S = \int d^4 x \sqrt{-g} \left[ \frac{R}{16\pi G_N} + \frac{1}{2} \sum_a \left( g^{\mu\nu} \partial_\mu \phi_a \partial_\nu \phi_a - m_a^2 \phi_a^2 \right) \right] \]

\[ \mathcal{M} = \sum_{L=0}^{+\infty} \mathcal{M}^{L\text{-loop}} \]

\[ \mathcal{M}^{L\text{-loop}} \sim O(G_N^{L+1}) \]

\[ p_1^\mu = (E_a, \vec{p}), \quad p_2^\mu = (E_a, \vec{p}'), \]
\[ p_3^\mu = (E_b, -\vec{p}), \quad p_4^\mu = (E_b, -\vec{p}') \]

\[ |\vec{p}| = |\vec{p}'| \quad q^\mu = p_1^\mu - p_2^\mu \]

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Tree level

\[ \mathcal{M}_{\text{tree}} = - \frac{4 \pi G_N}{E_a E_b} \frac{[2(p_1 \cdot p_3)^2 - m_a^2 m_b^2 - |\vec{q}|^2 (p_1 \cdot p_3)]}{|\vec{q}|^2} \]

\[ p_1 \cdot p_3 = E_a(p) E_b(p) + |\vec{p}|^2 \]

Newton’s law through Fourier transform

\[ V(r) = -\frac{G m_1 m_2}{r} \]
Result for the one-loop amplitude

1) Expand out traces
2) Reduce to scalar basis of integrals
3) Isolate coefficients
   (Bern, Dixon, Dunbar, Kosower, NEJB, Donoghue, Vanhove)
   (See also Cachazo and Guevara)

\[ M^{1\text{-loop}} = \frac{i16\pi^2 G_N^2}{E_a E_b} \left( c_- I_- + c_\times I_\times + c_\uparrow I_\uparrow + c_\downarrow I_\downarrow + \cdots \right) \]

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\[ \mathcal{M}^{1\text{-loop}} = \frac{i16\pi^2 G_N^2}{E_a E_b} \left( c_{\Box} \mathcal{I}_{\Box} + c_{\ast} \mathcal{I}_{\ast} + c_{\rightarrow} \mathcal{I}_{\rightarrow} + c_{\leftarrow} \mathcal{I}_{\leftarrow} + \cdots \right) \]

\[ \mathcal{I}_{\Box} = \int \frac{d^{d+1} \ell}{(2\pi)^{d+1}} \frac{1}{((\ell + p_1)^2 - m_a^2 + i\varepsilon)((\ell - p_3)^2 - m_b^2 + i\varepsilon)(\ell^2 + i\varepsilon)((\ell + q)^2 + i\varepsilon)} \]

\[ \mathcal{I}_{\ast} = \int \frac{d^{d+1} \ell}{(2\pi)^{d+1}} \frac{1}{((\ell + p_1)^2 - m_a^2 + i\varepsilon)((\ell + p_4)^2 - m_b^2 + i\varepsilon)(\ell^2 + i\varepsilon)((\ell + q)^2 + i\varepsilon)} \]

\[ \mathcal{I}_{\rightarrow} = \int \frac{d^{d+1} \ell}{(2\pi)^{d+1}} \frac{1}{((\ell + q)^2 + i\varepsilon)(\ell^2 + i\varepsilon)((\ell + p_1)^2 - m_a^2 + i\varepsilon)} \]

\[ \mathcal{I}_{\leftarrow} = \int \frac{d^{d+1} \ell}{(2\pi)^{d+1}} \frac{1}{((\ell - q)^2 + i\varepsilon)(\ell^2 + i\varepsilon)((\ell - p_3)^2 - m_b^2 + i\varepsilon)} \]
Classical pieces in loops

\[ \int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{\ell^2 + i\epsilon} \frac{1}{(\ell + q)^2 + i\epsilon} \frac{1}{(\ell + p_1)^2 - m_1^2 + i\epsilon} \]

\[ (\ell + p_1)^2 - m_1^2 = \ell^2 + 2\ell \cdot p_1 \approx 2m_1 \ell_0 \]

From scattering amplitudes to classical gravity
Classical pieces in loops

\[
\frac{1}{2m_1} \int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{\ell^2 + i\epsilon} \frac{1}{(\ell + q)^2 + i\epsilon} \frac{1}{\ell_0 + i\epsilon}
\]

Close contour

\[
\int_{|\vec{\ell}| \ll m} \frac{d^3 \vec{\ell}}{(2\pi)^3} \frac{i}{4m} \frac{1}{\ell^2} \frac{1}{(\ell + q)^2} = \frac{i}{32m|\vec{q}|}
\]

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One-loop level

Branch (explained by Weinberg)  Ignore quantum pieces

\[ M^{1-\text{loop}} = \frac{i16\pi^2G_N^2}{E_a E_b} \left( c_\Box \mathcal{I}_\Box + c_\otimes \mathcal{I}_\otimes + c_\Rightarrow \mathcal{I}_\Rightarrow + c_\Leftarrow \mathcal{I}_\Leftarrow + \cdots \right) \]

\[
\mathcal{I}_\Box = -\frac{i}{16\pi^2|\vec{q}|^2} \left( \frac{1}{m_a m_b} + \frac{m_a(m_a - m_b)}{3m_a^2 m_b^2} \right) + \frac{i\pi}{|p|E_p} \left( \frac{2}{3-d} - \log |\vec{q}|^2 \right) + \cdots
\]

\[
\mathcal{I}_\otimes = -\frac{i}{16\pi^2|\vec{q}|^2} \left( \frac{1}{m_a m_b} - \frac{m_a(m_a - m_b)}{3m_a^2 m_b^2} \right) \left( \frac{2}{3-d} - \log |\vec{q}|^2 \right) + \cdots
\]

\[
\mathcal{I}_\Rightarrow = -\frac{i}{32m_a |\vec{q}|} + \cdots
\]

\[
\mathcal{I}_\Leftarrow = -\frac{i}{32m_b |\vec{q}|} + \cdots
\]

\[ c_\Box = c_\otimes = 4\left( m_a^2 m_b^2 - 2(p_1 \cdot p_3)^2 \right)^2 \]

\[ c_\Rightarrow = 3m_a^2 \left( m_a^2 m_b^2 - 5(p_1 \cdot p_3)^2 \right) \]

\[ c_\Leftarrow = 3m_b^2 \left( m_a^2 m_b^2 - 5(p_1 \cdot p_3)^2 \right) \]
Computational setup

- We use the language of old-fashioned time-ordered perturbation theory
- In particular we eliminate by hand
  - Annihilation channels
  - Back-tracking diagrams
  - Anti-particle intermediate states

We will also assume (classical) long-distance scattering distances

(Cristofoli, Bjerrum-Bohr, Damgaard, Vanhove)
Relation to a potential

- One-loop amplitude after summing all contributions

\[ \mathcal{M}^{1-\text{loop}} = \frac{\pi^2 G_N^2}{E_p^2} \left[ \frac{1}{2|q'|} \left( \frac{c_\triangleright}{m_a} + \frac{c_\triangleleft}{m_b} \right) + \frac{i c_\Box}{E_p |\vec{p}|} \left( \frac{2}{3-d} - \log |q'|^2 \right) \right]. \]

Super-classical/singular

- How to relate to a classical potential?
  - Choice of coordinates
  - Born subtraction

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Einstein-Infeld-Hoffman Potential

- Solve for potential in non-relativistic limit,

\[ i\langle f|T|i \rangle = -2\pi i \delta(E - E') \times \left[ \langle f|\tilde{V}_{bs}(q)|i \rangle + \sum_n \frac{\langle f|\tilde{V}_{bs}(q)|n \rangle \langle n|\tilde{V}_{bs}(q)|i \rangle}{E - E_n + i\epsilon} + \ldots \right] \]

\[ \langle f|\tilde{V}_{bs}(q)|i \rangle = -\frac{Gm_1m_2}{r} \left[ 1 + 3\frac{G(m_1 + m_2)}{r} \right] \]

- Contact with Einstein-Infeld-Hoffmann Hamiltonian

\[ \tilde{V}_{bs}(r) = V(r) + \frac{7Gm_1m_2(m_1 + m_2)}{2c^2r^2} \]
Post-Newtonian interaction potentials

\[ H = \frac{p_1^2}{2m_1} + \frac{p_4^2}{2m_2} - \frac{p_1^4}{8m_1^3} - \frac{p_4^4}{8m_2^3} - \frac{Gm_1m_2}{r} \left( \frac{3p_1^2}{m_1^2} + \frac{3p_4^2}{m_2^2} - \frac{7p_1 \cdot p_4}{m_1m_2} - \frac{(p_1 \cdot \vec{r})(p_4 \cdot \vec{r})}{m_1m_2r^2} \right) \frac{G^2m_1m_2(m_1 + m_2)}{2r^2} \]

(Einstein-Infeld-Hoffman, Iwasaki)

Crucial subtraction of Born term to in order to get the correct PN potential

\((3 - 7/2 \rightarrow -1/2)\)

From scattering amplitudes to classical gravity
Relation to a relativistic PM potential

- Amplitude defined via perturbative expansion around a flat Minkowskian metric

- Now we need to relate the Scattering Amplitude to the potential for a bound state problem – alternative to matching (Cheung, Solon, Rothstein; Bern, Cheung, Roiban, Shen, Solon, Zeng)

- Starting point: the Hamiltonian of the relativistic Salpeter equation

\[
\hat{H} = \hat{H}_0 + \hat{V}, \quad \hat{H}_0 = \sqrt{\hat{k}^2 + m_a^2} + \sqrt{\hat{k}^2 + m_b^2}
\]
Relation to a potential

- Analysis involves solution of the Lippmann-Schwinger recursive equation:

\[
\mathcal{M}(p, p') = \langle p|V|p'\rangle + \int \frac{d^3 k}{(2\pi)^3} \frac{\langle p|V|k\rangle \mathcal{M}(k, p')}{E_p - E_k + i\varepsilon}
\]

\[
\langle p|V|p'\rangle = \mathcal{M}(p, p') - \int \frac{d^3 k}{(2\pi)^3} \frac{\mathcal{M}(p, k)\mathcal{M}(k, p')}{E_p - E_k + i\varepsilon} + \ldots
\]

\[
V(p, r) = \int \frac{d^3 q}{(2\pi)^3} e^{iq\cdot r} V(p, q)
\]
Tree level

\[ M_{\text{tree}} = \frac{4\pi G_N}{E_a E_b} \frac{[2(p_1 \cdot p_3)^2 - m_a^2 m_b^2 - |q|^2(p_1 \cdot p_3)]}{|q|^2} \]

\[ p_1 \cdot p_3 = E_a(p) E_b(p) + |\vec{p}|^2 \]

Same result as from matching (Cheung, Solon, Rothstein; Bern, Cheung, Roiban, Shen, Solon, Zeng)

\[ V_{1PM}(p, r) = \frac{1}{E_p^2 \xi} \frac{G_N c_1(p^2)}{r} + \cdots \]

\[ c_1(p^2) \equiv m_a^2 m_b^2 - 2(p_1 \cdot p_3)^2 \]

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One-loop

\[ M^{\text{Iterated}} = -\frac{16\pi^2 G_N^2}{E_a(p^2) E_b(p^2)} \int \frac{d^d k}{(2\pi)^d} \frac{A(\vec{p}, \vec{k})}{|\vec{p} - \vec{k}|^2} \frac{A(\vec{k}, \vec{p}')}{|\vec{p}' - \vec{k}|^2} \frac{G(p^2, k^2)}{E_a(k^2) E_b(k^2)} \]

\[ G(p^2, k^2) = \frac{1}{E_p - E_k + i\epsilon} \]

\[ M^{\text{Iterated}} = \frac{32\pi^2 G_N^2}{E_p^3 \xi} c_1^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{|\vec{p} - \vec{k}|^2 |\vec{p}' - \vec{k}|^2 (k^2 - p^2)} \]

\[ -\frac{16\pi^2 G_N^2}{E_p^3 \xi^2} \left( \frac{c_1^2(1 - \xi)}{2E_p^2 \xi} + 4c_1 p_1 \cdot p_3 \right) \int \frac{d^d k}{(2\pi)^d} \frac{1}{|\vec{p} - \vec{k}|^2 |\vec{p}' - \vec{k}|^2} + \cdots \]

From scattering amplitudes to classical gravity
One-loop

\[ M^{\text{Iterated}} = \frac{i\pi G_N^2}{E_p^3 \xi} \frac{4c_1^2}{|p|} \left( \log |q|^2 - \frac{2}{3-d} \right) + \frac{2\pi^2 G_N^2}{E_p^2 \xi^2 |q|} \left( \frac{c_1^2 (\xi - 1)}{2E_p^2 \xi} - 4c_1 p_1 \cdot p_3 \right) \]

\[ M^{1-\text{loop}} = \frac{\pi^2 G_N^2}{E_p^3 \xi} \left[ \frac{1}{2|q|} \left( \frac{c_\triangledown}{m_a} + \frac{c_\triangleleft}{m_b} \right) + \frac{i}{E_p |p|} \frac{c_\Box}{\pi |q|^2} \left( \frac{2}{3-d} - \log |q|^2 \right) \right] \]

\[ V_{2PM}(p, q) = M^{1-\text{loop}} + M^{\text{Iterated}} \]

Again same result as from matching, no singular term
Effective potential

In fact we do not have to go through either matching procedure or solving Lippmann-Schwinger to derive observables such as the scattering angle.

Energy relation makes everything simple:

\[
p^2 = p^2_\infty - 2E\xi \left[ \tilde{M}^{\text{cl.}}_{\text{tree}}(p^2_\infty, r) + \tilde{M}^{\text{cl.}}_{1-\text{loop}}(p^2_\infty, r) \right]
\]

(Damour; Bern, Cheung, Roiban, She, Solon, Zeng; Kalin, Porto; NEJB, Damgaard, Cristofoli)
Effective potential

Thus given the classical amplitude

\[
\tilde{M}^{\text{cl.}}(p, r) \equiv - \frac{1}{2E\xi} \sum_{n=1}^{\infty} \frac{G_N^m \tilde{c}_{(n-1)-\text{loop}}(p)}{r^n}
\]

\[
f_n(E) = \tilde{c}_{(n-1)-\text{loop}}(p_{\infty}) \quad V_{\text{eff}}(r) \equiv - \sum_{n=1}^{\infty} \frac{G_N^m f_n(E)}{r^n}
\]

Non-relativistic Hamiltonian with effective potential

\[
\hat{H} = \hat{p}^2 + V_{\text{eff}}(r)
\]
Scattering angle all orders

\[ \chi = \sum_{k=1}^{\infty} \tilde{\chi}_k(b) , \quad \tilde{\chi}(b) \equiv \frac{2b}{k!} \int_0^{+\infty} du \left( \frac{d}{du^2} \right)^k \frac{V^k_{\text{eff}}(r)}{p_{\infty}^{2k}} \]

(Kalin, Porto; NEJB, Damgaard, Cristofoli)

\[ p_r = \sqrt{p_{\infty}^2 - \frac{L^2}{r^2} - V_{\text{eff}}(r)} \]

\[ \frac{\chi}{2} = - \int_{r_m}^{+\infty} dr \frac{\partial p_r}{\partial L} - \frac{\pi}{2} \]

Corrects ‘Bohm’s formula’ + no reference minimal distance
post-Minkowskian expansion

Will use similar eikonal setup as for bending of light (extended to massive case):

\[
M(\vec{b}) = \int d^2 \vec{q} e^{-i \vec{q} \cdot \vec{b}} M(\vec{q})
\]

\[
M(\vec{b}) = 4p(E_1 + E_2)(e^{i \chi(\vec{b})} - 1)
\]

Eikonal phase

\[
\vec{p}_1 = -\vec{p}_4
\]

\[
b \equiv |\vec{b}|
\]

b orthogonal and

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post-Minkowskian expansion

Stationary phase condition (leading order in q)

\[
2 \sin(\theta/2) = \frac{-2M}{\sqrt{\hat{M}^4 - 4m_1^2 m_2^2}} \frac{\partial}{\partial b} (\chi_1(b) + \chi_2(b))
\]

\[
\chi_1(b) = 2G \frac{\hat{M}^4 - 2m_1^2 m_2^2}{\sqrt{\hat{M}^4 - 4m_1^2 m_2^2}} \left( \frac{1}{d-2} - \log \left( \frac{b}{2} \right) - \gamma_E \right)
\]

\[
\chi_2(b) = \frac{3\pi G^2}{8\sqrt{\hat{M}^4 - 4m_1^2 m_2^2}} \frac{m_1 + m_2}{b} \left( 5\hat{M}^4 - 4m_1^2 m_2^2 \right)
\]
Final result becomes

\[ 2 \sin \left( \frac{\theta}{2} \right) = \frac{4GM}{b} \left( \frac{\hat{M}^4 - 2m_1^2m_2^2}{\hat{M}^4 - 4m_1^2m_2^2} \right) + \frac{3\pi}{16} \frac{G(m_1 + m_2)}{b} \left( \frac{5\hat{M}^4 - 4m_1^2m_2^2}{\hat{M}^4 - 4m_1^2m_2^2} \right) \]

Agrees with (Westpfahl)

Light-like limit

\[ \theta = \frac{4Gm_1}{b} + \frac{15\pi}{4} \frac{G^2m_1^2}{b^2} \]
Any PM order given amplitude...

<table>
<thead>
<tr>
<th>PM</th>
<th>$\chi^{PM}/(\frac{G_{N}}{p_{\infty}}L)^{PM}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$f_1$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{2}\pi p_{\infty}^2 f_2$</td>
</tr>
<tr>
<td>3</td>
<td>$2f_3p_{\infty}^4 + f_1 f_2 p_{\infty}^2 - \frac{f_1^3}{12}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{3}{8}\pi p_{\infty}^4 (2f_4 p_{\infty}^2 + f_2^2 + 2f_1 f_3)$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{8}{3}f_5p_{\infty}^8 + 4(f_2 f_3 + f_1 f_4)p_{\infty}^6 + f_1(f_2^2 + f_1 f_3)p_{\infty}^4 - \frac{1}{6}f_1^3 f_2 p_{\infty}^2 + \frac{f_1^5}{80}$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{5}{16}\pi p_{\infty}^6 (3f_6 p_{\infty}^4 + 3(f_3^2 + 2f_2 f_4 + 2f_1 f_5)p_{\infty}^2 + f_2^3 + 6f_1 f_2 f_3 + 3f_1^2 f_4)$</td>
</tr>
<tr>
<td>7</td>
<td>$\frac{16}{5}f_7 p_{\infty}^{10} + 8(f_3 f_4 + f_2 f_5 + f_1 f_6)p_{\infty}^8 + 6(f_3 f_2^2 + 2f_1 f_4 f_2 + f_1(f_3^2 + f_1 f_5))p_{\infty}^6 - \frac{1}{8}f_1^3 (2f_2^2 + f_1 f_3)p_{\infty}^4 + \frac{3}{80}f_1^5 f_2 p_{\infty}^2 - \frac{f_1^7}{448}$</td>
</tr>
<tr>
<td>8</td>
<td>$\frac{35}{128}\pi p_{\infty}^8 (4f_8 p_{\infty}^6 + 6(f_4^2 + 2(f_3 f_5 + f_2 f_6 + f_1 f_7))p_{\infty}^4 + 12(f_4 f_2^2 + (f_3^2 + 2f_1 f_5) f_2 + f_1(2f_3 f_4 + f_1 f_6))p_{\infty}^2 + f_2^4 + 6f_1 f_2^2 + 12f_1^2 f_2 f_3 + 12f_1^2 f_2 f_4 + 4f_1^3 f_5$</td>
</tr>
</tbody>
</table>

Confirmation of 3PM & 4PM

Bern, Cheung, Roiban, Shen, Solon, Zeng) )

From scattering amplitudes to classical gravity
Outlook

- Amplitude toolbox for computations already provided new efficient methods for computation:
- Double-copy and KLT clearly helps simplify computations
- Amplitude tools can provide compact trees for unitarity computations
- Very impressive computations by (Bern, Cheung, Roiban, Shen, Solon, Zeng, and many others) + much more to come...
- Endless tasks ahead / open questions regarding spin, radiation, quantum terms, high order curvature terms etc
- Clearly much more physics to learn....