SUSY decomposition and BCJ numerators from forward limits

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Based on 1703.01269 and 1703.05717 with Bo Feng/Yi-Jian Du

and work in progress with Alex Edison, Song He and Oliver Schlotterer

Outline

(We consider D dimensions, arbitrary matter contents, but only external gluons at one loop.)

CHY formalism at tree level and one loop

Explicit construction of master numerators at tree level and one loop

SUSY decomposition from representation theory

Cachazo-He-Yuan formalism at tree level

$$\mathcal{M}^{(0)}_{L\otimes R} = \int \frac{d\sigma_1 \cdots d\sigma_n}{\mathrm{Vol}[\mathrm{SL}(2,\mathbb{C})]} \left[\prod_{a=1}^{n}{}' \delta(E_a)\right] \mathcal{I}^{(0)}_L(k,\epsilon,\sigma)\, \mathcal{I}^{(0)}_R(k,\tilde{\epsilon},\sigma)$$

$$\equiv \int d\mu_n^{\mathrm{tree}}\, \mathcal{I}^{(0)}_L(k,\epsilon,\sigma)\, \mathcal{I}^{(0)}_R(k,\tilde{\epsilon},\sigma)$$

Cachazo, He, Yuan
1307.2199, ...

- The delta function imposes $n-3$ independent scattering equations:

$$\prod_{a=1}^{n}{}' \delta(E_a) \equiv \sigma_{ij}\sigma_{jk}\sigma_{ki} \prod_{a\neq i,j,k} \delta(E_a)$$

$$E_a = \sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} \equiv \sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_{ab}} = 0$$

- The $\mathrm{SL}(2,\mathbb{C})$ redundancy is fixed by:

$$\frac{d\sigma_1 \cdots d\sigma_n}{\mathrm{Vol}[\mathrm{SL}(2,\mathbb{C})]} = \sigma_{pq}\sigma_{qr}\sigma_{rp} \prod_{c\neq p,q,r}^{n} d\sigma_c$$

Pure gluon integrand

$$\mathcal{I}_L^{(0)} = \mathcal{I}_{\text{color}}^{(0)} = \frac{\text{Tr}\left(T^{a_1}T^{a_2}\ldots T^{a_n}\right)}{\sigma_{12}\sigma_{23}\ldots\sigma_{n1}} + \text{perm}\,(2,3,\ldots,n)$$
$$\equiv \text{PT}\,(1,2,\ldots,n)\,\text{Tr}\left(T^{a_1}T^{a_2}\ldots T^{a_n}\right) + \text{perm}\,(2,3,\ldots,n)$$
$$\mathcal{I}_R^{(0)} = \mathcal{I}_{\text{gluon}}^{(0)} = \text{Pf}'(\Psi) \equiv \frac{(-1)^{i+j}}{\sigma_{ij}}\text{Pf}\,(\Psi_{ij}^{ij})$$

Ψ is a $2n \times 2n$ anti-symmetric matrix:

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}$$

The $n \times n$ blocks A, B and C are defined as

	A_{ab}	B_{ab}	C_{ab}
$a \neq b$	$\frac{k_a \cdot k_b}{\sigma_{ab}}$	$\frac{\epsilon_a \cdot \epsilon_b}{\sigma_{ab}}$	$\frac{\epsilon_a \cdot k_b}{\sigma_{ab}}$
$a = b$	0	0	$-\sum_{c \neq a} \frac{\epsilon_a \cdot k_c}{\sigma_{ac}}$

Pure gluon integrand

On the support of scattering equations:

$$\mathrm{Pf}'(\Psi) = \sum_{\mathsf{A}\cup\mathsf{B}=\{2,3,\ldots,n-1\}} \mathrm{Pf}\,(\Psi_{\mathsf{A}})(-1)^{|\mathsf{A}|} \sum_{\rho\in S_{\mathsf{B}}} W^{(0)}_{\mathrm{gluon}}(1,\rho,n)\,\mathrm{PT}\,(1,\rho,n)$$

The baseline factor for the pure gluon case is

$$W^{(0)}_{\mathrm{gluon}}(1,\rho,n) = \epsilon_1\cdot f_{\rho(b_1)}\cdot f_{\rho(b_2)}\ldots f_{\rho(b_{|\mathsf{B}|})}\cdot\epsilon_n\,,$$

where $f_i^{\mu\nu} = k_i^\mu\epsilon_i^\nu - k_i^\nu\epsilon_i^\mu$ is the linearized field strength.

Fu, Du, Huang, Feng
1702.08158
Du, **FT**
1703.05717

Two-fermion and two-scalar integrand

- If 1 and n are scalars, $W_{2\mathrm{s}}(1,n)=1$ and zero otherwise,

Cachazo, He, Yuan 1309.0885

$$\mathcal{I}_{2\mathrm{s}}^{(0)} = (-1)^{n-2}\,\mathrm{Pf}\,(\Psi_{\{2,3,\ldots,n-1\}})\,\mathrm{PT}(1,n)$$

- If 1 and n are fermions,

$$\mathcal{I}_{2\mathrm{f}}^{(0)} = \sum_{\mathsf{A}\cup\mathsf{B}=\{2,3,\ldots,n-1\}} \mathrm{Pf}\,(\Psi_{\mathsf{A}})(-1)^{|\mathsf{A}|}\sum_{\rho\in S_{\mathsf{B}}} W_{2\mathrm{f}}^{(0)}(1,\rho,n)\,\mathrm{PT}\,(1,\rho,n)$$

- The baseline factor for the two-fermion case is

He, Schlotterer 2015 unpublished

$$W_{2\mathrm{f}}^{(0)}(1,\rho,n) = (\chi_1 \not{f}_{\rho_1}\not{f}_{\rho_2}\ldots\not{f}_{\rho_{|\mathsf{B}|}}\xi_n) \ \text{ where } \ \not{f}_i \equiv \frac{1}{8}f_i^{\mu\nu}[\gamma_\mu,\gamma_\nu] = \frac{1}{2}\not{k}_i\not{\epsilon}_i$$

- On-shell fermionic wavefunctions:

$$k_\mu\gamma^\mu_{\alpha\beta}\chi^\beta = \not{k}_{\alpha\beta}\chi^\beta = 0\,, \qquad k^\mu\gamma_\mu^{\alpha\beta}\xi_\beta = \not{k}^{\alpha\beta}\xi_\beta = \chi^\alpha$$

$$\text{e.g. from 10d to 4d } \chi^\alpha \to (\langle k|^\alpha, [k|_{\dot\alpha})^A\,, \quad \xi_\alpha \to (\tfrac{|q\rangle_\alpha}{\langle kq\rangle}, \tfrac{|q]^{\dot\alpha}}{[kq]})_A$$

- Fermion vertex operators are in $(-\frac{1}{2},-\frac{3}{2})$ ghost picture

Roehrig, Skinner 1709.03262

Forward limits, one-loop CHY integrand

One-loop scattering equations localize the integrand to $\tau \to i\infty$, where the torus degenerates to a nodal sphere,

$$\mathcal{M}^{(1)}_{L\otimes R} = \int \frac{d^D\ell}{\ell^2} \int d\mu_n^{\text{nodal}}\, \mathcal{I}^{(1)}_L(k,\epsilon,\sigma)\, \mathcal{I}^{(1)}_R(k,\tilde{\epsilon},\sigma)$$

- Nodal sphere measure: $d\mu^{\text{tree}}_{n+2}$ at forward limit $k_\pm \to \pm\ell$,

$$d\mu_n^{\text{nodal}} = \frac{d\sigma_+ d\sigma_- d\sigma_1 d\sigma_2 \ldots d\sigma_n}{\text{Vol}[\text{SL}(2,\mathbb{C})]} \prod_a{}' \delta(E_a)$$

$$E_+ = \sum_{i=1}^n \frac{\ell\cdot k_i}{\sigma_{+i}}\,, \qquad E_- = -\sum_{i=1}^n \frac{\ell\cdot k_i}{\sigma_{-i}}\,, \qquad E_i = \frac{k_i\cdot\ell}{\sigma_{i+}} - \frac{k_i\cdot\ell}{\sigma_{i-}} + \sum_{j\neq i} \frac{k_i\cdot k_j}{\sigma_{ij}}$$

Geyer, Mason, Monteiro, Tourkine
1507.00321, 1511.06315
Adamo, Casali, Skinner
1312.3828

- For color-ordered amplitudes (single trace)

$$\mathcal{I}^{(1)}_L = \mathcal{I}^{(1)}_{\text{color}} = \text{PT}(+,1,2,\ldots,n,-) + \text{cyclic}\,(1,2,\ldots,n)$$

- Integrands are given in linearized propagators

Forward limits, one-loop CHY integrand

- Forward limit: a map from tree level to one loop

Geyer, Monteiro 1711.09923
He, Schlotterer 1612.00417

Agerskov, Bjerrum-Bohr, Gomez, Lopez-Arcos 1910.03602

$$\text{fwl}: \quad \begin{aligned} & k_\pm \to \pm\ell \\ & \epsilon_\mu(k_+)\epsilon_\nu(k_-) \to \eta_{\mu\nu} - (\ell_\mu\bar{\ell}_\nu + \ell_\nu\bar{\ell}_\mu) \ \text{ where } \ \ell\cdot\bar{\ell} = 1 \\ & \chi^\alpha(k_+)\xi_\beta(k_-) \to \frac{1}{2}\delta^\alpha_\beta \end{aligned}$$

Roehrig, Skinner 1709.03262

- SUSY decomposition at one-loop

$$\begin{aligned} & \mathcal{I}^{(1)}_{(n_\mathrm{v},n_\mathrm{f},n_\mathrm{s},D)}(1,2,\ldots,n) \\ &= \Big[n_\mathrm{v}\,\mathcal{I}^{(0)}_\mathrm{gluon}(+,1,2,\ldots,n,-) - n_\mathrm{f}\,\mathcal{I}^{(0)}_\mathrm{2f}(+,1,2,\ldots,n,-) \\ &\quad + n_\mathrm{s}\,\mathcal{I}^{(0)}_\mathrm{2s}(+,1,2,\ldots,n,-) \Big]_\mathrm{fwl} \\ &= \sum_{\mathsf{A}\cup\mathsf{B}=\{1,2,\ldots,n\}} \underbrace{\mathrm{Pf}\,(\Psi_\mathsf{A})(-1)^{|\mathsf{A}|}}_{\mathcal{O}(\ell^{n-|\rho|})} \sum_{\rho\in S_\mathsf{B}} \mathrm{PT}(+,\rho,-)\, \underbrace{W^{(1)}_{(n_\mathrm{v},n_\mathrm{f},n_\mathrm{s},D)}(\rho)}_{\text{independent of } \ell} \end{aligned}$$

- The one-loop baseline factor:

$$\begin{aligned} & W^{(1)}_{(n_\mathrm{v},n_\mathrm{f},n_\mathrm{s},D)}(\rho) \\ &= \begin{cases} n_\mathrm{v}(D-2) + n_\mathrm{s} - n_\mathrm{f}\,2^{D/2-2} & \mathsf{B} = \emptyset \\ n_\mathrm{v}\mathrm{Tr}_\mathrm{v}(f_{\rho(b_1)}f_{\rho(b_2)}\ldots f_{\rho(b_{|\mathsf{B}|})}) - \dfrac{n_\mathrm{f}}{2}\mathrm{Tr}_\mathrm{s}(f_{\rho(b_1)}f_{\rho(b_2)}\ldots f_{\rho(b_{|\mathsf{B}|})}) & \mathsf{B} \neq \emptyset \end{cases} \end{aligned}$$

The master numerators (DDM basis)

$$\mathcal{I}^{(0)}_{\text{gluon/2f/2s}} = \sum_{\mathsf{A}\cup\mathsf{B}=\{2,3,\ldots,n-1\}} \mathrm{Pf}\,(\Psi_{\mathsf{A}})(-1)^{|\mathsf{A}|} \sum_{\rho\in S_{\mathsf{B}}} \mathrm{PT}\,(1,\rho,n) W^{(0)}_{\text{gluon/2f/2s}}(1,\rho,n)$$

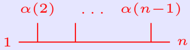
$$\mathcal{I}^{(1)}_{(n_{\rm v},n_{\rm f},n_{\rm s},D)} = \sum_{\mathsf{A}\cup\mathsf{B}=\{1,2,\ldots,n\}} \mathrm{Pf}\,(\Psi_{\mathsf{A}})(-1)^{|\mathsf{A}|} \sum_{\rho\in S_{\mathsf{B}}} \mathrm{PT}(+,\rho,-) W^{(1)}_{(n_{\rm v},n_{\rm f},n_{\rm s},D)}(\rho)$$

$\downarrow$

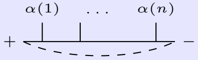
Cachazo, He, Yuan
1309.0885

Del Duca, Dixon, Maltoni
hep-th/9910563

$$\mathcal{I}^{(0)}_{\text{gluon/2f/2s}}(1,2,\ldots,n) = \sum_{\alpha\in S_{n-2}} \mathrm{PT}(1,\alpha,n)\; \underbrace{N^{(0)}_{\text{gluon/2f/2s}}(1,\alpha,n)}$$



$$\mathcal{I}^{(1)}_{(n_{\rm v},n_{\rm f},n_{\rm s},D)}(1,2,\ldots,n) = \sum_{\alpha\in S_{n}} \mathrm{PT}(+,\alpha,-)\; \underbrace{N^{(0,\text{fwl})}_{(n_{\rm v},n_{\rm f},n_{\rm s},D)}[+,\alpha,-]}$$



DDM basis numerators

On the support of scattering equations,

$$\mathcal{I}^{(0)}_{\text{gluon/2f/2s}} = \sum_{\alpha \in S_{n-2}} \text{PT}(1,\alpha,n)\, N^{(0)}_{\text{gluon/2f/2s}}[1,\alpha,n]$$

Construct local n-point numerators

Du, **FT**
1703.05717
FT, Feng
1703.01269

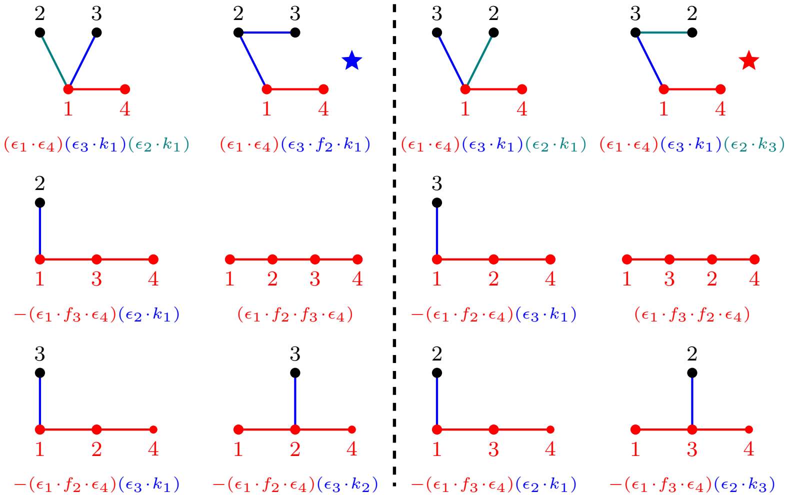
Fu, Du, Huang, Feng
1702.08158

$$N^{(0)}[1,\alpha(2,\ldots,n-1),n] = \sum_{T \in \text{IT}(\alpha)} N^{(0)}_{\text{RO}}[T]$$

Other methods, see [Bjerrum-Bohr, Bourjaily, Damgaard, Feng, 1608.00006]

Tree-level DDM basis numerators

$N^{(0)}_{\text{gluon}}(1,2,3,4)$ RO $= (3,2)$ $N^{(0)}_{\text{gluon}}(1,3,2,4)$



$$N^{(0)}[1,\alpha,n] = \sum_{T\in \mathrm{IT}(\alpha)} N^{(0)}_{\text{RO}}[T]$$

★ (blue): $(\epsilon_1\cdot\epsilon_4)(\epsilon_3\cdot k_2)(\epsilon_2\cdot k_1)$ if RO $= (2,3)$

★ (red): $(\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot f_3\cdot k_1)$ if RO $= (2,3)$

Improvement: crossing symmetric numerators

- Crossing symmetry is not manifest due to the reference order,

$$N^{(0)}[1, \alpha(2, \ldots, n{-}1), n] \neq N^{(0)}[1, 2, \ldots, n{-}1, n]\Big|_{2\to\alpha(2),\ldots,n-1\to\alpha(n-1)}$$

- crossing symmetric: average over reference orders

$$\text{complexity:} \quad \underbrace{(n-1)!}_{\text{increasing trees}} \times \underbrace{(n-2)!}_{\text{reference orders}} + \text{relabeling}$$

- **Goal:** traverse each tree once and obtain the crossing symmetric result

$$\text{complexity:} \quad \underbrace{(n-1)!}_{\text{increasing trees}} + \text{relabeling}$$

Improvement: crossing symmetric numerators

- The baseline factors are crossing symmetric, so we only need to consider the structure planted on top.
- First, if i is a leaf, we have 

$\rightarrow V_i^\mu = \epsilon_i^\mu$.
- If a vertex j has only one branch:

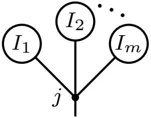
$$\rightarrow V^\mu_{I\cup\{j\}} = \alpha(V_I \cdot k_j)\epsilon_j^\mu + \beta(V_I \cdot f_j)^\mu$$

- The first term is contributed by the reference orders in which j is before the entire set I, while the second term is contributed by the others.
- Thus only the relative orderings matter:

$$\alpha = \frac{|I|!}{(|I|+1)!} = \frac{1}{|I|+1}\,, \qquad \beta = \frac{|I|\times|I|!}{(|I|+1)!} = \frac{|I|}{|I|+1}$$

Improvement: crossing symmetric numerators

- If there are multiple branches connected to j (with $I = \bigcup_{i=1}^m I_i$):



$$\to V^\mu_{I\cup\{j\}} = \alpha\,\epsilon_j^\mu \prod_{i=1}^{m} V_{I_i}\cdot k_j + \sum_{i=1}^{m} \beta_i\,(V_{I_i}\cdot f_j)^\mu \prod_{l\neq i} V_{I_l}\cdot k_j$$

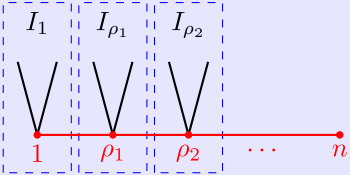
- α: (normalized) number of reference orders in which j is before I
- β_i: (normalized) number of reference orders in which the first element in the sub-order of $I \cup \{j\}$ belongs to I_i

$$\alpha = \frac{|I|!}{(|I|+1)!} = \frac{1}{|I|+1}\,, \qquad \beta_i = \frac{|I_i|\times|I|!}{(|I|+1)!} = \frac{|I_i|}{|I|+1}$$

Tree-level master numerators

DDM basis numerator: $$N^{(0)}_{\text{gluon/2f/2s}}(1,\alpha,n) = \sum_{T\in \mathrm{IT}(\alpha)} N^{(0)}_{\text{gluon/2f/2s}}[T]\,,$$

where for $T =$



$$N^{(0)}_{\text{gluon/2f/2s}}[T] = (-1)^{n-|\rho|} W^{(0)}_{\text{gluon/2f/2s}}(1,\rho,n) \prod_{j\in\rho\cup\{1\}} V_{I_j}\cdot k_j$$

The master numerators (DDM basis)

$$\mathcal{I}^{(0)}_{\text{gluon}/2\text{f}/2\text{s}} = \sum_{\mathsf{A}\cup\mathsf{B}=\{2,3,\ldots,n-1\}} \text{Pf}\,(\Psi_{\mathsf{A}})(-1)^{|\mathsf{A}|} \sum_{\rho\in S_{\mathsf{B}}} \text{PT}\,(1,\rho,n) W^{(0)}_{\text{gluon}/2\text{f}/2\text{s}}(1,\rho,n)$$

$$\mathcal{I}^{(1)}_{(n_{\text{v}},n_{\text{f}},n_{\text{s}},D)} = \sum_{\mathsf{A}\cup\mathsf{B}=\{1,2,\ldots,n\}} \underbrace{\text{Pf}\,(\Psi_{\mathsf{A}})(-1)^{|\mathsf{A}|}}_{\mathcal{O}(\ell^{n-|\rho|})} \sum_{\rho\in S_{\mathsf{B}}} \text{PT}(+,\rho,-) \underbrace{W^{(1)}_{(n_{\text{v}},n_{\text{f}},n_{\text{s}},D)}(\rho)}_{\text{independent of } \ell}$$

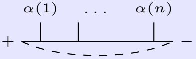
↓

Cachazo, He, Yuan
1309.0885

Del Duca, Dixon, Maltoni
hep-th/9910563

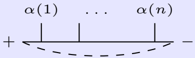
$$\mathcal{I}^{(0)}_{\text{gluon}/2\text{f}/2\text{s}}(1,2,\ldots,n) = \sum_{\alpha\in S_{n-2}} \text{PT}(1,\alpha,n)\ \underbrace{N^{(0)}_{\text{gluon}/2\text{f}/2\text{s}}(1,\alpha,n)}$$

$$\mathcal{I}^{(1)}_{(n_{\text{v}},n_{\text{f}},n_{\text{s}},D)}(1,2,\ldots,n) = \sum_{\alpha\in S_n} \text{PT}(+,\alpha,-)\ \underbrace{N^{(0,\text{fwl})}_{(n_{\text{v}},n_{\text{f}},n_{\text{s}},D)}[+,\alpha,-]}$$



One-loop master numerators

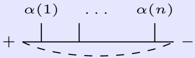
$$N^{(1)}_{(n_{\rm v},n_{\rm f},n_{\rm s},D)}[\alpha;\ell] = \underbrace{N^{(0,\text{fwl})}_{(n_{\rm v},n_{\rm f},n_{\rm s},D)}[+,\alpha,-]} = \sum_{T\in \text{IT}(\alpha)} N^{(0,\text{fwl})}_{(n_{\rm v},n_{\rm f},n_{\rm s},D)}[T]$$



$$N^{(0,\text{fwl})}_{(n_{\rm v},n_{\rm f},n_{\rm s},D)}[T] = (-1)^{n-|\rho|} \overbrace{W^{(1)}_{(n_{\rm v},n_{\rm f},n_{\rm s},D)}(\rho)}^{\text{independent of } \ell} \overbrace{\prod_{j\in\rho\cup\{\ell\}} V_{I_j}\cdot k_j}^{\mathcal{O}(\ell^{n-|\rho|})}$$

One-loop master numerators

$$N^{(1)}_{(n_\mathrm{v},n_\mathrm{f},n_\mathrm{s},D)}[\alpha;\ell] = \underbrace{N^{(0,\mathrm{fwl})}_{(n_\mathrm{v},n_\mathrm{f},n_\mathrm{s},D)}[+,\alpha,-]} = \sum_{T\in\mathrm{IT}(\alpha)} N^{(0,\mathrm{fwl})}_{(n_\mathrm{v},n_\mathrm{f},n_\mathrm{s},D)}[T]$$



$$N^{(0,\mathrm{fwl})}_{(n_\mathrm{v},n_\mathrm{f},n_\mathrm{s},D)}[T] = (-1)^{n-|\rho|} \overbrace{W^{(1)}_{(n_\mathrm{v},n_\mathrm{f},n_\mathrm{s},D)}(\rho)}^{\text{independent of } \ell} \overbrace{\prod_{j\in\rho\cup\{\ell\}} V_{I_j}\cdot k_j}^{\mathcal{O}(\ell^{n-|\rho|})}$$

$$W^{(1)}_{(n_\mathrm{v},n_\mathrm{f},n_\mathrm{s},D)}(\rho) = \begin{cases} n_\mathrm{v}(D-2) + n_\mathrm{s} - n_\mathrm{f}\, 2^{D/2-2} & \mathsf{B} = \emptyset \\ n_\mathrm{v}\mathrm{Tr}_\mathrm{v}(f_{\rho(b_1)} f_{\rho(b_2)} \ldots f_{\rho(b_{|\mathsf{B}|})}) - \frac{n_\mathrm{f}}{2}\mathrm{Tr}_\mathrm{s}(f_{\rho(b_1)} f_{\rho(b_2)} \ldots f_{\rho(b_{|\mathsf{B}|})}) & \mathsf{B} \neq \emptyset \end{cases}$$

Power of loop momentum

- Assuming local numerators
- For B $=\emptyset$, the baseline factor $W^{(1)}_{(n_{\rm v},n_{\rm f},n_{\rm s},D)}(\emptyset)$ vanishes if

	maximal	half-maximal		minimal	
	vector	vector	hyper	vector	hyper
$(n_{\rm v}, n_{\rm f}, n_{\rm s}, D)$	$(1,1,0,10)$	$(1,2,0,6)$	$(0,1,2,6)$	$(1,2,0,4)$	$(0,1,1,4)$

- In addition, for $|\mathrm{B}|=1$:

$$W^{(1)}_{(n_{\rm v},n_{\rm f},n_{\rm s},D)}(b_1) = n_{\rm v}\mathrm{Tr}_{\rm v}(f_{b_1}) - \frac{n_{\rm f}}{2}\mathrm{Tr}_{\rm s}(f_{b_1}) = 0 \quad (f_i \text{ is antisymmetric})$$

- For $|\mathrm{B}|=2$ and 3,

$$\mathrm{Tr}_{\rm s}(f_1 f_2) = 2^{D/2-4}\mathrm{Tr}_{\rm v}(f_1 f_2)\,, \quad \mathrm{Tr}_{\rm s}(f_1 f_2 f_3) = 2^{D/2-4}\mathrm{Tr}_{\rm v}(f_1 f_2 f_3)$$

- In the presence of SUSY, $\mathrm{order}(\ell) \leqslant n-2$ in the numerators
- In the presence of maximal SUSY (16 supercharges), $\mathrm{order}(\ell) \leqslant n-4$ in the numerators

Gamma trace to Lorentz trace

The parity even part of the gamma trace is given by

$$\mathrm{Tr}_{\mathrm{s}}(1,2,\ldots,n)\Big|_{\substack{\text{parity}\\ \text{even}}} = 2^{D/2-1-n}\sum_{m=1}^{n/2}\frac{1}{2^m m!}\sum_{\substack{\mathsf{A}_1\cup\mathsf{A}_2\cup\ldots\cup\mathsf{A}_m\\ =\{1,2,\ldots,n\}}}\left[\prod_{i=1}^{m}\sum_{\rho\in S_{\mathsf{A}_i}/Z_{\mathsf{A}_i}}\mathrm{Tr}_{\mathrm{v}}(\rho)\,\mathrm{ord}_\rho^{\mathrm{id}}\right],$$

where $\mathrm{ord}_\rho^{\mathrm{id}}$ is a sign function depending on the number of descents in ρ, assuming ρ_1 is the smallest element,

$$\mathrm{ord}_\rho^{\mathrm{id}} = (-1)^{\text{number of descents in }\rho}\,.$$

For example, $\mathrm{ord}_{1243}^{\mathrm{id}} = -1$ and $\mathrm{ord}_{1432}^{\mathrm{id}} = 1$.

Maximal SUSY

- $n = 4$: the baseline factor reproduces the permutation-symmetric t_8-tensor as the numerator, Green, Schwarz, Brink 1982

$$N^{(1)}_{\text{max}}[1,2,3,4] = \mathrm{Tr}_{\mathrm{v}}(f_1 f_2 f_3 f_4) - \frac{1}{2}\mathrm{Tr}_{\mathrm{s}}(f_1 f_2 f_3 f_4) = \frac{1}{2} t_8(1,2,3,4)$$

$$t_8(1,2,3,4) = \mathrm{Tr}_{\mathrm{v}}(f_1 f_2 f_3 f_4) - \frac{1}{4}\mathrm{Tr}_{\mathrm{v}}(f_1 f_2)\mathrm{Tr}_{\mathrm{v}}(f_3 f_4) + \mathrm{cyclic}(2,3,4)$$

- $n = 5$: introduce two-particle polarization and field strength,

$$\epsilon^{\mu}_{12} = (\epsilon_1 \cdot k_2)\epsilon^{\mu}_2 - (\epsilon_2 \cdot k_1)\epsilon^{\mu}_1 + \frac{1}{2}(\epsilon_1 \cdot \epsilon_2)(k_1 - k_2)^{\mu}$$

$$f^{\mu\nu}_{12} = (\epsilon_1 \cdot k_2) f^{\mu\nu}_2 - (\epsilon_2 \cdot k_1) f^{\mu\nu}_1 + [f_1, f_2]^{\mu\nu}$$
$$= k^{\mu}_{12}\epsilon^{\nu}_{12} - k^{\nu}_{12}\epsilon^{\mu}_{12} - (k_1 \cdot k_2)[\epsilon^{\mu}_1, \epsilon^{\nu}_2]$$

He, Schlotterer, Zhang 1706.00640
Mafra, Schlotterer 1410.0668

- The numerator is given by

$$N^{(1)}_{\text{max}}[1,2,3,4,5;\ell] = \ell_{\mu} t^{\mu}_8(1,2,3,4,5) - \frac{1}{2}\Big[t_8(12,3,4,5) + (12|1,2,3,4,5)\Big]$$

$$t^{\mu}_8(\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D},\mathsf{E}) = \epsilon^{\mu}_{\mathsf{A}} t_8(\mathsf{B},\mathsf{C},\mathsf{D},\mathsf{E}) + (\mathsf{A} \leftrightarrow \mathsf{B},\mathsf{C},\mathsf{D},\mathsf{E})$$

- The five-point numerators satisfy $N^{(1)}_{\text{max}}[2,3,4,5,1;\ell] = N^{(1)}_{\text{max}}[1,2,3,4,5;\ell - k_1]$

Six-point numerator with maximal SUSY

He, Schlotterer, Zhang
1706.00640
Mafra, Schlotterer
1410.0668

$$N^{(1)}_{\mathsf{max}}[1,2,3,4,5,6;\ell]$$
$$= \ell_\mu \ell_\nu t_8^{\mu\nu}(1,2,3,4,5,6) - \Big[\ell_\mu t_8^\mu(12,3,4,5,6) + (12|1,2,3,4,5,6)\Big]$$
$$+ \frac{1}{2}\Big[t_8(12,34,5,6) + (12|34|1,2,3,4,5,6)\Big]$$
$$+ \frac{1}{3}\Big[t_8(123,4,5,6) + t_8(321,4,5,6) + (123|1,2,3,4,5,6)\Big]$$
$$+ \frac{1}{6}\,\hat{t}_{12}(1,2,3,4,5,6)$$

on linearized propagators

$$t_8^{\mu\nu}(\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D},\mathsf{E},\mathsf{F}) = \epsilon_{\mathsf{A}}^{(\mu}\epsilon_{\mathsf{B}}^{\nu)} t_8(\mathsf{C},\mathsf{D},\mathsf{E},\mathsf{F}) + (\mathsf{AB}|\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D},\mathsf{E},\mathsf{F})$$
$$f_{123}^{\mu\nu} = k_{123}^{[\mu}\epsilon_{123}^{\nu]} - (k_{12}\cdot k_3)\epsilon_{12}^{[\mu}\epsilon_3^{\nu]} - (k_1\cdot k_2)(\epsilon_1^{[\mu}\epsilon_{23}^{\nu]} + \epsilon_{13}^{[\mu}\epsilon_2^{\nu]})$$
$$\epsilon_{123}^\mu = \frac{1}{2}\Big[(\epsilon_{12}\cdot k_3)\epsilon_3^\mu - (\epsilon_3\cdot k_{12})\epsilon_{12}^\mu + (\epsilon_{12}\cdot f_3)^\mu - (\epsilon_3\cdot f_{12})^\mu\Big]$$
$$\hat{t}_{12} = \frac{1}{6}\Big[(k_2-k_1)_\mu t_8^\mu(12,3,4,5,6) + (12|1,2,3,4,5,6)\Big]$$
$$-\frac{1}{12}\sum_{i=1}^{6} k_{i\mu}k_{i\nu}t_8^{\mu\nu}(1,2,3,4,5,6)$$

Conclusion and outlook

What we have achieved:

- SUSY decomposition from vector $\leftrightarrow$ spinor representations
- Full control of generic matter content in the loop
- Efficient generation crossing symmetric BCJ numerators at one-loop

What we want to understand

- Higher-point numerators in terms of multi-particle pol. and field strength
- Parity odd sector, hexagon anomaly, perhaps from $(-\frac{1}{2}, -\frac{1}{2})$ picture
- Improving conversion to quadratic propagators

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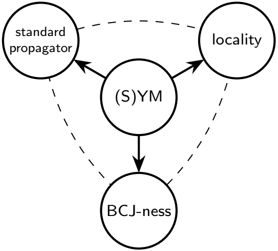

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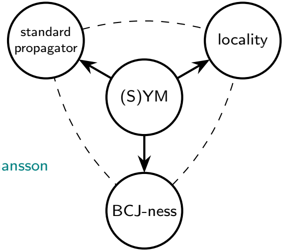

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Thanks for your attention!

Back-up slides

- Draw all the increasing trees w.r.t the color order $1 \prec \alpha(2) \ldots \prec \alpha(n-1) \prec n$

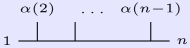
$$\mathrm{IT}(\alpha) = \{T \in \mathbf{T}_1 | \text{ All edges } i \bullet\!\!\rightarrow\!\!\bullet\, j \text{ satisfy } i \prec j \text{ in } (1, \alpha, n)\}$$

- For each tree, identify the "baseline" $(1, \rho, n)$: the path connecting 1 and n
- For each tree, identify "ordered splitting paths" w.r.t a reference order RO:
    1. Draw a path from the first element of RO towards the baseline, which will end on the baseline or a previously traversed ordered splitting path
    2. Repeat until all vertices are traversed.
- $N^{(0)}_{\text{RO}}[T]$ is obtained by dressing the paths:
    1. Baseline: $(-1)^{n-|\rho|} W^{(0)}_{\text{gluon/2f/2s}}(1, \rho, n)$
    2. Ordered splitting path $(i, j, \ldots, k, l)$: $\epsilon_i \cdot f_j \ldots f_k \cdot k_l$

Back-up slides

Result:

$$A^{(0)}_{\text{gluon/2f/2s}}(1,2,\dots,n) = \sum_{\alpha\in S_{n-2}} \underbrace{m(1,2,\dots,n|1,\alpha,n)}_{\text{bi-adjoint scalar amp.}} \underbrace{N^{(0)}_{\text{gluon/2f/2s}}(1,\alpha,n)}$$



$$A^{(1)}(1,2,\dots,n) = \int \frac{d^D\ell}{\ell^2} \sum_{\alpha\in S_n} \Big[m^{\text{fwl}}(+,1,2,\dots,n,-|+,\alpha,-)N^{(1)}[\alpha;\ell] + \text{cyclic}(1,2,\dots,n) \Big]$$