QCD meets gravity 2019

@ IPAM, UCLA

Differential equations for one-loop string integrals

Oliver Schlotterer (Uppsala University)

based on 1908.09848, 1908.10830 with C. Mafra

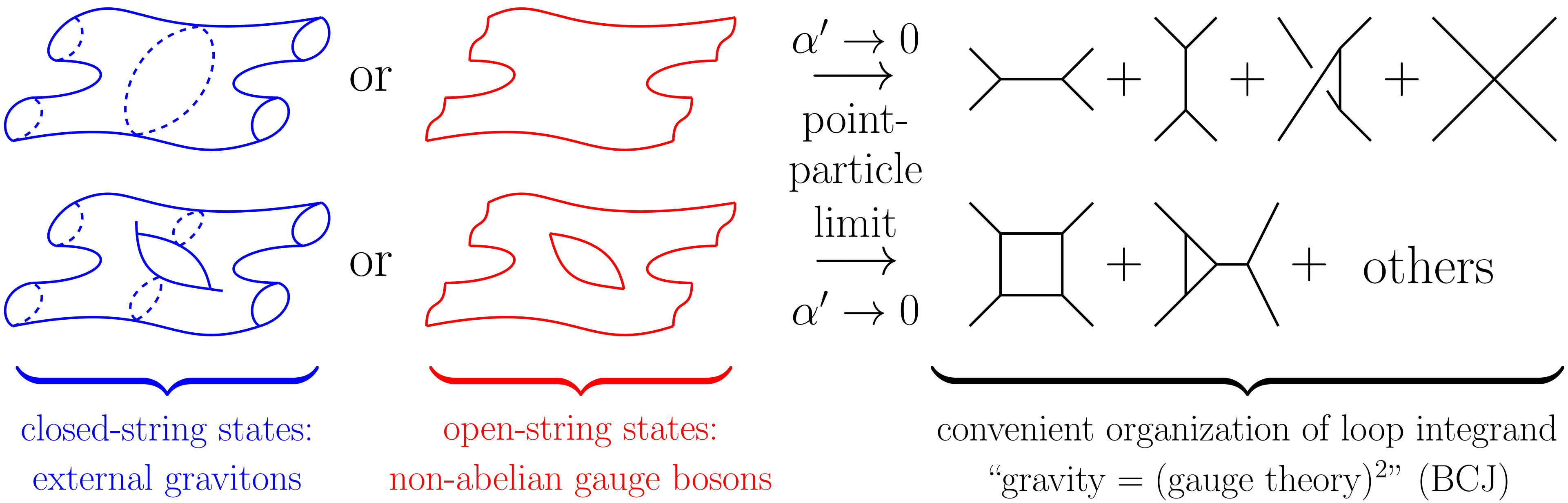
and 1911.03476 with J. Gerken & A. Kleinschmidt

09.12.2019

Intro I – String perturbation theory

String amplitudes $\longleftrightarrow$ worldsheets as “fattened” Feynman diag’s

loop order in perturbation theory = genus of the worldsheet



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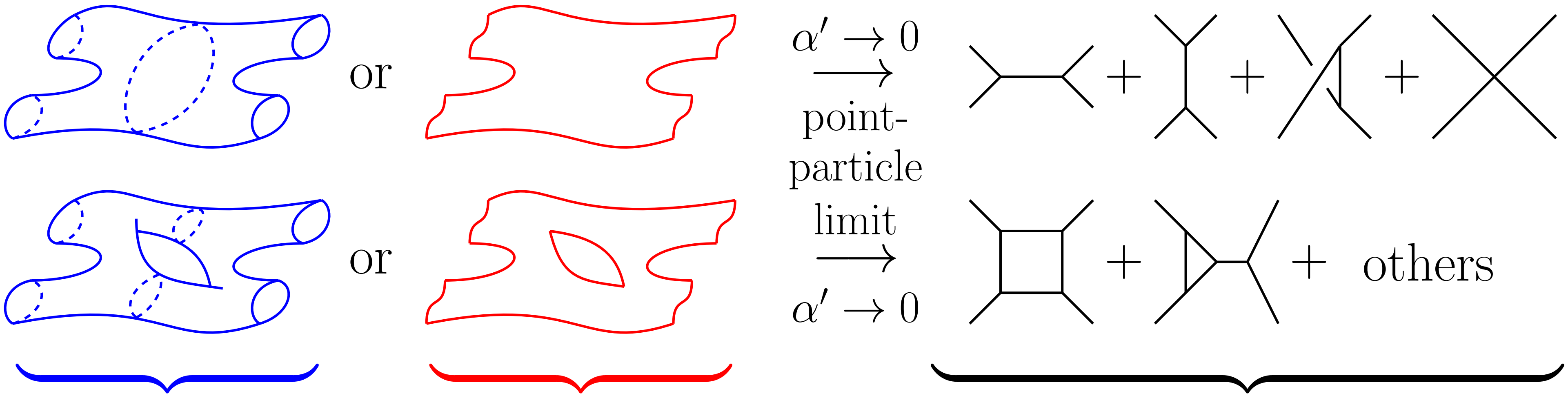


closed-string states: external gravitons

open-string states: non-abelian gauge bosons

convenient organization of loop integrand "gravity = (gauge theory)^2" (BCJ)

This talk: Study corrections to field theory $\sim$ inverse string tension α'

$\Longrightarrow$ rewarding laboratory for (elliptic) multiple zeta values & modular forms

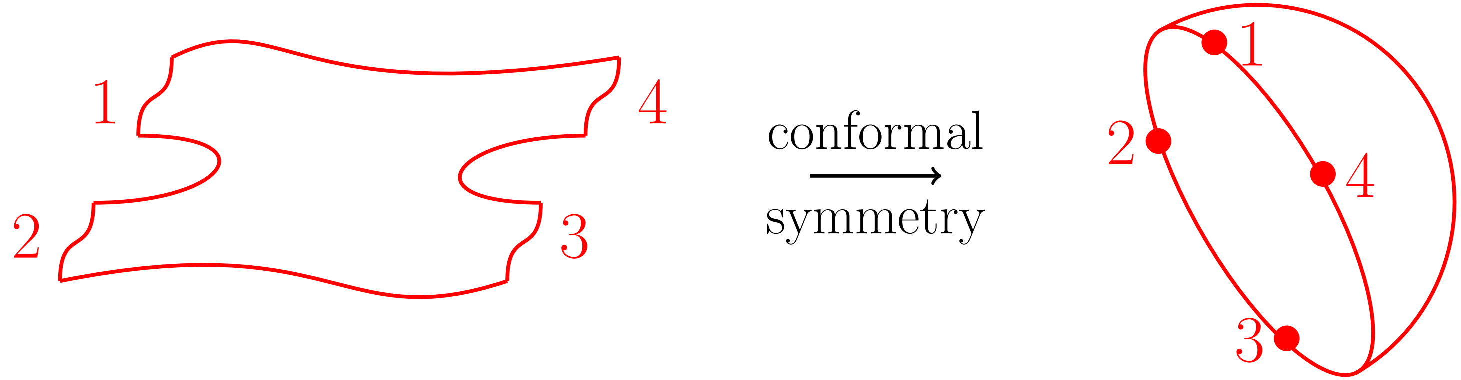
governed by differential equations similar to those of Feynman integrals

Intro I – String perturbation theory

Map external states to punctures • on the worldsheet, e.g.

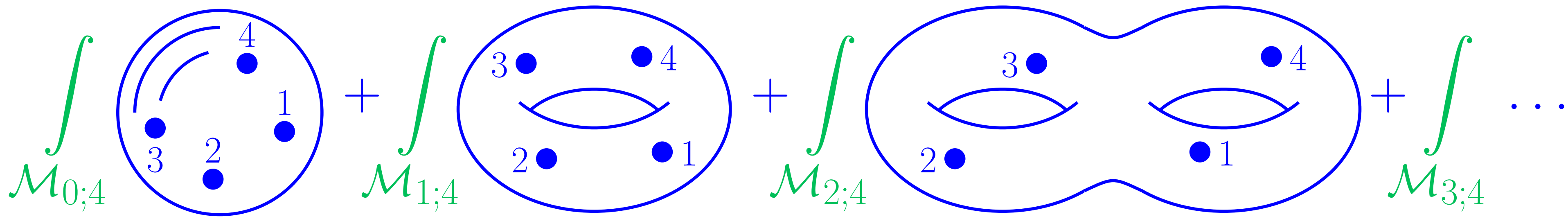
open strings

at tree level:



String amplitudes (n points, g loop) $\leftrightarrow$ integrals over moduli spaces $\mathcal{M}_{g;n}$

of n-punctured worldsheets of genus g (with / without boundary),



α'-expansions $\leftrightarrow$ generating series for (large classes of) periods of $\mathcal{M}_{g;n}$.

Intro II – From worldsheet cartoons to integral bases

Universal integrand: α'-dependent "Koba–Nielsen factor"

"Koba–Nielsen" $\;\;\; = -\frac{\alpha'}{2} k_i \cdot k_j$

$$\mathrm{KN}^{\tau}_{g,n} = \exp\Big(\sum_{1\leq i<j}^{n} s_{ij} \, \underbrace{G_g(z_i, z_j, \tau)}\Big)$$

punctures $z_{i=1,2,\ldots,n}$

moduli τ @ genus $g>0$

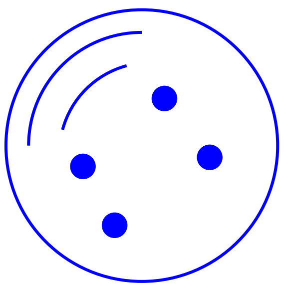
extra $\frac{1}{2}$ for open strings

Green function, e.g. $-\log |z_{ij}|^2$ at tree level

At tree level: additionally, Parke–Taylor factors $\in$ integrand

$$\mathrm{PT}(1,2,3,\ldots,n) = \frac{1}{z_{12}z_{23}\cdots z_{n-1,n}z_{n,1}} \, , \qquad z_{ij} = z_i - z_j$$

Closed-string tree amplitudes $\leftrightarrow$ basis of integrals over spheres ($\sigma, \rho \in S_n$)



$$W^{\mathrm{tree}}(\sigma(1,2,\ldots,n)\,|\,\rho(1,2,\ldots,n)) = \int_{\mathbb{C}} \frac{\mathrm{d}^2 z_1 \ldots \mathrm{d}^2 z_n}{\mathrm{vol}\,\mathrm{SL}_2(\mathbb{C})}$$
$$\times \, \mathrm{KN}_{0,n} \, \overline{\mathrm{PT}(\sigma(1,2,\ldots,n))} \, \mathrm{PT}(\rho(1,2,\ldots,n))$$

Intro II – From worldsheet cartoons to integral bases

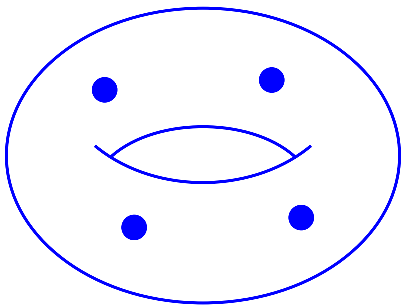
Goals of this talk:

- propose genus-one analogues of Parke–Taylor factors

$$\varphi(1,2,\ldots,n|\tau) \ \leftrightarrow \ \text{function on torus with poles } (z_{12}z_{23}\ldots z_{n-1,n})^{-1}$$

[Mafra, OS 1908.09848, 1908.10830]

- conjectural basis of torus integrals in one-loop string amplitudes



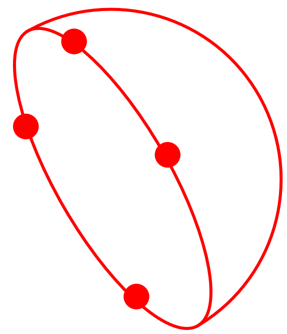
$$W^\tau(\sigma(1,2,\ldots,n)\,|\,(1,2,\ldots,n)) \ = \int_{\text{torus}} \prod_{j=1}^{n} \frac{\mathrm{d}^2 z_j}{\operatorname{Im}\tau} \times \mathrm{KN}^\tau_{1,n}\, \overline{\varphi(\sigma(1,2,\ldots,n)|\tau)}\, \varphi(\rho(1,2,\ldots,n)|\tau)$$

universal to bosonic, heterotic & supersymmetric theories.

[Gerken, Kleinschmidt, OS 1911.03476]

- homogeneous first-order differential equation w.r.t. τ for W^τ

Motivation I – Double copy



At tree level: Parke–Taylor basis revealed double copy

$$\begin{pmatrix}\text{open}\\ \text{superstring}\end{pmatrix} = \begin{pmatrix}\text{super}\\ \text{Yang Mills}\end{pmatrix} \otimes \begin{pmatrix}\text{disk- or}\\ Z\text{-integrals}\end{pmatrix}$$

$$Z^{\text{tree}}(\sigma(\text{cycle})\,|\,\rho(1,2,\ldots,n)) = \int_{\sigma\{-\infty<z_1<z_2<\ldots<z_n<\infty\}} \frac{\mathrm{d}z_1 \ldots \mathrm{d}z_n}{\text{vol}\,\mathrm{SL}_2(\mathbb{R})} \mathrm{KN}_{0,n}\,\mathrm{PT}(\rho(1,2,\ldots,n))$$

- double copy manifest by KLT-type formula (field-theory kernel $S[\alpha|\beta]$)

$$A^{\text{tree}}_{\text{open}}(\sigma) = \sum_{\alpha,\beta\in S_{n-3}} A^{\text{tree}}_{\text{SYM}}(1,\alpha,n,n{-}1)\,S[\alpha|\beta]\,Z^{\text{tree}}(\sigma\,|\,1,\beta,n{-}1,n)$$

[Mafra, OS, Stieberger 1106.2645
& Brödel, OS, Stieberger 1304.7267]

- both $\mathrm{PT}(\ldots)$ and $A^{\text{tree}}_{\text{SYM}}(\ldots)$ fall into $(n{-}3)!$ bases

integration by parts $\Longrightarrow$ $\{\,\mathrm{PT}(1,2,\ldots,n)\ \&\ \mathrm{perm}(2,3,\ldots,n{-}2)\,\}$.

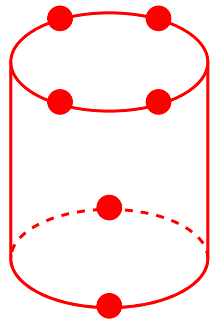
Motivation I – Double copy

One-loop generalization of $(n-3)!$ Parke–Taylors:

$\longrightarrow$ conjectural $(n-1)!$ basis of Kronecker–Eisenstein integrands

integration by parts $\implies$ $\{ \varphi(1,2,3,\ldots,n|\tau) \ \&\ \mathrm{perm}(2,3,\ldots,n) \}$.

- open strings: induces basis of cylinder- & Möbius-strip integrals



$$Z^\tau(\sigma(\text{cycle})\,|\,\rho(1,2,\ldots,n)) = \int_{\sigma(\text{cycle})} \mathrm{KN}^\tau_{1,n}\, \varphi(\rho(1,2,\ldots,n)|\tau)$$

Eduardo's & Piotr's talk:
monodromy rel's among cycles

this talk: conjecturally
$(n-1)!$ (twisted) cocycles

- long-term goal: one-loop double-copy construction for various theories:

$$\begin{pmatrix}\text{one-loop bosonic / he-}\\ \text{terotic / SUSY strings}\end{pmatrix} = \begin{pmatrix}\text{some field}\\ \text{theory}\end{pmatrix} \otimes \begin{pmatrix}\text{above one-loop}\\ Z^\tau\text{-integrals}\end{pmatrix}$$

Motivation II – All-order α'-expansion

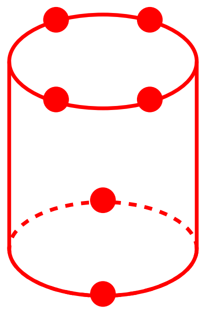
Expanding $Z^{\text{tree}}, W^{\text{tree}}$ in $s_{ij} = -\frac{\alpha'}{2} k_i \cdot k_j \; \Rightarrow$ multiple zeta values (MZVs)

$$\zeta_{n_1,n_2,\ldots,n_r} = \sum_{0<k_1<k_2<\ldots<k_r}^{\infty} k_1^{-n_1} k_2^{-n_2} \ldots k_r^{-n_r} \, , \qquad n_r \geq 2$$

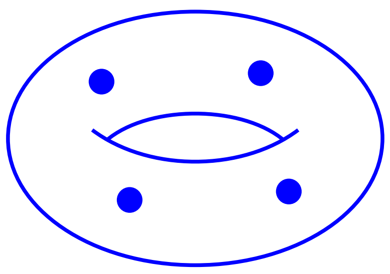
@ uniform transcendentality: weight $n_1+n_2+\ldots+n_r$ matches order in α'

[Terasoma 9908045; Broedel, OS, Stieberger, Terasoma 1304.7304]

Analogous α'-expansion at genus one $\rightarrow$ functions of τ (integrate later)



$\longleftrightarrow \quad Z^{\tau}(\cdot|\cdot) \quad \Longrightarrow$ $\left\{ \right.$ elliptic MZVs **[Enriquez 1301.3042]** **[Brödel, Mafra, Matthes, OS 1412.5535]**



$\longleftrightarrow \quad W^{\tau}(\cdot|\cdot) \quad \Longrightarrow$ $\left\{ \right.$ modular (graph) forms **[D'Hoker, Green, Gürdogan, Vanhove 1512.06779]** **[D'Hoker, Green 1603.00839]**

Motivation II – All-order α'-expansion

Generate one-loop α'-expansion from homogeneous differential eq. in τ:

$$2\pi i \frac{\partial}{\partial \tau} Z^{\tau}(*|1,\rho(2,3,\ldots,n)) \;=\; \sum_{\alpha\in S_{n-1}} D^{\tau}(\rho|\alpha)\, Z^{\tau}(*|1,\alpha(2,3,\ldots,n))$$

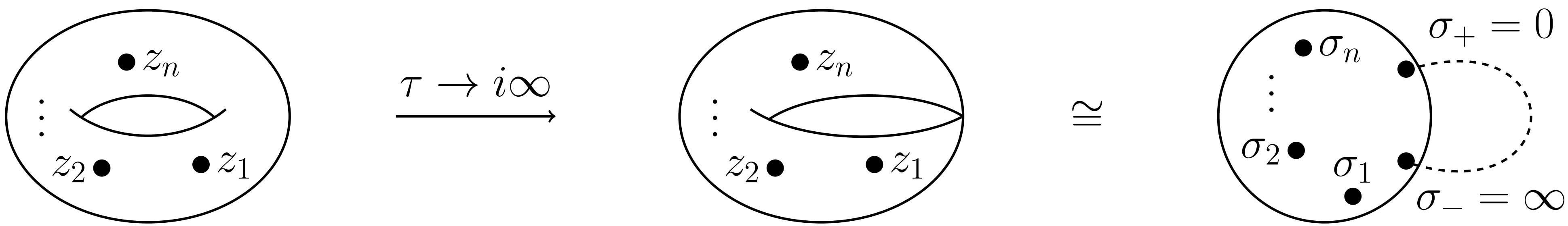
[Mafra, OS 1908.09848, 1908.10830]

Clue: matrix $D^{\tau}(\rho|\alpha)$ acting on integrands is *linear in* α'

$\Longrightarrow$ solution via path-ordered exponential has uniform transcendentality!

$$Z^{\tau}(*|1,\rho) \;=\; \sum_{\alpha\in S_{n-1}} \underbrace{\exp\left\{ \int_{i\infty}^{\tau} \frac{d\tau'}{2\pi i} D^{\tau'}(\rho|\alpha) \right\}}_{\text{generates eMZVs}} Z^{i\infty}(*|1,\alpha)$$

$\nwarrow$ Z^{tree} at $(n+2)$ points



Motivation II – All-order α'-expansion

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$$2\pi i \frac{\partial}{\partial \tau} Z^\tau(*|1,\rho(2,3,\ldots,n)) = \sum_{\alpha \in S_{n-1}} D^\tau(\rho|\alpha)\, Z^\tau(*|1,\alpha(2,3,\ldots,n))$$

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$\nwarrow$ Z^{tree} at $(n+2)$ points

- resembles ϵ-form of diff. eq. for Feynman integrals in $D = 4 - 2\epsilon$ dim

[e.g. Henn 1304.1806; Adams, Weinzierl 1802.05020]

- work in progress: similar expansion techniques for closed-string $W^\tau(\cdot|\cdot)$

Outline: some more details on the results

I. The Kronecker–Eisenstein integrands

II. Open-string differential equations

III. Closed-string integrals and their differential equations

IV. Summary & Outlook

[Mafra, OS 1908.09848 & 1908.10830,

Gerken, Kleinschmidt, OS 1911.03476]

Results I – The Kronecker–Eisenstein integrands

Parke–Taylor factors are related by partial fraction ($z_{ij} = z_i - z_j$)

$$\frac{1}{z_{12}z_{13}} = \frac{1}{z_{12}z_{23}} + \frac{1}{z_{13}z_{32}} \quad \Longrightarrow \quad \text{KK relations among PT}(\ldots)$$

Naive genus-1 generalization of z_{ij}^{-1}: odd Jacobi theta function

$$\partial_z \log \theta_1(z_{ij}|\tau) \ = \ \frac{1}{z_{ij}} + \left(\begin{array}{c} \text{quasi-periodic completion} \\ \text{w.r.t. } z{\rightarrow}z{+}1 \ \& \ z{\rightarrow}z{+}\tau \end{array} \right),$$

... more specifically:

$$\theta_1(z|\tau) \ = \ 2e^{i\pi\tau/4} \sin(\pi z) \prod_{n=1}^{\infty} \big(1{-}e^{2\pi i n\tau}\big)\big(1{-}e^{2\pi i(n\tau+z)}\big)\big(1{-}e^{2\pi i(n\tau-z)}\big)$$

<u>Problem</u>: quasi-periodic completion spoils partial fraction:

$$\partial_z \log \theta_1(z_{12}|\tau)\partial_z \log \theta_1(z_{13}|\tau) \ \neq \ \partial_z \log \theta_1(z_{12}|\tau)\partial_z \log \theta_1(z_{23}|\tau) \ + \ \partial_z \log \theta_1(z_{13}|\tau)\partial_z \log \theta_1(z_{32}|\tau)$$

Results I – The Kronecker–Eisenstein integrands

Parke–Taylor factors are related by partial fraction $(z_{ij} = z_i - z_j)$

$$\frac{1}{z_{12}z_{13}} = \frac{1}{z_{12}z_{23}} + \frac{1}{z_{13}z_{32}} \quad \Longrightarrow \quad \text{KK relations among PT}(\ldots)$$

genus-1 generalization of z_{ij}^{-1}: doubly-periodic Kronecker–Eisenstein series

$$\Omega(z,\eta,\tau) \;=\; \exp\left(2\pi i\eta\,\frac{\text{Im}\,z}{\text{Im}\,\tau}\right)\frac{\theta_1'(0|\tau)\theta_1(z+\eta|\tau)}{\theta_1(z|\tau)\theta_1(\eta|\tau)}$$

Partial fraction generalizes to Fay identity involving auxiliary var's η

$$\Omega(z_{12},\eta_2,\tau)\,\Omega(z_{13},\eta_3,\tau) \;=\; \Omega(z_{12},\eta_2{+}\eta_3,\tau)\,\Omega(z_{23},\eta_3,\tau) + \Omega(z_{13},\eta_2{+}\eta_3,\tau)\,\Omega(z_{32},\eta_2,\tau)$$

Kronecker–Eisenstein integrand at n points: $n{-}1$ auxiliary var's $\eta_2,\eta_3,\ldots,\eta_n$

$$\varphi^{\tau}_{\vec{\eta}}(1,2,\ldots,n) \;=\; \prod_{j=2}^{n}\Omega(z_{j-1,j},\eta_j{+}\eta_{j+1}{+}\ldots{+}\eta_n,\tau)$$

Results I – The Kronecker–Eisenstein integrands

Fay identity among doubly-periodic Kronecker–Eisenstein series ...

$$\Omega(z,\eta,\tau) \ = \ \exp\left(2\pi i\eta \frac{\text{Im } z}{\text{Im }\tau}\right) \frac{\theta_1'(0|\tau)\theta_1(z+\eta|\tau)}{\theta_1(z|\tau)\theta_1(\eta|\tau)}$$

... propagates to Kronecker–Eisenstein integrands

$$\varphi^\tau_{\vec{\eta}}(1,2,\ldots,n) \ = \ \prod_{j=2}^{n} \Omega(z_{j-1,j},\eta_j{+}\eta_{j+1}{+}\ldots{+}\eta_n,\tau)$$

$\Longrightarrow$ KK relations leaving only $(n{-}1)!$ independent permutations

$$\varphi^\tau_{\vec{\eta}}(a_1,\ldots,a_p,1,b_1,\ldots,b_q) \ = \ (-1)^p\, \varphi^\tau_{\vec{\eta}}\big(1,(a_p,\ldots,a_1) \sqcup (b_1,\ldots,b_q)\big)$$

Same counting for conjectural one-loop basis integrals: $\rho \in S_{n-1}$ basis

$$Z^\tau_{\vec{\eta}}(\sigma(\text{cycle})\,|\,1,\rho(2,3,\ldots,n)) \ = \ \int_{\sigma(\text{cycle})} \text{KN}^\tau_{1,n}\, \varphi^\tau_{\vec{\eta}}(1,\rho(2,3,\ldots,n))\,.$$

Results I – The Kronecker–Eisenstein integrands

Example at four points: 6 permutations $\rho \in S_3$ of $(z_j, \eta_j) \in$ integrand

$$Z^{\tau}_{\eta_2,\eta_3,\eta_4}(\sigma(\text{cycle})\,|\,1,\rho(2,3,4)) \;=\; \int_{\sigma(\text{cycle})} \text{KN}^{\tau}_{1,4}$$

$$\times\; \rho\Big\{\Omega(z_{12},\eta_2{+}\eta_3{+}\eta_4,\tau)\Omega(z_{23},\eta_3{+}\eta_4,\tau)\Omega(z_{34},\eta_4,\tau)\Big\}$$

- open superstring: 4pt integrand is 1 in place of Ω^3 $\Rightarrow$ pick η_j^{-3} order

from product of $$\Omega(z,\eta,\tau) \;=\; \frac{1}{\eta} + \partial_z \log\theta_1(z|\tau) \;+\; \mathcal{O}(\eta)$$

- open bos. string: 4pt integrand $\sim \partial^4_{z_i}$ of $(\log\theta_1)$'s $\Rightarrow$ pick η_j^{+1} order

In fact, $Z^{\tau}_{\vec{\eta}}$ are generating series of genus-one integrals in string amplitudes:

different orders in η_j $\longleftrightarrow$ different string theories / amounts of SUSY

Results II – Open-string differential equations

Another benefit of η_j-dependent $\Omega(z,\eta,\tau)$ in the integrand of

$$Z^\tau_{\vec\eta}(\sigma(\text{cycle})\,|\,1,\rho(2,3,\ldots,n)) \ = \int_{\sigma(\text{cycle})} \text{KN}^\tau_{1,n} \times \rho\Big\{ \prod_{j=2}^n \Omega(z_{j-1,j},\eta_j{+}\eta_{j+1}{+}\ldots{+}\eta_n,\tau)\Big\} \, .$$

$\Longrightarrow$ $(n{-}1)!$-family $\rho \in S_{n-1}$ closes under τ-derivative

$$2\pi i \, \frac{\partial}{\partial \tau} \, Z^\tau_{\vec\eta}(*|1,\rho(2,3,\ldots,n)) \ = \sum_{\alpha \in S_{n-1}} D^\tau_{\vec\eta}(\rho|\alpha)\, Z^\tau_{\vec\eta}(*|1,\alpha(2,3,\ldots,n))$$

with $(n{-}1)! \times (n{-}1)!$ matrix $D^\tau_{\vec\eta}(\rho|\alpha)$ linear in α' (i.e. $s_{ij} = -\frac{\alpha'}{2} k_i \cdot k_j$).

- in comparison to Feynman integrals, α' is the new dim-reg ϵ

- closure under ∂_τ is strong evidence the $Z^\tau_{\vec\eta}(*\,|\,1,\rho)$ furnish a basis

Results II – Open-string differential equations

Two-point example: "1×1 matrix" $D^\tau_{\eta_2}(2|2)$

$$2\pi i \, \frac{\partial}{\partial \tau} Z^\tau_{\eta_2}(*|1,2) \; = \; \underbrace{s_{12}\Big(\frac{1}{2}\,\frac{\partial^2}{\partial \eta_2^2} - \wp(\eta_2,\tau) - 2\zeta_2\Big)}_{D^\tau_{\eta_2}(2|2)} \, Z^\tau_{\eta_2}(*|1,2)$$

with Weierstraß function generating holomorphic Eisenstein series

$$\wp(\eta,\tau) \; = \; \frac{1}{\eta^2} + \sum_{k=4}^{\infty} (k{-}1)\,\eta^{k-2}\,\mathrm{G}_k(\tau) \; , \qquad \mathrm{G}_k(\tau) \; = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau{+}n)^k}$$

All-order α'-expansion from path-ordered exponential

$$Z^\tau_{\eta_2}(*|1,2) \; = \; \underbrace{\exp\Big\{ \int_{i\infty}^{\tau} \frac{\mathrm{d}\tau'}{2\pi i} \, D^{\tau'}_{\eta_2}(2|2) \Big\}}_{\text{eMZVs as iterated Eisenstein integrals}} \, Z^{i\infty}(*|1,2)$$

$$Z^{i\infty}(*|1,2) = \pi \cot(\pi\eta_2) \, \frac{\Gamma(1-s_{12})}{\big[\Gamma(1-\frac{s_{12}}{2})\big]^2}$$

Results II – Open-string differential equations

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Three-point example: 2×2 differential operator $D^\tau_{\eta_2,\eta_3}(2,3|\alpha(2,3))$

$$\begin{aligned}2\pi i\partial_\tau Z^\tau_{\eta_2,\eta_3}(*|1,2,3) \;=\; &\Big(s_{12}\big[\tfrac{1}{2}\partial^2_{\eta_2}-\wp(\eta_2{+}\eta_3,\tau)\big]+s_{23}\big[\tfrac{1}{2}(\partial_{\eta_2}{-}\partial_{\eta_3})^2-\wp(\eta_3,\tau)\big]\\ &+s_{13}\big[\tfrac{1}{2}\partial^2_{\eta_3}-\wp(\eta_3,\tau)\big]-2\zeta_2 s_{123}\Big)Z^\tau_{\eta_2,\eta_3}(*|1,2,3)\\ &+s_{13}\big[\wp(\eta_2{+}\eta_3,\tau)-\wp(\eta_3,\tau)\big]Z^\tau_{\eta_2,\eta_3}(*|1,3,2)\\ =&\sum_{\alpha\in S_2} D^\tau_{\eta_2,\eta_3}(2,3|\alpha(2,3))\,Z^\tau_{\eta_2,\eta_3}(*|1,\alpha(2,3))\,.\end{aligned}$$

In general, all τ-dependence of $D^\tau_{\vec\eta}$ occurs via $\wp(\eta,\tau)$ & hence $\mathrm{G}_k(\tau)$.

Results III – Closed-string differential equations

Conjectural basis of closed-string integrals in bos / het / SUSY theories

$$W^{\tau}_{\vec\eta}(1,\sigma(2,\ldots,n)\,|\,1,\rho(2,\ldots,n)) \;=\; \int_{\rm torus} \Big(\prod_{j=1}^{n} \frac{{\rm d}^2 z_j}{{\rm Im}\,\tau} \Big)\, {\rm KN}^{\tau}_{1,n}$$

$$\times \prod_{j=2}^{n} \sigma\Big\{ \overline{\Omega(z_{j-1,j},\eta_j{+}\eta_{j+1}{+}\ldots{+}\eta_n,\tau)} \Big\}\, \rho\Big\{ \Omega(z_{j-1,j},\eta_j{+}\eta_{j+1}{+}\ldots{+}\eta_n,\tau) \Big\}$$

Expansion in $\eta_j,\bar\eta_j$ and α' generates modular graph forms

[D'Hoker, Green, Gürdogan, Vanhove 1512.06779; D'Hoker, Green 1603.00839]

Differential eq. involves modular version of $\frac{\partial}{\partial\tau}$ "Maaß operators"

$$\nabla_{\vec\eta} \;=\; (\tau{-}\bar\tau)\,\frac{\partial}{\partial\tau} + n - 1 + \sum_{j=2}^{n} \eta_j\,\frac{\partial}{\partial\eta_j}$$

$$\overline{\nabla}_{\vec\eta} \;=\; (\bar\tau{-}\tau)\,\frac{\partial}{\partial\bar\tau} + n - 1 + \sum_{j=2}^{n} \bar\eta_j\,\frac{\partial}{\partial\bar\eta_j}$$

Results III – Closed-string differential equations

Largely recycle differential operators $D^{\tau}_{\vec{\eta}}$ from open string ...

$$2\pi i \frac{\partial}{\partial \tau} Z^{\tau}_{\vec{\eta}}(*|1,\rho(2,3,\ldots,n)) \; = \sum_{\alpha \in S_{n-1}} D^{\tau}_{\vec{\eta}}(\rho|\alpha)\, Z^{\tau}_{\vec{\eta}}(*|1,\alpha(2,3,\ldots,n))$$

... but drop the ζ_2 term (indicated by "sv" notation)

e.g. $\quad \mathrm{sv}\, D_{\eta_2}(2|2) = s_{12}\Big(\frac{1}{2}\partial^2_{\eta_2} - \wp(\eta_2,\tau) \cancel{-2\zeta_2}\Big)$ @ 2pt

$$\mathrm{sv}\, D_{\eta_2,\eta_3}(2,3|2,3) = s_{12}\big[\tfrac{1}{2}\partial^2_{\eta_2} - \wp(\eta_2{+}\eta_3,\tau)\big] + s_{23}\big[\tfrac{1}{2}(\partial_{\eta_2}{-}\partial_{\eta_3})^2 - \wp(\eta_3,\tau)\big]$$

$$+ s_{13}\big[\tfrac{1}{2}\partial^2_{\eta_3} - \wp(\eta_3,\tau)\big] \cancel{-2\zeta_2\, s_{123}}$$ @ 3pt

Results III – Closed-string differential equations

Largely recycle differential operators $D^\tau_{\vec\eta}$ from open string ...

$$2\pi i \, \frac{\partial}{\partial \tau} \, Z^\tau_{\vec\eta}(*|1,\rho(2,3,\ldots,n)) \; = \sum_{\alpha \in S_{n-1}} D^\tau_{\vec\eta}(\rho|\alpha) \, Z^\tau_{\vec\eta}(*|1,\alpha(2,3,\ldots,n))$$

... but drop the ζ_2 term (indicated by "sv" notation)

$$\text{e.g.} \quad \text{sv}\, D_{\eta_2}(2|2) = s_{12}\Big(\frac{1}{2}\partial^2_{\eta_2} - \wp(\eta_2,\tau) \cancel{- 2\zeta_2}\Big) \qquad @ \text{ 2pt}$$

Holomorphic differential only acts on 2nd entry ρ:

$$2\pi i \nabla_{\vec\eta} W^\tau_{\vec\eta}(1,\sigma|1,\rho) = (\tau - \bar\tau) \sum_{\alpha \in S_{n-1}} \text{sv}\, D^\tau_{\vec\eta}(\rho|\alpha) \, W^\tau_{\vec\eta}(1,\sigma|1,\alpha)$$

$$+ \, 2\pi i \sum_{j=2}^{n} \bar\eta_j \partial_{\eta_j} W^\tau_{\vec\eta}(1,\sigma|1,\rho)$$

no open-string analogue:
mild interaction between
left & right after loop integral

Results III – Closed-string differential equations

Also, Laplace operator closes on $W^\tau_{\vec\eta}$: simply mixes component integrals

$$\Delta_{\vec\eta} = (\overline{\nabla}_{\vec\eta} - 1)\nabla_{\vec\eta} - \Big(n - 1 + \sum_{j=2}^{n} \eta_j \partial_{\eta_j}\Big)\Big(n - 2 + \sum_{j=2}^{n} \bar\eta_j \partial_{\bar\eta_j}\Big)$$

generate Laplace equations for all modular graph forms ($\partial_{\eta_1} = 0$):

$$\begin{aligned}(2\pi i)^2 \Delta_{\vec\eta} W^\tau_{\vec\eta}(1,\sigma|1,\rho) = \sum_{\alpha,\beta\in S_{n-1}} \Big\{ &\delta_{\sigma,\alpha}\delta_{\rho,\beta}\Big[(2\pi i)^2(2-n)\Big(n-1+\sum_{i=2}^{n}(\eta_i\partial_{\eta_i}+\bar\eta_i\partial_{\bar\eta_i})\Big) \\ &+ (2\pi i)^2 \sum_{2\leq i<j}^{n} (\eta_i\bar\eta_j - \eta_j\bar\eta_i)(\partial_{\eta_j}\partial_{\bar\eta_i} - \partial_{\eta_i}\partial_{\bar\eta_j}) \\ &+ 2\pi i(\tau-\bar\tau) \sum_{1\leq i<j\leq n} s_{ij}(\partial_{\eta_j}-\partial_{\eta_i})(\partial_{\bar\eta_j}-\partial_{\bar\eta_i})\Big] \\ &+ 2\pi i(\tau-\bar\tau)\Big[\delta_{\sigma,\alpha}\sum_{i=2}^{n}\eta_i\partial_{\bar\eta_i}\text{sv}\, D^\tau_{\vec\eta}(\rho|\beta) + \delta_{\rho,\beta}\sum_{i=2}^{n}\bar\eta_i\partial_{\eta_i}\overline{\text{sv}\, D^\tau_{\vec\eta}}(\sigma|\alpha)\Big] \\ &+ (\tau-\bar\tau)^2 \text{sv}\, D^\tau_{\vec\eta}(\sigma|\alpha)\,\overline{\text{sv}\, D^\tau_{\vec\eta}}(\rho|\beta)\Big\} W^\tau_{\vec\eta}(1,\alpha|1,\beta)\,.\end{aligned}$$

same sv $D^\tau_{\vec\eta}$

as in $\nabla_{\vec\eta} W^\tau_{\vec\eta}$

Summary & Outlook

- proposed universal $(n-1)!$ basis of torus integrands $\varphi^{\tau}_{\vec{\eta}}(1,2,\ldots,n)$

  for one-loop string amplitudes in bosonic / heterotic / SUSY theories

- basis integrals satisfy homogeneous first-order differential eq. in τ

- broader picture: closed-string amplitudes from single-valued open strings

  $\longrightarrow$ 1-loop input from comparing & solving differential equations

  [Brown 1707.01230 & 1708.03354; Gerken, Kleinschmidt, Mafra, OS: in progress]

- search for similar differential eq's for higher-genus modular graph forms

  [D'Hoker, Green, Pioline 1712.06135 & 1806.02691]

Thank you for your attention !