The Eikonal Limit and Post-Minkowskian Scattering

Talk by P.H. Damgaard at
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Overview

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- The Eikonal versus Potential Method

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- Scattering Angle: Agreement

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- Super-Classical–Classical Identities

Eikonal versus Potential Method

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Natural formalism for *classical* scattering using QFT

Example:

Two-to-two scattering of massive point particles in perturbation theory

[Kabat, Ortiz (1992); Akhoury, Saotome, Sterman (2013); Bjerrum-Bohr, PHD, Festuccia, Planté, Vanhove (2018); Collado, Di Vecchia, Russo, Thomas (2018)]

Eikonal versus Potential Method

The exponentiation. First tree level:

$$\mathcal{M}_1(\vec{q}) = \frac{8\pi G}{\vec{q}^2}\left((s - m_1^2 - m_2^2)^2 - 2m_1^2 m_2^2\right)$$

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In impact-parameter space:

$$\mathcal{M}(\vec{b}) \ \equiv \ \int d^2\vec{q} e^{-i\vec{q}\cdot\vec{b}} \mathcal{M}(\vec{q})$$

Eikonal versus Potential Method

More convenient variables (tree level: $i{=}1$)

$$\chi_i(b) = \frac{1}{2\sqrt{(s-m_1^2-m_2^2)^2-4m_1^2m_2^2}}\int \frac{d^2\vec{q}}{(2\pi)^2}\, e^{-i\vec{q}\cdot\vec{b}}\mathcal{M}_i(\vec{q})$$

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Then

$$\mathcal{M}_1^{\text{sum}}(q) = 4p(E_1+E_2) \int d^2 b_\perp e^{-iq\cdot b_\perp} \left(e^{i\chi_1(b)} - 1\right)$$

is the sum of all boxes and crossed boxes in the eikonal limit.

Eikonal versus Potential Method

To 2PM order it still exponentiates:

$$\mathcal{M}_1^{\text{sum}}(q) + \mathcal{M}_2^{\text{sum}}(q) = 4p(E_1 + E_2) \int d^2 b_\perp e^{-iq\cdot b_\perp} \left(e^{i(\chi_1(b)+\chi_2(b))} - 1 \right)$$

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Now take saddle point

$$2\sin(\theta/2) = \frac{-2\sqrt{s}}{\sqrt{(s-m_1^2-m_2^2)^2 - 4m_1^2 m_2^2}} \frac{\partial}{\partial b} \left(\chi_1(b) + \chi_2(b)\right)$$

to get the scattering angle

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Let us try to reconcile the two

The Effective Potential

Relativistic Salpeter equation

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{V} = \sum_{i=1}^{2} \sqrt{\hat{p}^2 + m_i^2} + \hat{V}$$

[Cheung, Rothstein, Solon (2018); Bern, Cheung, Roiban, She, Solon, Zeng (2019)]

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Two ways to fix potential V:

- Matching with effective low-q^2 theory

- Solving the Lippmann-Schwinger equation

[Bjerrum-Bohr, Critstofoli, PHD, Vanhove (2019)]

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The Effective Potential

Lippmann-Schwinger equation

$$\mathcal{M}(p,p') \;=\; \tilde{V}(p,p') \;+\; \int \frac{d^3k}{(2\pi)^3} \frac{\tilde{V}(k,p)\,\mathcal{M}(k,p')}{E_p - E_k + i\epsilon}$$

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Invert it and iterate

$$\tilde{V}(p,p') \;=\; \mathcal{M}(p,p') - \int \frac{d^3k}{(2\pi)^3} \frac{\mathcal{M}(p,k)\,\mathcal{M}(k,p')}{E_p - E_k + i\epsilon} + \ldots$$

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The Effective Potential

Retain only classical pieces and Fourier transform in $D{=}4$:

$$\tilde{\mathcal{M}}^{cl.}(r,p) \;=\; V - 2E\xi\, V\partial_{p^2}V + \left(\frac{3\xi-1}{2E\xi}\right)V^2 + \ldots$$

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Surprisingly, the same functional relation is encoded in the energy relation

$$\sum_{i=1}^{2}\sqrt{p^2+m_i^2} + V(p,r) \;=\; E\;, \quad V(p,r) \;=\; \sum_{n=1}^{\infty}\left(\frac{G_N}{r}\right)^n c_n(p^2)$$

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The Effective Potential

From the inverse function theorem

$$p^2 \;=\; p_\infty^2 + \sum_{k=1}^{\infty} \frac{G_N^k}{k!} \frac{d^k p^2}{dG_N^k}\bigg|_{G_N=0} \,, \qquad p_\infty^2 = \frac{(m_1^2+m_2^2-E^2)^2-4m_1^2m_2^2}{4E^2}$$

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Solving it

$$p^2 = p_\infty^2 + \sum_{n=0}^{\infty} \frac{G_N^n f_n(E)}{r^n}$$

The Effective Potential

Compactly, in $D{=}4$:

$$p^2 = p_\infty^2 - 2E\xi\tilde{\mathcal{M}}(p_\infty, r)$$

[Damour (2017); Bern, Cheung, Roiban, She, Solon, Zeng (2019); Kalin, Porto (2019), Bjerrum-Bohr,PHD,Cristofoli (2019); Damour (2019)]

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Only the classical part of the scattering amplitude enters in the energy relation

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At 2PM order:

$$p^2 = p_\infty^2 - 2E\xi\left(\tilde{\mathcal{M}}_{tree}(r, p_\infty) + \tilde{\mathcal{M}}_{1-loop}(r, p_\infty) - \tilde{\mathcal{M}}_{tree}^2(r, p_\infty)\frac{\xi E}{p_\infty^2}\frac{\Gamma(d-2)}{\Gamma(d-3)}\right)$$

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How can we reconcile the two?

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New piece to potential V from sum of box and crossed-box diagrams:

$$\frac{(8\pi G_N)^2\gamma_{(p)}^2(m_1+m_2)}{4E^4p^2\xi(4\pi)^{\frac{D-1}{2}}}\Gamma\left(\frac{5-D}{2}\right)\frac{\Gamma^2(\frac{D-3}{2})}{\Gamma(D-4)}(q^2)^{\frac{D-5}{2}}$$

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It *almost* cancels by a Born subtraction, leaving

$$\frac{(8\pi G_N)^2\gamma_{(p)}^2(m_1+m_2-E)}{4E^4p^2\xi(4\pi)^{\frac{D-1}{2}}}\Gamma\left(\frac{5-D}{2}\right)\frac{\Gamma^2(\frac{D-3}{2})}{\Gamma(D-4)}(q^2)^{\frac{D-5}{2}}$$

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In general ($r^2 = u^2 + b^2$),

$$\theta = \sum_{k=1}^{\infty} \frac{2b}{k!} \int_0^{\infty} du \left(\frac{d}{db^2} \right)^k \frac{(V_{eff}(r))^k r^{2(k-1)}}{p_\infty^{2k}} \ , \quad V_{eff}(r) \ = \ -\sum_{n=1}^{\infty} \frac{G_N^n f_n^D(p_\infty^2)}{r^{n(D-3)}}$$

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The new term:

$$\frac{b}{p_\infty^4} \int_0^{+\infty} du \left(\frac{d}{db^2}\right)^2 \left[r^2 V_{eff}^2(r)\right]$$

Scattering Angle

This new term conspires with the quadratic amplitude correction to yield

$$
-\frac{2G_N^2\Gamma(D-\frac{5}{2})\Gamma^2(\frac{D-3}{2})}{p_\infty^2 E b^{2D-6}\pi^{D-\frac{7}{2}}}\times
\left[\frac{m_1+m_2}{\Gamma(D-3)}\left((s-m_1^2-m_2^2)^2-\frac{4m_1^2m_2^2}{(D-2)^2}-\frac{D-3}{4(D-2)^2}[(s-m_1^2-m_2^2)^2-4m_1^2m_2^2]\right)
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$$

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$$\left.+\frac{\gamma^2}{E^2p_\infty^2}\frac{m_1+m_2}{\Gamma(D-4)}\right]$$

This agrees with the eikonal calculation of Collado, Di Veccia, Russo, Thomas (2018)

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In impact-parameter space

$$\mathcal{M}_1^{\rm sum}(q) + \mathcal{M}_2^{\rm sum}(q) = 4p(E_1 + E_2) \int d^2 b_\perp e^{-iq\cdot b_\perp} \left(e^{i(\chi_1(b)+\chi_2(b))} - 1\right)$$

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Expand the exponent

$$\mathcal{M}_1^{\rm sum}(q)+\mathcal{M}_2^{\rm sum}(q) = 4p(E_1+E_2) \int d^2 b_\perp e^{-iq\cdot b_\perp} \left(i(\chi_1(b) + \chi_2(b)) - \frac{1}{2}\chi_1(b)^2 + \dots\right)$$

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At n-loop order we get identities linking $1/\hbar^n$-pieces to powers of lower-order amplitudes

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