MHV Amplitudes on Self-Dual Plane Waves

Tim Adamo
University of Edinburgh

QCD Meets Gravity

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Work in progress with Lionel Mason & Atul Sharma
(also work with E. Casali, A. Ilderton, S. Nekovar)

Motivation

Many reasons to be interested in perturbative QFT in *strong* (non-trivial) background fields

- Practical (strong field QED/QCD, cosmology, GWs, holography, non-perturbative effects)
- Theoretical (probe robustness of structures in pQFT)

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Challenges cut across both!

Example: tree-level gauge theory and gravity

- Flat background: full tree-level S-matrix
- Even simplest strong backgrounds: only 3- or (sometimes) 4-points

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$$\delta^3_{+,\perp}\left(\sum_{i=1}^{n} k_i\right) \frac{\langle i\, j\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle n\,1\rangle} \int_{-\infty}^{+\infty} \mathrm{d}x^- \, \mathrm{e}^{\mathrm{i}\mathcal{F}_n(x^-)}$$

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- Full tree-level S-matrix conjecture for Yang-Mills on such backgrounds

Plane waves

Solution to vacuum equations (in d dim.) with:

- covariantly constant null symmetry n,
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[Trautman, Basler-Hadicke, TA-Casali-Mason-Nekovar]

$$\mathrm{d}s^2 = 2\mathrm{d}x^+\,\mathrm{d}x^- - (\mathrm{d}x^\perp)^2\,, \qquad A = x^\perp\,\dot{a}_\perp(x^-)\,\mathrm{d}x^-$$

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$\dot{a}_\perp(x^-)$ compactly supported $\leftrightarrow$ well-defined S-matrix [Schwinger, TA-Casali-Mason-Nekovar]

Self-dual plane waves

In $d = 4$, complexify $\mathbb{R}^{1,3}$ to $\mathbb{C}^4$: $\mathrm{d}s^2 = 2\,(\mathrm{d}x^+\,\mathrm{d}x^- - \mathrm{d}z\,\mathrm{d}\tilde{z})$.
Require PW & self-duality:

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Propagation direction of wave: $n = \frac{\partial}{\partial x^+}$

$$\text{Since } n^2 = 0\,, \qquad n^{\alpha\dot{\alpha}} = \iota^\alpha\,\tilde{\iota}^{\dot{\alpha}}\,, \qquad \iota^\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \tilde{\iota}^{\dot{\alpha}}$$

Result:

$$A = \tilde{z}\,\dot{f}(x^-)\,\mathrm{d}x^- = \tilde{z}\,\dot{f}(x^-)\,\iota_\alpha\tilde{\iota}_{\dot{\alpha}}\,\mathrm{d}x^{\alpha\dot{\alpha}}$$

$$F = \dot{f}(x^-)\,\mathrm{d}\tilde{z}\wedge\mathrm{d}x^- = \dot{f}\,\tilde{\iota}_{\dot{\alpha}}\tilde{\iota}_{\dot{\beta}}\,\mathrm{d}x_\alpha{}^{\dot{\alpha}}\wedge\mathrm{d}x^{\alpha\dot{\beta}}$$

SDPW kinematics

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SDPW have *chiral* on-shell kinematics

Gluon with incoming momentum $k_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}$: $\mathsf{T}^{\mathsf{a}}\, \mathcal{E}^{\pm}_{\alpha\dot\alpha}(x^-)\, \mathrm{e}^{\mathrm{i}\phi_k}$

$$\phi_k = k\cdot x + e\tilde{z}\, f(x^-) + \frac{k}{k_+}\int^{x^-} \mathrm{d}t\, e\, f(t)$$

On-shell kinematics:

$$K_{\alpha\dot\alpha}(x^-) = \lambda_\alpha\, \tilde\Lambda_{\dot\alpha}\,, \qquad \tilde\Lambda_{\dot\alpha} := \tilde\lambda_{\dot\alpha} + \frac{e}{\sqrt{k_+}}\,\tilde\iota_{\dot\alpha}\, f(x^-)$$

$$\mathcal{E}^{-}_{\alpha\dot\alpha} = \frac{\lambda_\alpha\,\tilde\iota_{\dot\alpha}}{[\tilde\iota\,\tilde\lambda]}\,, \qquad \mathcal{E}^{+}_{\alpha\dot\alpha} = \frac{\iota_\alpha\,\tilde\Lambda_{\dot\alpha}}{\langle\iota\,\lambda\rangle}$$

Twistor theory

Twistor space: $Z^A = (\mu^{\dot{\alpha}}, \lambda_\alpha)$ homog. coords. on $\mathbb{CP}^3$

$$\mathbb{PT} = \mathbb{CP}^3 \setminus \{\lambda_\alpha = 0\}$$

$x \in \mathbb{C}^4$ given by $X \cong \mathbb{CP}^1 \subset \mathbb{PT}$ via $\mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\lambda_\alpha$

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Familiar applications from flat background:

- Massless free fields $\leftrightarrow$ cohomology on $\mathbb{PT}$ [Penrose, Sparling, Eastwood-Penrose-Wells]
- Representation for on-shell scattering kinematics [Hodges]
- Full tree-level S-matrix of $\mathcal{N} = 4$ SYM [Witten, Berkovits, Roiban-Spradlin-Volovich]
- Full tree-level S-matrix of $\mathcal{N} = 8$ SUGRA [Cachazo-Skinner]

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Theorem [Ward, 1977]

There is a 1:1 correspondence between:

- SD SU(N) Yang-Mills fields on $\mathbb{C}^4$, and
- rank N holomorphic vector bundles $E \to \mathbb{PT}$ trivial on every $X \subset \mathbb{PT}$ (+ technical conditions)

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Upshot: twistor theory *trivializes* the SD sector

SDPWs in Twistor Space

Can construct $E \to \mathbb{PT}$ explicitly; holomorphicity encoded by partial connection on E:

$$\bar{D} = \bar{\partial} + \mathsf{A}\,, \qquad \mathsf{A} = \int_{\mathbb{C}^*} \frac{\mathrm{d}s}{s}\,\bar{\delta}^2(\iota - s\,\lambda) \int^{s\,[\tilde{\iota}\,\mu]} \mathrm{d}t\, f(t)$$

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Penrose transform: gluons encoded by E-twisted cohomology on $\mathbb{PT}$

$$\text{- helicity} \;\leftrightarrow H^{0,1}_{\bar{D}}(\mathbb{PT}, \mathcal{O}(-4) \otimes E)$$

$$+\ \text{helicity} \;\leftrightarrow H^{0,1}_{\bar{D}}(\mathbb{PT}, \mathcal{O} \otimes E)$$

So what?

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Can give perturbative formulation of SDPW background field Yang-Mills in $\mathbb{PT}$ [Mason, Boels, TA-Mason-Sharma]

Properties:

- Perturbative around SD sector
- Generating functional for MHV interactions localized on lines $X \subset \mathbb{PT}$
- Explicit expansion using holomorphic trivialization of $E|_X$

MHV amplitude

Evaluating this expansion gives:

$$\boxed{\delta^3_{+,\perp}\left(\sum_{i=1}^{n} k_i\right) \frac{\langle i\, j\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle} \int\limits_{-\infty}^{+\infty} \mathrm{d}x^-\, \mathrm{e}^{\mathrm{i}\mathcal{F}_n(x^-)}}$$

for *Volkov exponent*

$$\mathbb{K}^{\alpha\dot\alpha}(x^-) := \sum_{i=1}^{n-1} K_i^{\alpha\dot\alpha}(x^-)\,, \qquad \boxed{\mathcal{F}_n(x^-) := \frac{1}{\mathbb{K}_+}\int^{x^-} \mathrm{d}t\, \mathbb{K}^2(t)}$$

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Other sanity checks & features:

- Explicit checks at 3- and 4-points
- Perturbative limit ($\text{MHV}_n +$ background $\rightarrow \text{MHV}_{n+1}$)
- Flat background limit
- Generalization to $\mathcal{N}=4$ SYM

Full tree-level S-matrix?

Easy guess for N^{k}MHV, based on holomorphic maps $Z : \mathbb{CP}^1 \to \mathbb{PT}$

$$\int \frac{\prod_{a=0}^{k+1} \mathrm{d}^{4|4} U_a}{\mathrm{vol}\,\mathrm{GL}(2,\mathbb{C})} \,\mathrm{tr}\left(\prod_{i=1}^{n} \frac{\gamma_i^{-1}\, \mathcal{A}_i\, \gamma_i\, \mathrm{d}\sigma_i}{\sigma_i - \sigma_{i+1}} \right)$$

where:

- $Z(\sigma) = \sum_{a=0}^{k+1} U_a\, \sigma^a$ is a degree $k+1$ holomorphic map
- $\{\sigma_i\} \subset \mathbb{CP}^1$ punctures on $\mathbb{CP}^1$
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Currently just a conjecture...

Summary

Upshot: it *is* possible to make all-multiplicity statements in strong backgrounds!

Many exciting things to do:

- Prove/correct $\mathrm{N}^k\mathrm{MHV}$ conjecture
- Gravitational SDPWs
- Double copy for full tree-level SDPW S-matrix
- Generalize to generic PW backgrounds