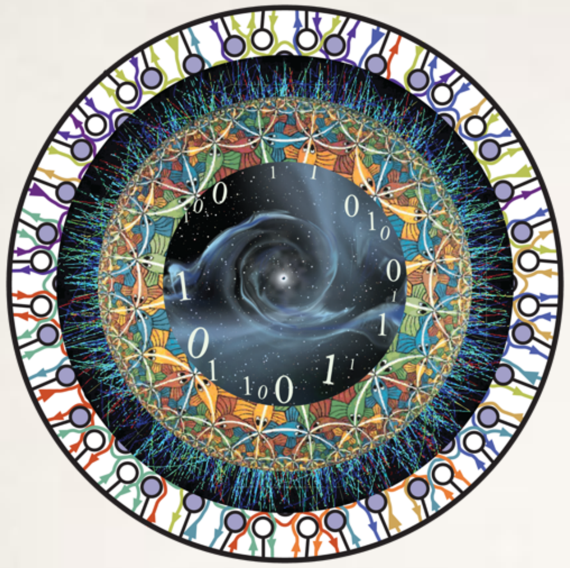
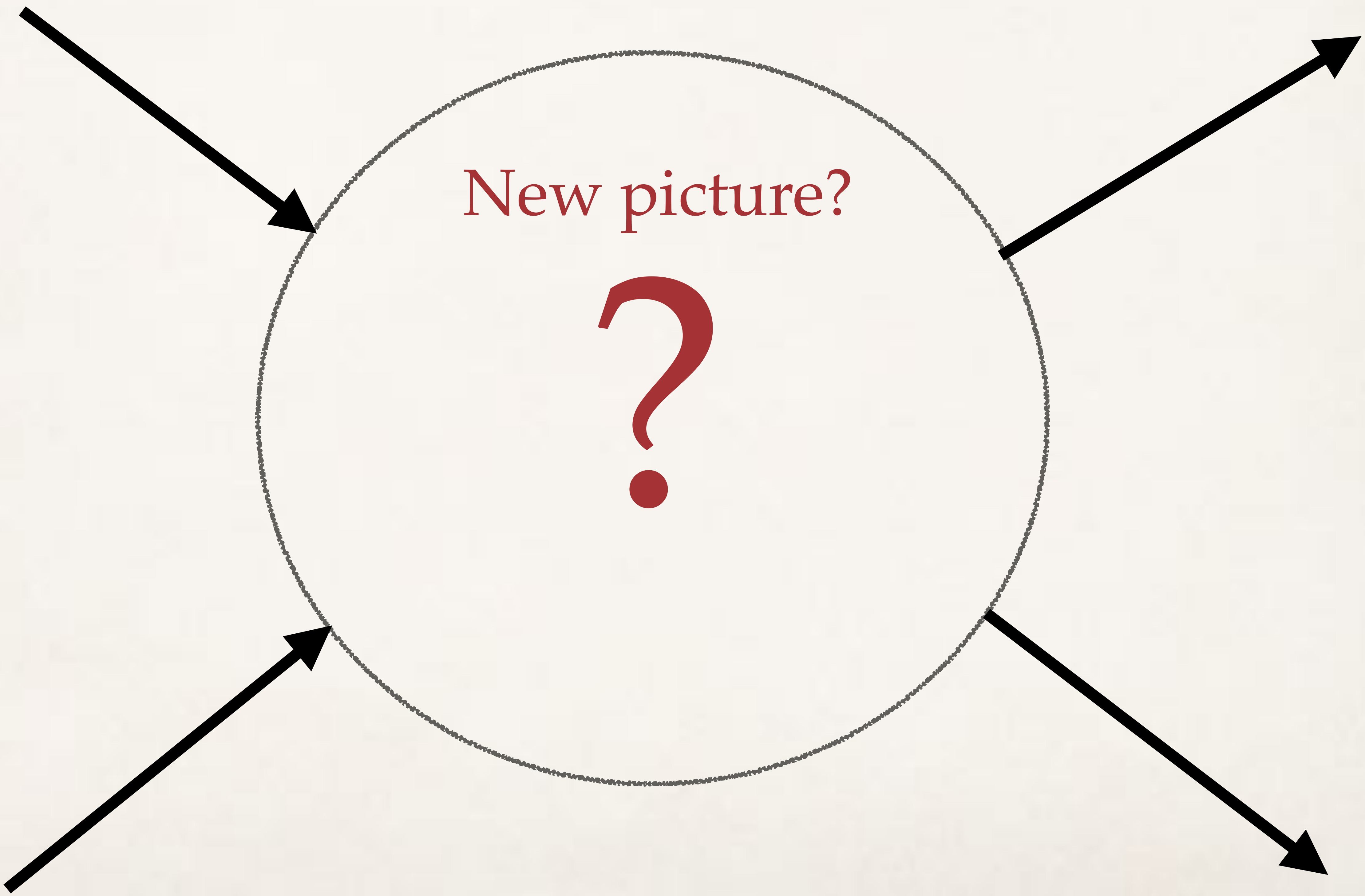


Towards the Gravituhedron

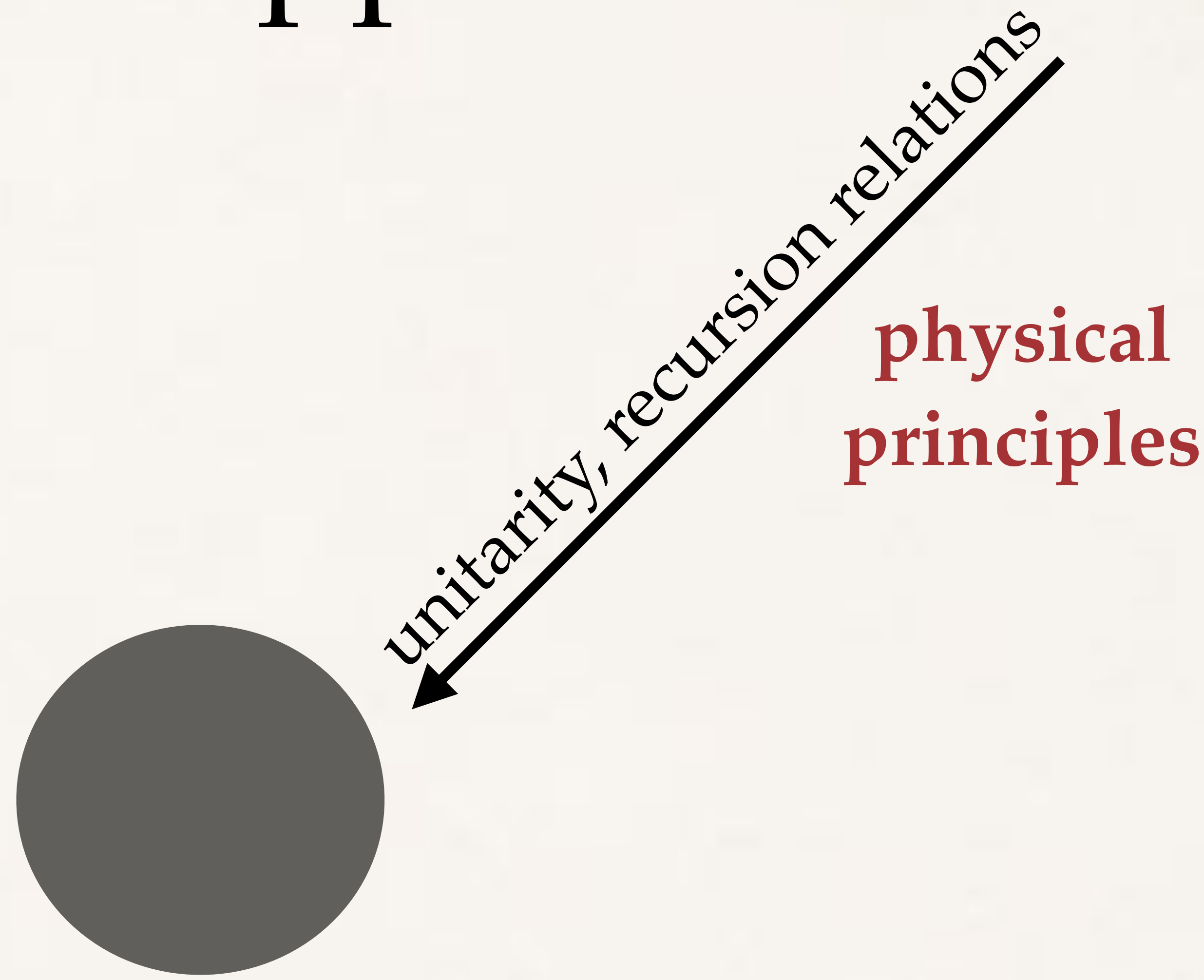
Jaroslav Trnka
Center for Quantum Mathematics and Physics (QMAP)
University of California, Davis

QCD meets Gravity, UCLA, December 2019

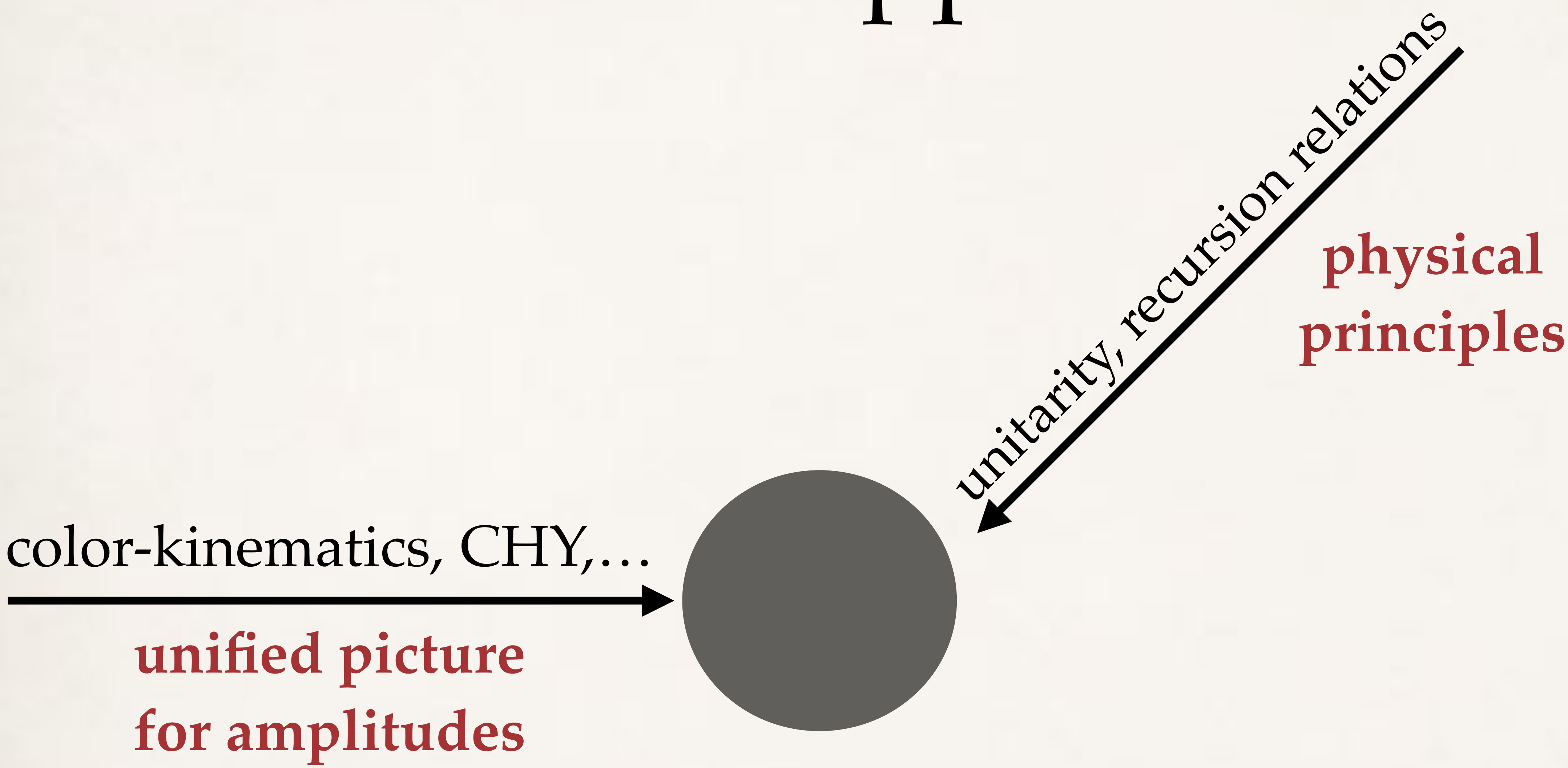
What is the scattering amplitude?



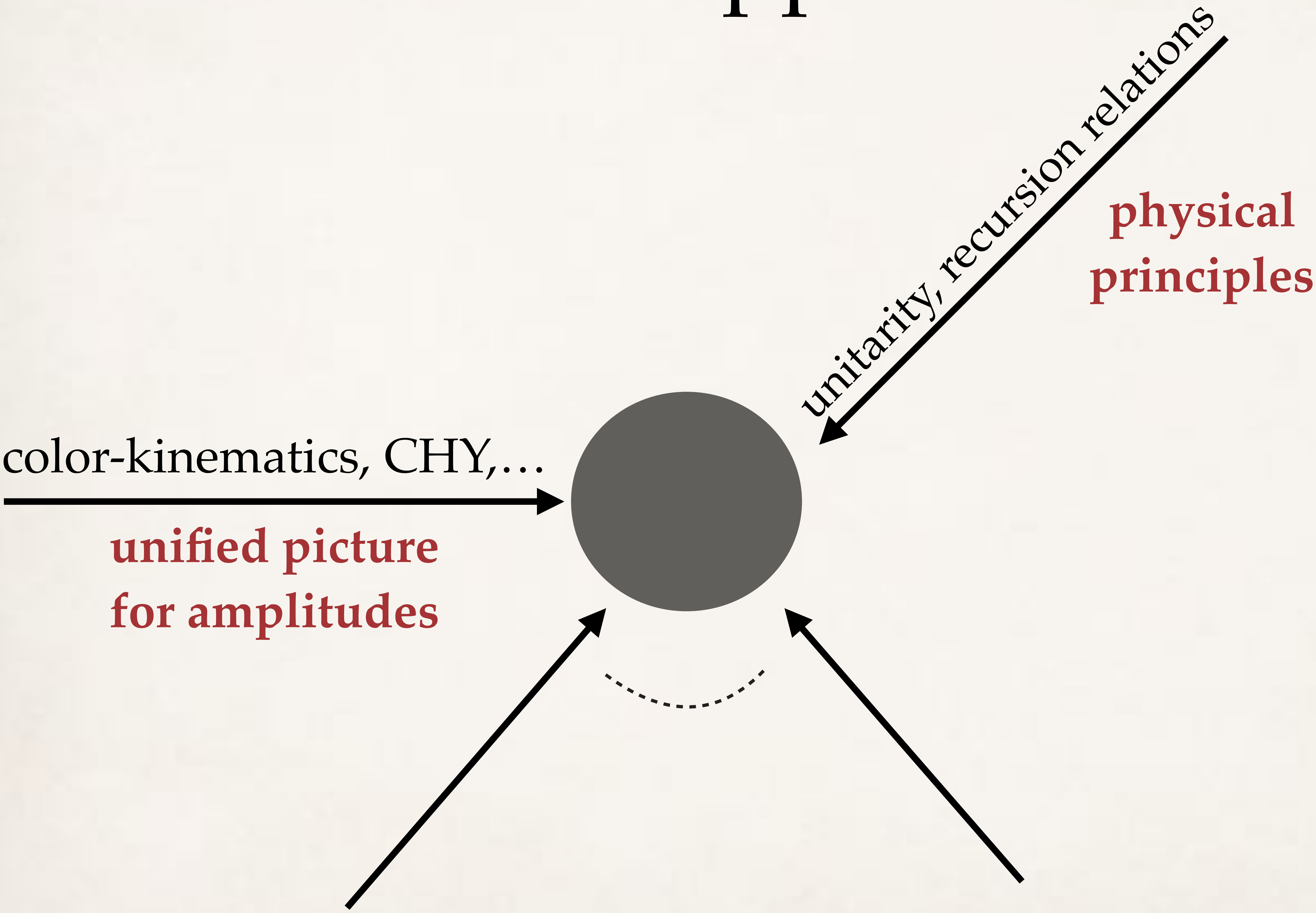
Various approaches



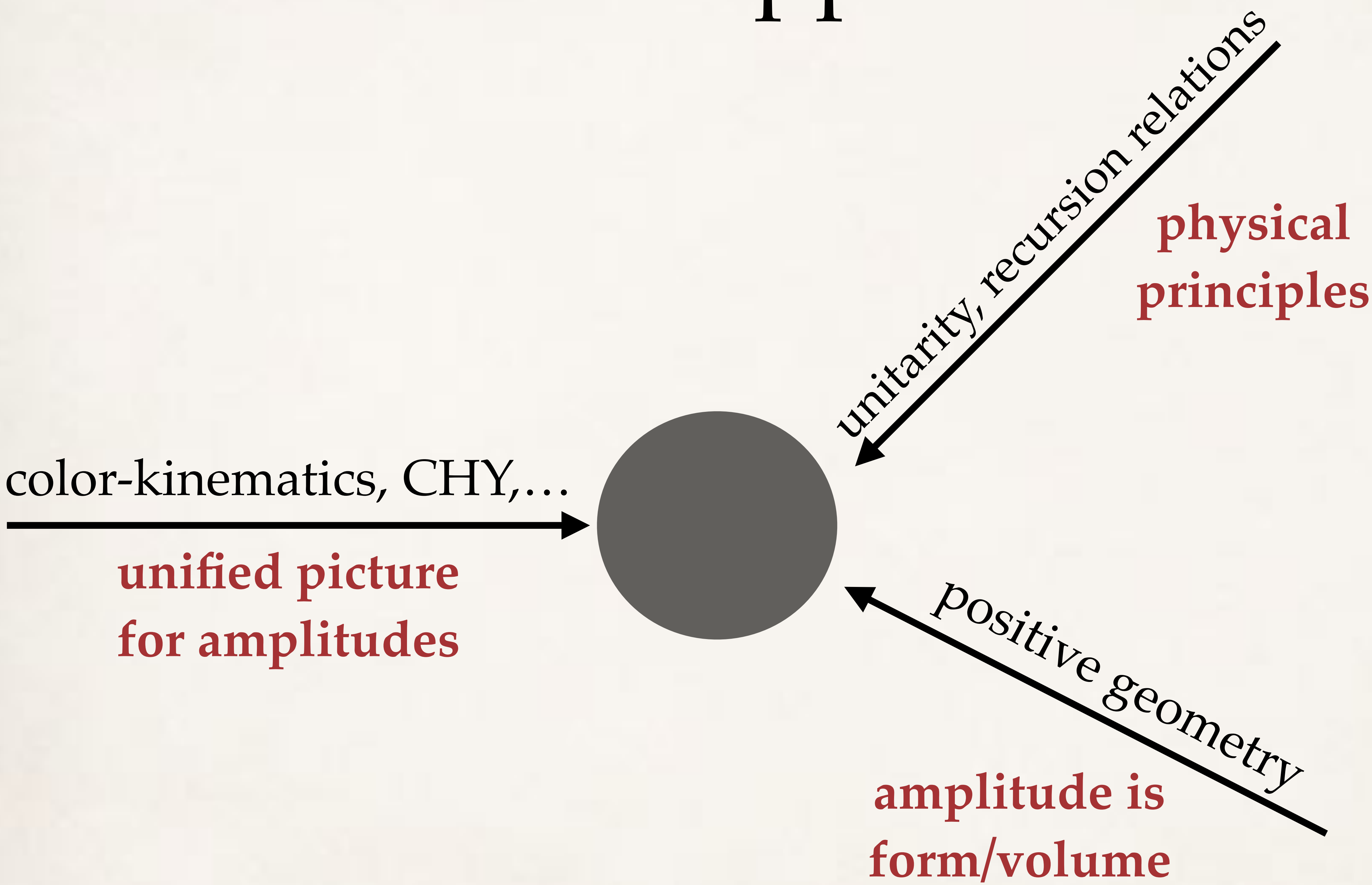
Various approaches



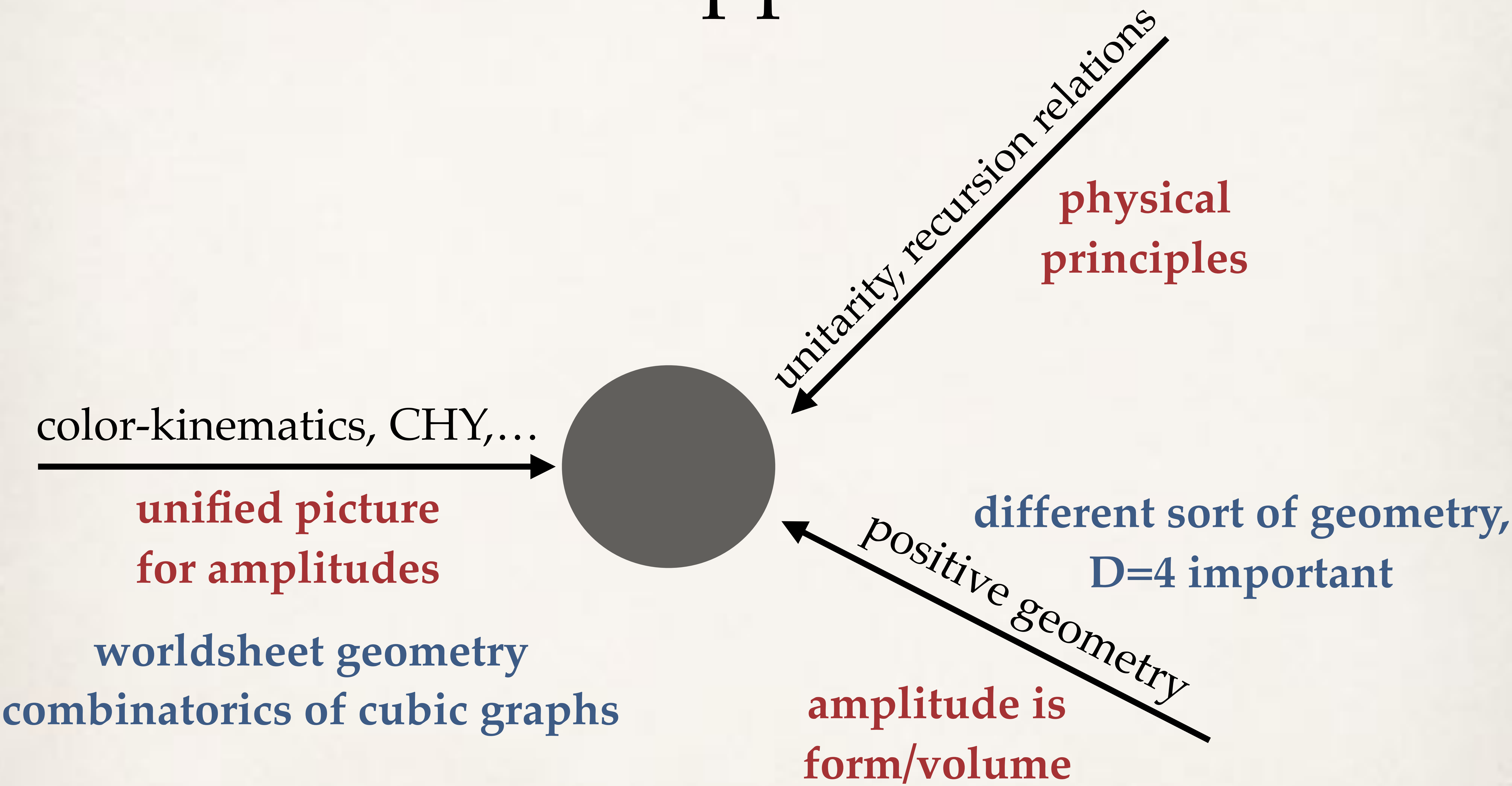
Various approaches



Various approaches



Various approaches



Geometry of gluon amplitudes

Tree-level gluon amplitudes

- Simplest amplitude: MHV amplitude

(Parke, Taylor)
(Nair)

$$A_n = \frac{\delta(Q)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\ldots\langle n1\rangle}$$

color-ordered amplitudes: cyclic symmetry

- General N=4 SYM tree-level amplitude

$$A_{n,k} = A_n \times R_{n,k}(Z_j)$$

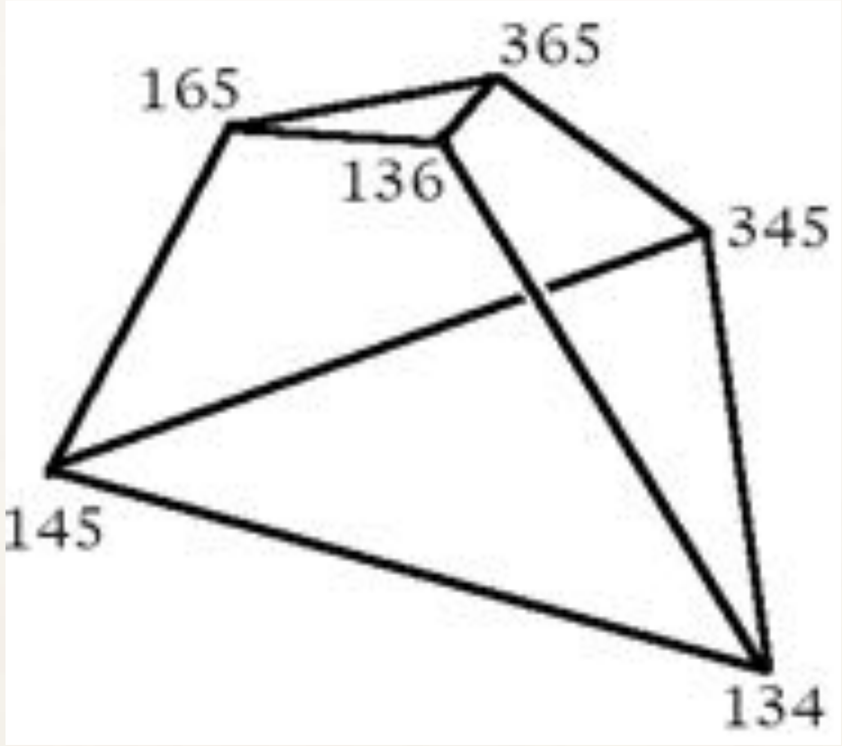
dual conformal symmetry
momentum twistors
geometric interpretation

Amplitude as volume

- Focus on 6pt NMHV amplitude $1^{-}2^{-}3^{-}4^{+}5^{+}6^{+}$

(Hodges 2009)

$$\boxed{A_6 = \int_P dV}$$

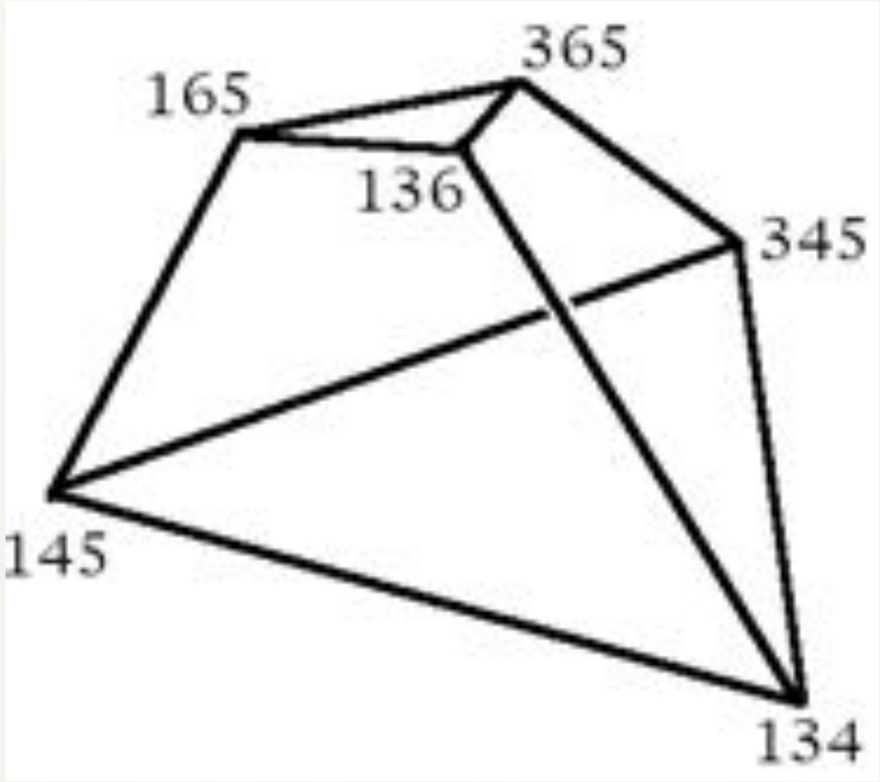


3-d polytope in momentum twistor space

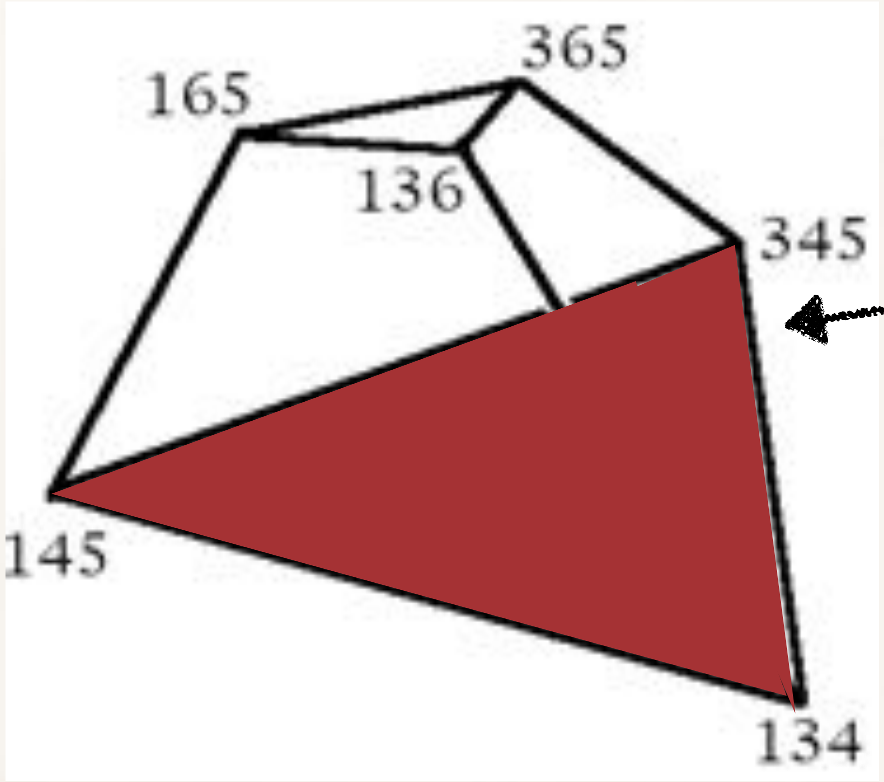
"face" of 4-d polytope

superamplitude

Amplitude as volume

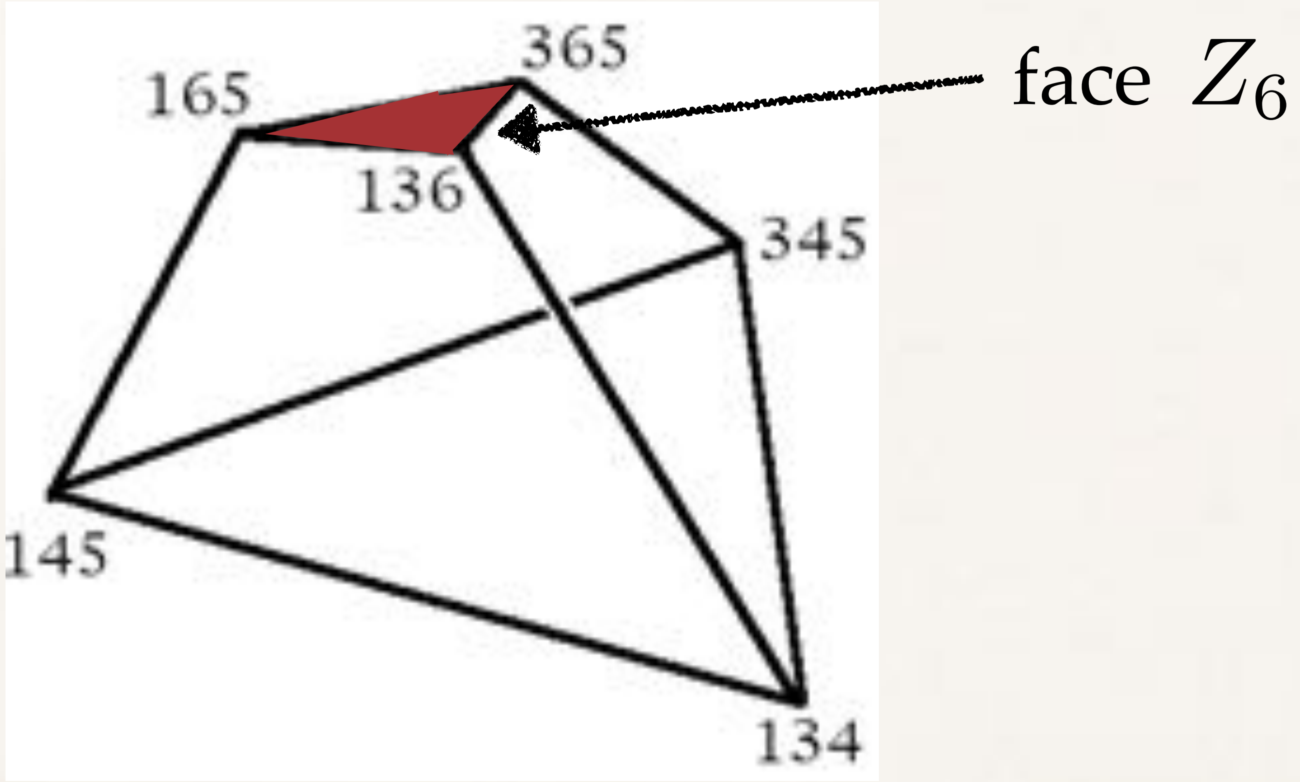


Amplitude as volume

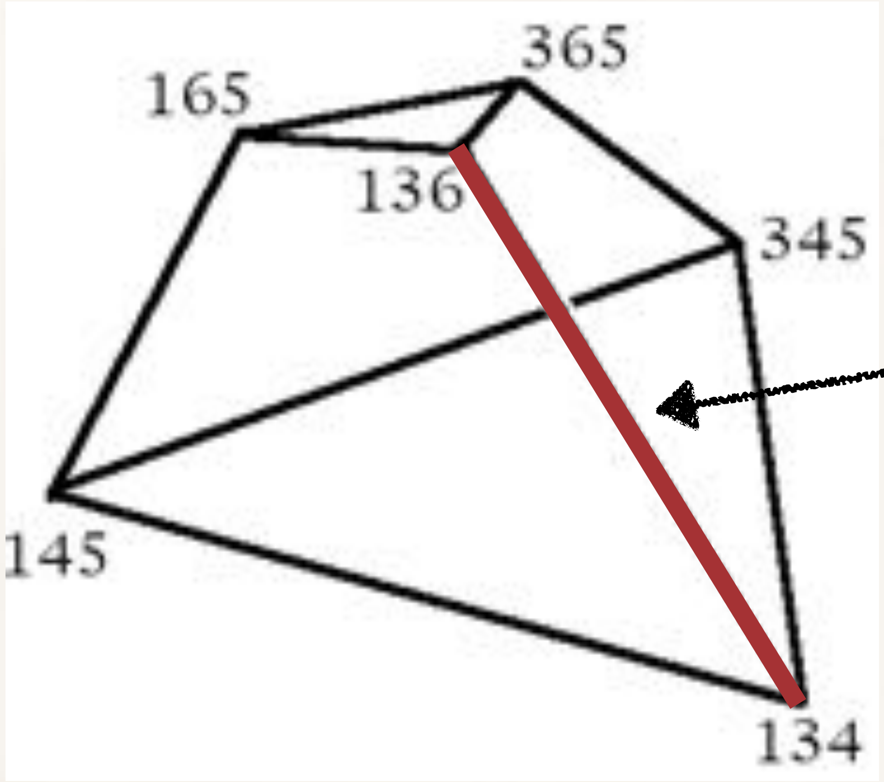


face Z_4

Amplitude as volume



Amplitude as volume

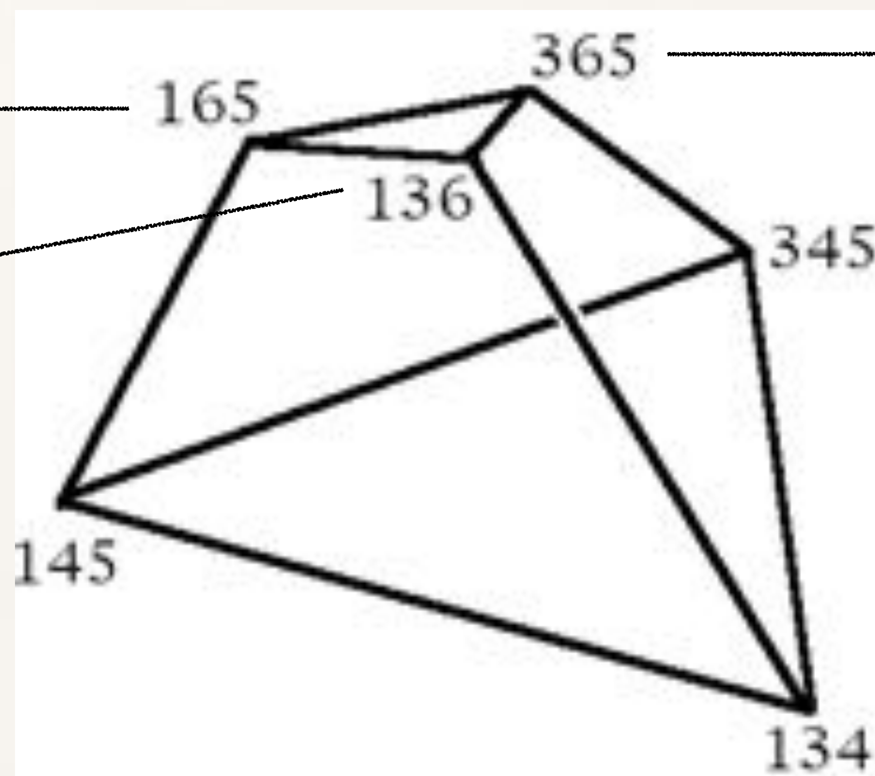


edge 13 $\equiv Z_1 Z_3$

Amplitude as volume

vertices correspond to poles in the amplitude

$\langle 1256 \rangle$ $\langle 6123 \rangle$ $\langle 1245 \rangle$



$\langle 2356 \rangle$ $\langle 2345 \rangle$ $\langle 1234 \rangle$

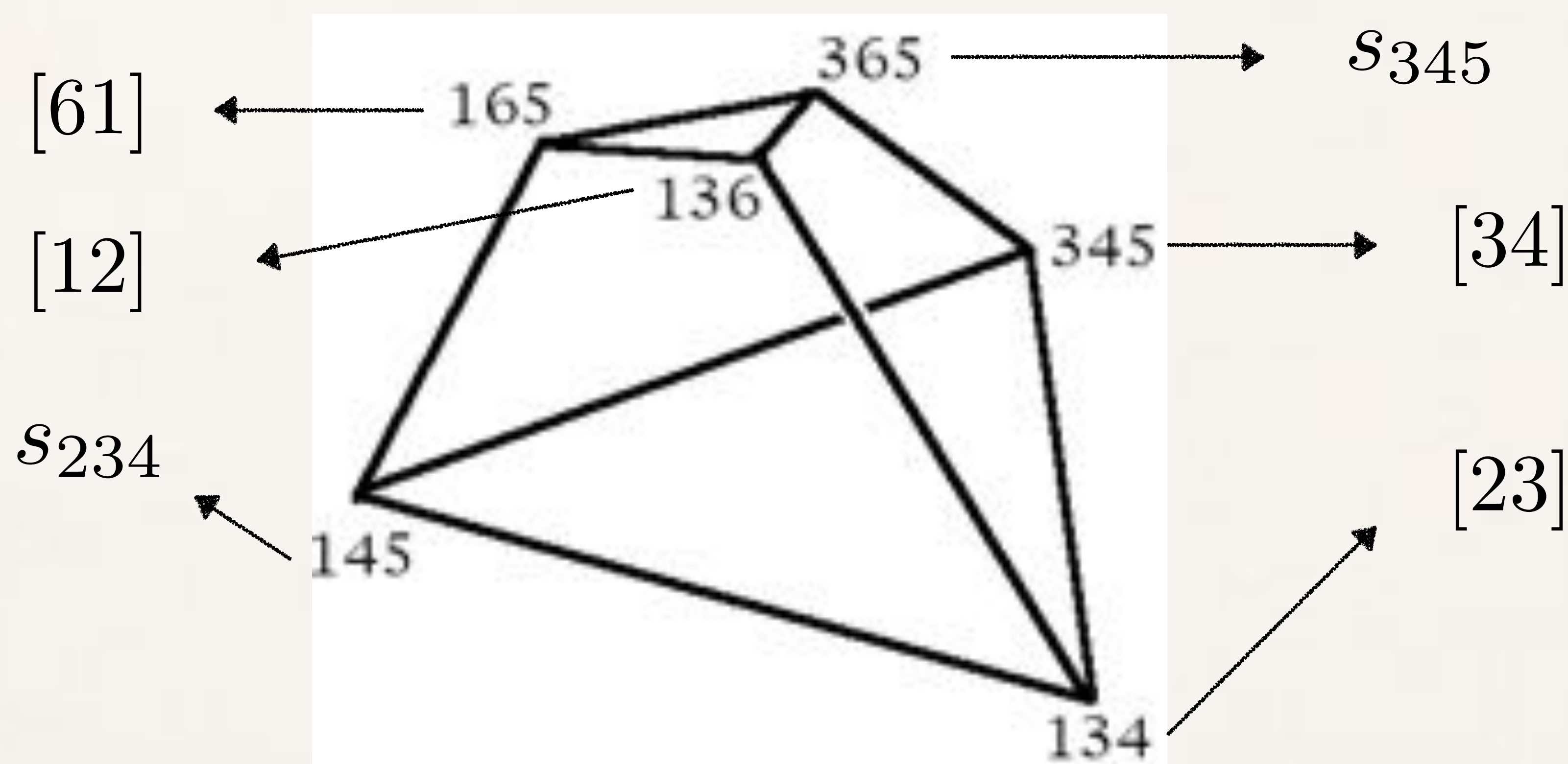
$$\langle i i+1 j j+1 \rangle \sim s_{i+1 \ldots j}$$

"2" is special

$1^- 2^- 3^- 4^+ 5^+ 6^+$

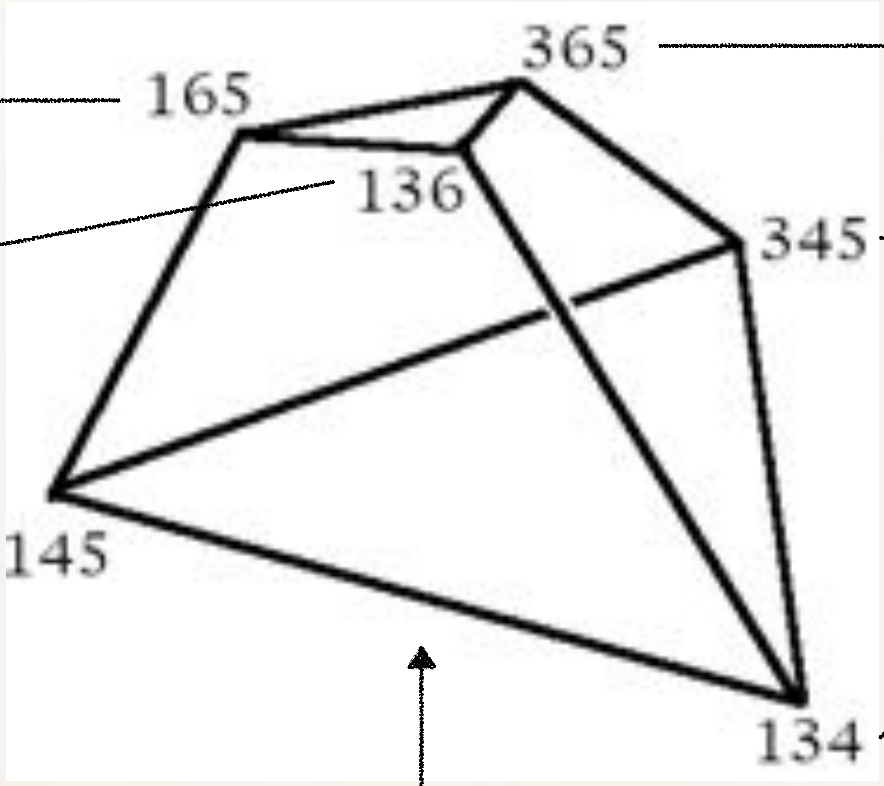
Amplitude as volume

vertices correspond to poles in the amplitude



Amplitude as volume

vertices correspond to poles in the amplitude



$[61]$

$[12]$

s_{345}

$[34]$

s_{234}

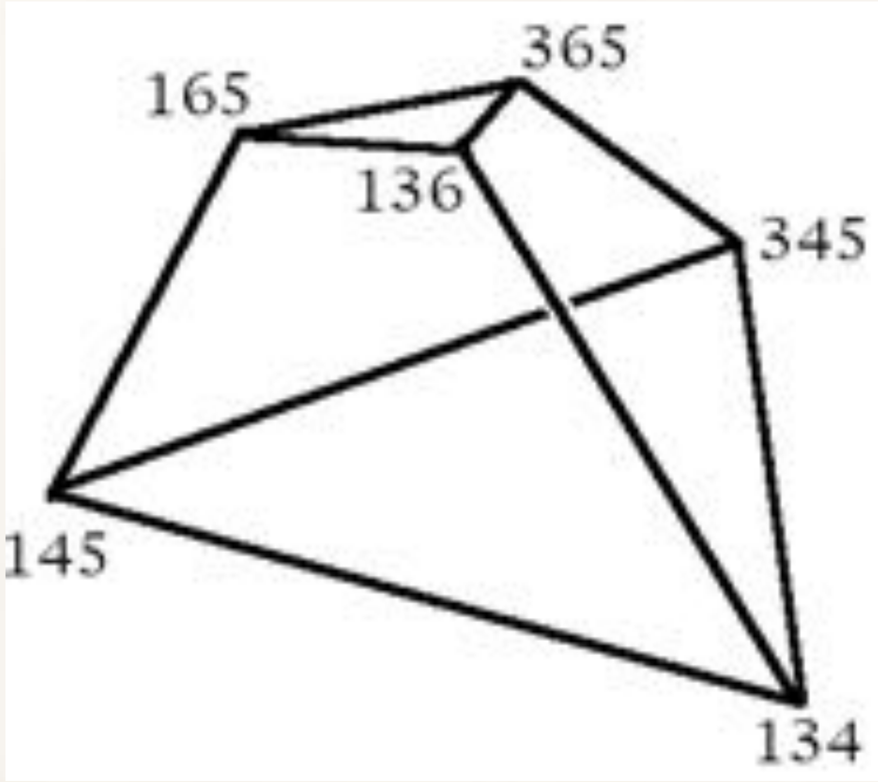
$[23]$

only edges, faces correspond to consistent factorizations / soft limits

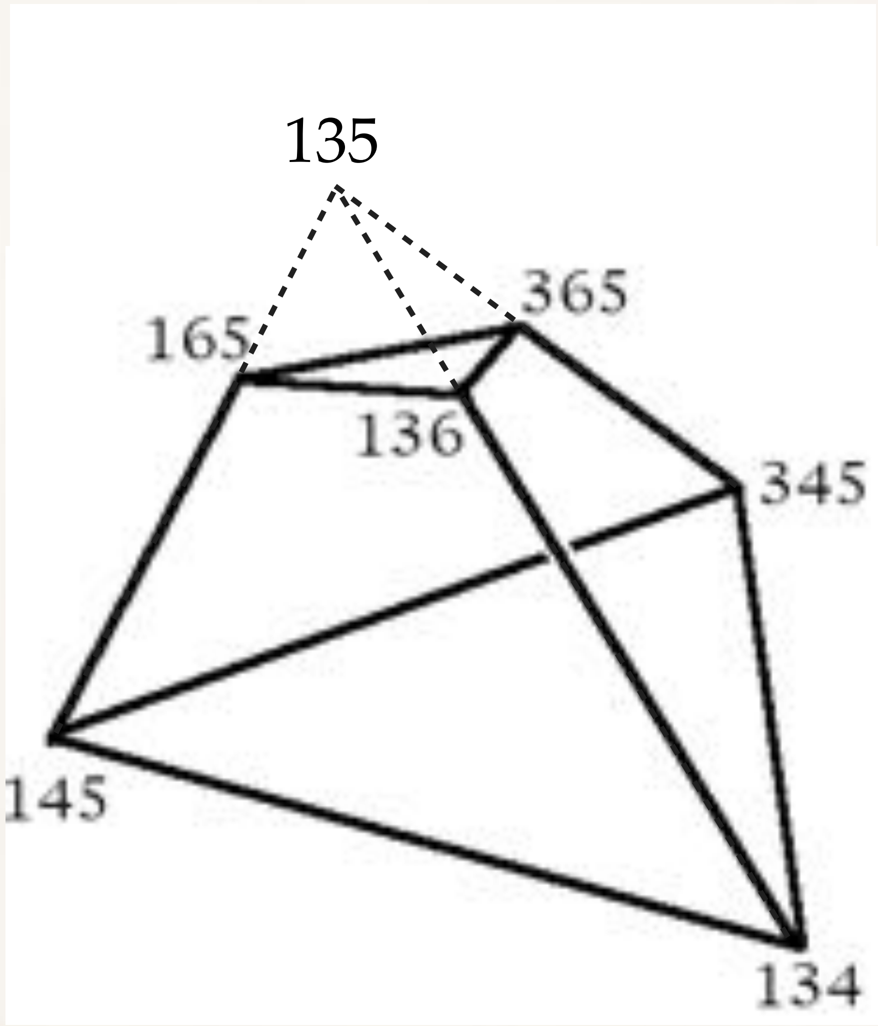
$s_{234} = [23] = 0$

Amplitude as volume

How to calculate amplitude from this picture?

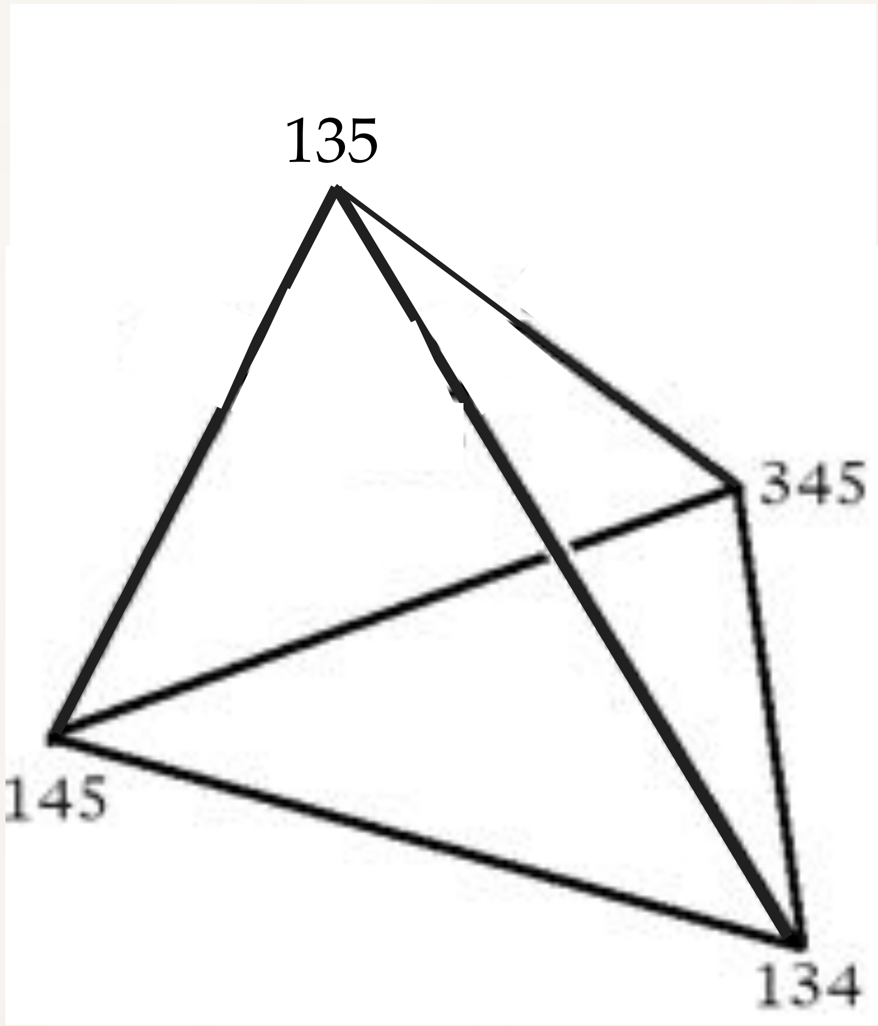


Triangulation



Triangulation

Volume of tetrahedron



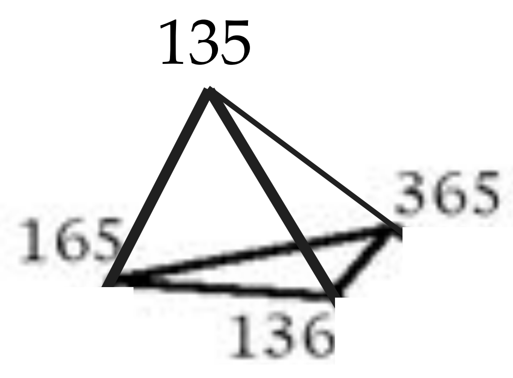
numerator fixed by weights

$$\frac{\langle 1345\rangle^3}{\langle 1245\rangle\langle 2345\rangle\langle 1234\rangle\langle 1235\rangle}$$

poles are vertices

Triangulation

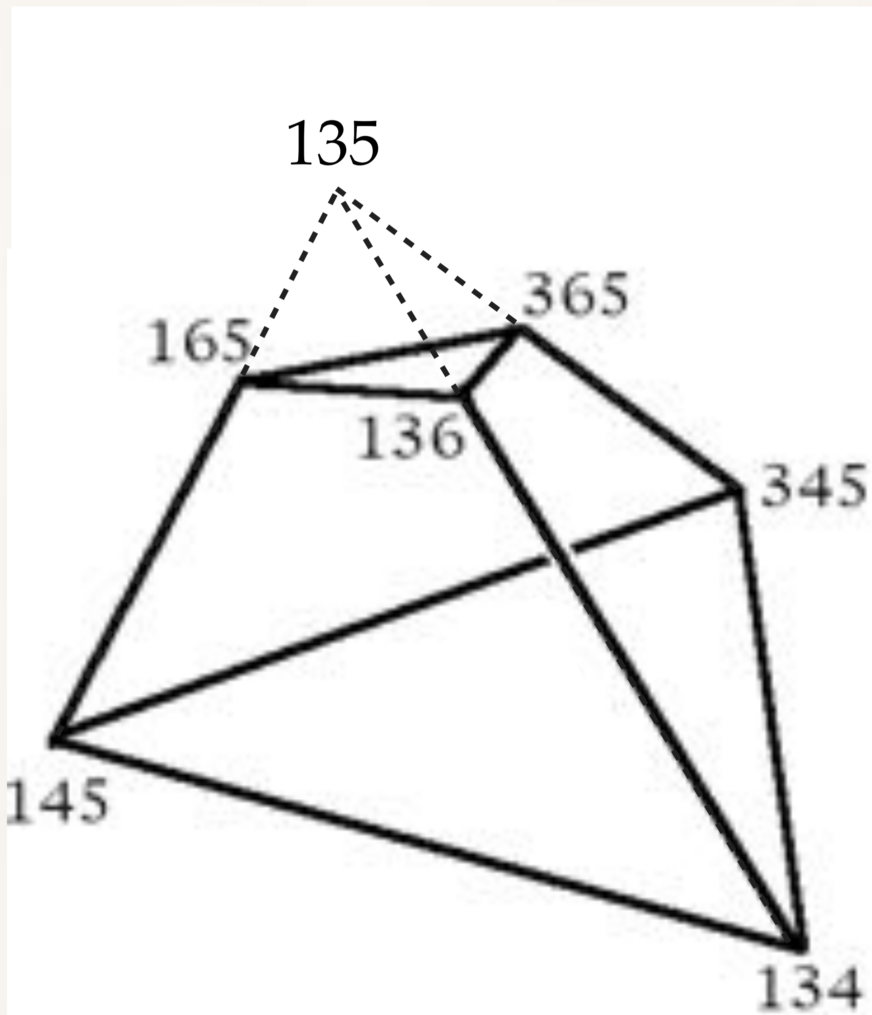
Volume of tetrahedron



$$\frac{\langle 1356\rangle^3}{\langle 1256\rangle\langle 6123\rangle\langle 2356\rangle\langle 1235\rangle}$$

Triangulation

$$R_{6,3} =$$

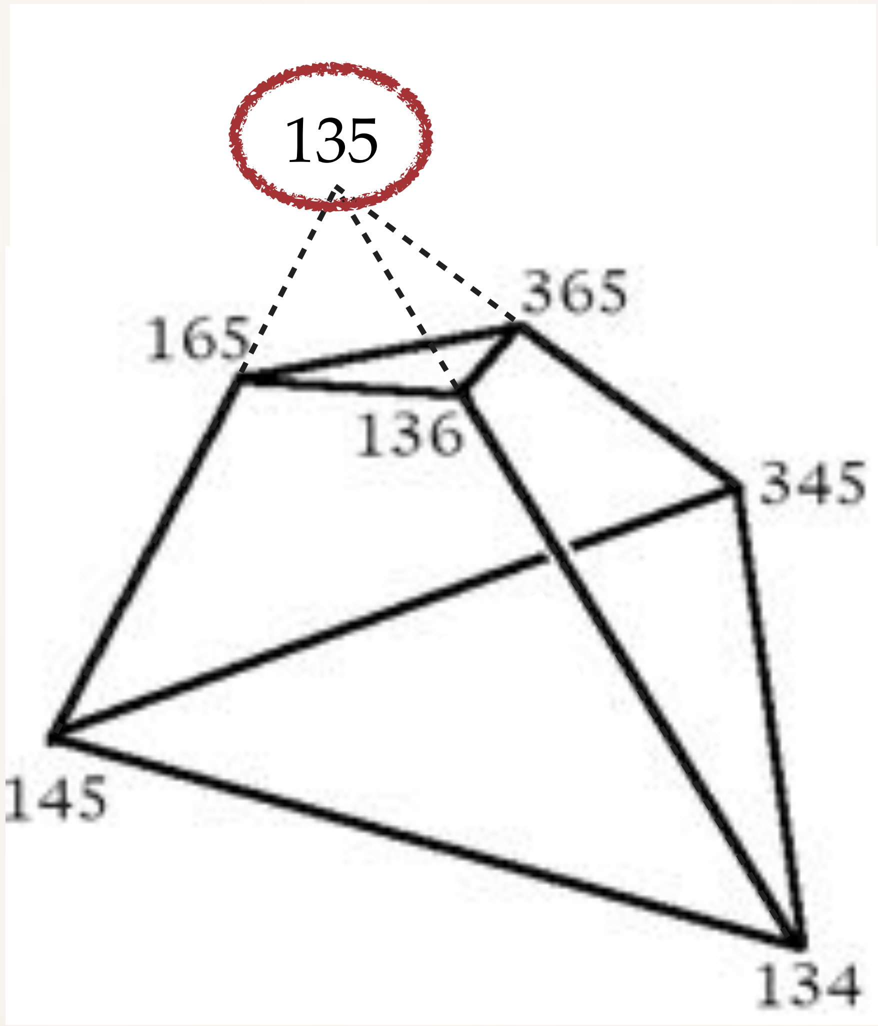


$$\frac{\langle 1345\rangle^3}{\langle 1245\rangle\langle 2345\rangle\langle 1234\rangle\langle 1235\rangle} - \frac{\langle 1356\rangle^3}{\langle 1256\rangle\langle 6123\rangle\langle 2356\rangle\langle 1235\rangle}$$

Volume of polyhedron

Triangulation

$$R_{6,3} =$$



$$\frac{\langle 1345\rangle^3}{\langle 1245\rangle\langle 2345\rangle\langle 1234\rangle\langle 1235\rangle}$$

-

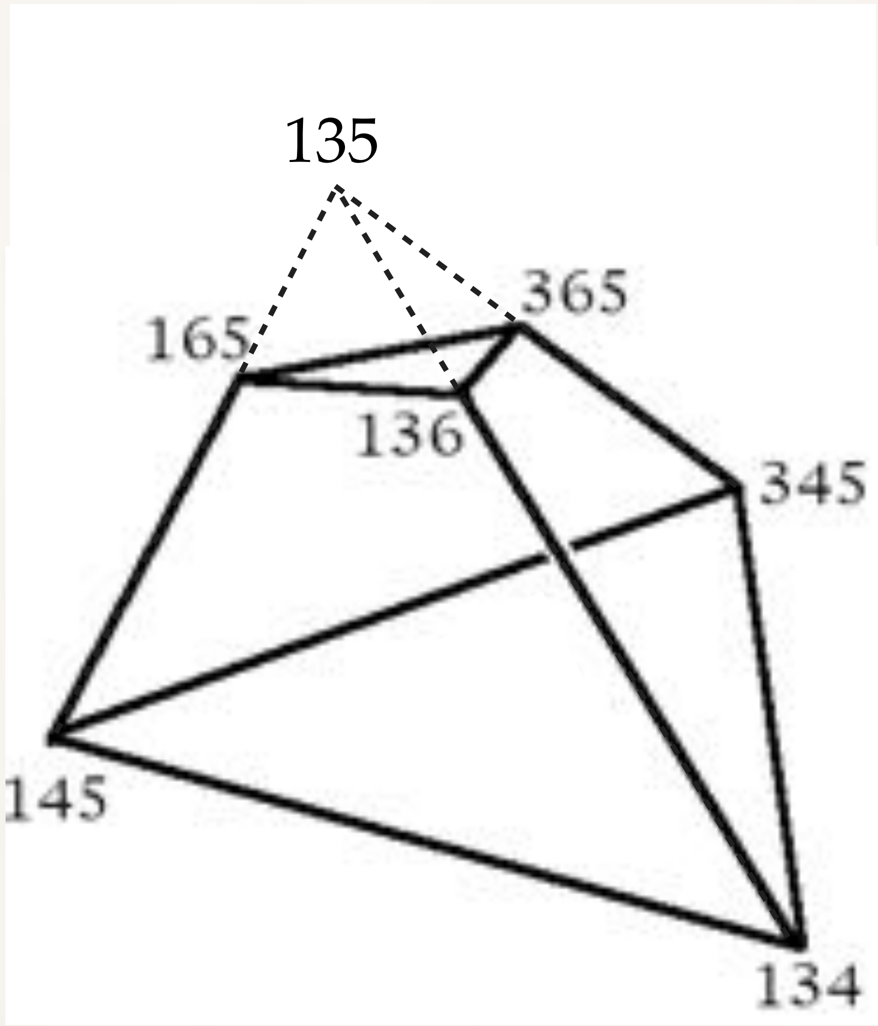
$$\frac{\langle 1356\rangle^3}{\langle 1256\rangle\langle 6123\rangle\langle 2356\rangle\langle 1235\rangle}$$

Volume of polyhedron

spurious pole: cancels

Triangulation

$$R_{6,3} =$$



$$\frac{\langle 1345\rangle^3}{\langle 1245\rangle\langle 2345\rangle\langle 1234\rangle\langle 1235\rangle}$$

-

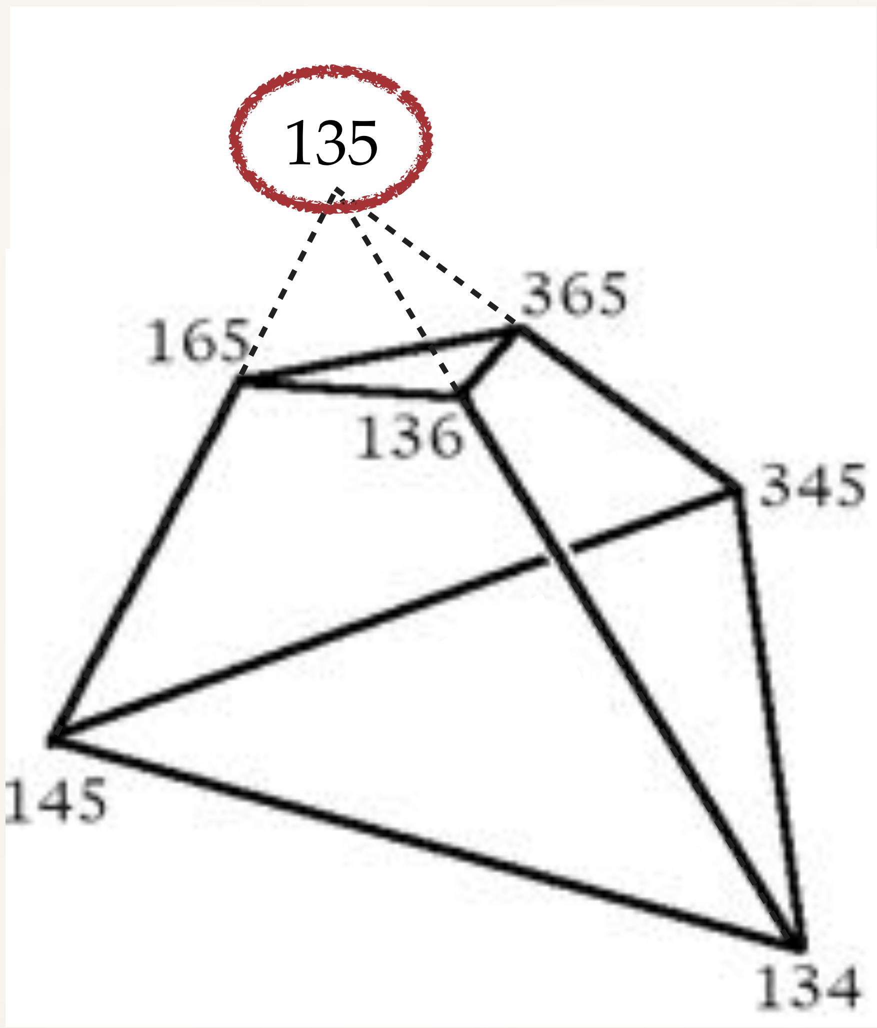
$$\frac{\langle 1356\rangle^3}{\langle 1256\rangle\langle 6123\rangle\langle 2356\rangle\langle 1235\rangle}$$

In momentum space

$$A_{6,3} = \frac{\langle 1|2+3|4]^3}{s_{234}[23][34]\langle 56\rangle\langle 61\rangle\langle 5|3+4|2]} + \frac{\langle 3|4+5|6]^3}{s_{345}[61][12]\langle 34\rangle\langle 45\rangle\langle 5|3+4|2]}$$

Triangulation

$$R_{6,3} =$$



$$\frac{\langle 1345\rangle^3}{\langle 1245\rangle\langle 2345\rangle\langle 1234\rangle\langle 1235\rangle}$$

-

$$\frac{\langle 1356\rangle^3}{\langle 1256\rangle\langle 6123\rangle\langle 2356\rangle\langle 1235\rangle}$$

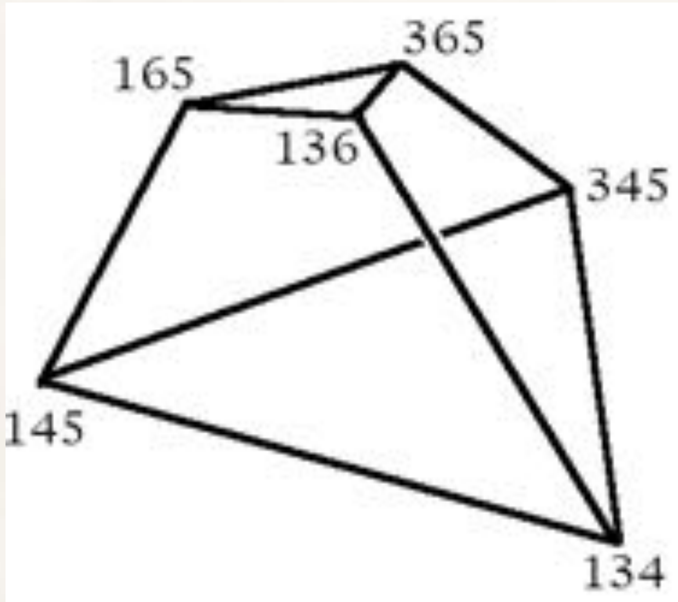
In momentum space

$$A_{6,3} = \frac{\langle 1|2+3|4]^3}{s_{234}[23][34]\langle 56\rangle\langle 61\rangle\langle 5|3+4|2]} + \frac{\langle 3|4+5|6]^3}{s_{345}[61][12]\langle 34\rangle\langle 45\rangle\langle 5|3+4|2]}$$

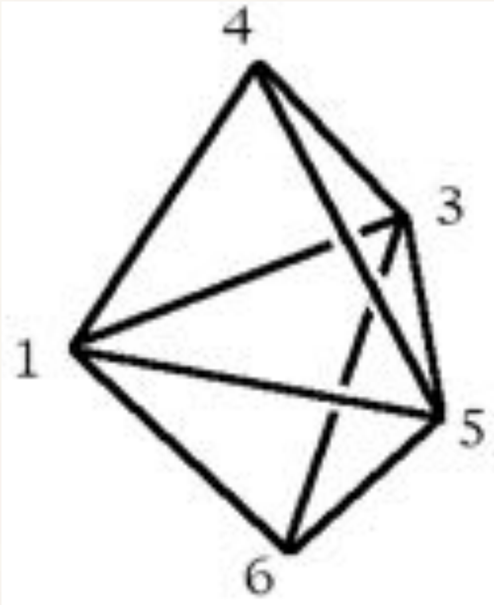
Amplituhedron

(Arkani-Hamed, JT 2013) (Arkani-Hamed, Thomas, JT 2017)

- This volume picture does not generalize



dual



A = volume

A = dlog form

- Amplituhedron: generalization to Grassmannians, beyond
- Search for "dual Amplituhedron" still ongoing

(Arkani-Hamed, Hodges, JT 2015) (Herrmann, Langer, Zheng, JT, in progress)

BCFW recursion relations

- Calculate 6pt amplitude using BCFW recursion relations

$$1^-2^-3^-4^+5^+6^+$$

many different shifts λ $\widetilde{\lambda}$

BCFW recursion relations

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many different shifts λ $\widetilde{\lambda}$

BCFW recursion relations

- Calculate 6pt amplitude using BCFW recursion relations

$$1^-2^-3^-4^+5^+6^+$$

many different shifts λ $\widetilde{\lambda}$

We **always** get one of two formulas:

$$R_{6,3} = (1) + (3) + (5) = (2) + (4) + (6)$$

R-invariants

$$R[a,b,c,d,e] = \frac{(\langle abcd\rangle\eta_e + \langle bcde\rangle\eta_a + \cdots + \langle eabc\rangle\eta_d)^4}{\langle abcd\rangle\langle bcde\rangle\langle cdea\rangle\langle deab\rangle\langle eabc\rangle}$$

make manifest **dual conformal** (and also Yangian) **symmetry**

where $(1) = R[2,3,4,5,6]$ etc

R-invariants

(Drummond, Henn, Korchemsky, Sokatchev) (Arkani-Hamed, Cachazo, Cheung, Kaplan) (Mason, Skinner)

$$R_{6,3} = (1) + (3) + (5) = (\cancel{2}) + (4) + (6)$$

$$\frac{\langle 1356\rangle^3}{\langle 1256\rangle\langle 6123\rangle\langle 2356\rangle\langle 1235\rangle} + \frac{\langle 1345\rangle^3}{\langle 1245\rangle\langle 2345\rangle\langle 1234\rangle\langle 1235\rangle}$$

$$\frac{\langle 3456\rangle^3}{\langle 2345\rangle\langle 2356\rangle\langle 2346\rangle\langle 2456\rangle} + \frac{\langle 1456\rangle^3}{\langle 1245\rangle\langle 1256\rangle\langle 2456\rangle\langle 1246\rangle} + \frac{\langle 1346\rangle^3}{\langle 1234\rangle\langle 6123\rangle\langle 1246\rangle\langle 2346\rangle}$$

For helicity amplitude $1^-2^-3^-4^+5^+6^+$

R-invariants

(Drummond, Henn, Korchemsky, Sokatchev) (Arkani-Hamed, Cachazo, Cheung, Kaplan) (Mason, Skinner)

$$R_{6,3} = (1) + (3) + (5) = \cancel{(2)} + (4) + (6)$$

$$\frac{\langle 1356\rangle^3}{\langle 1256\rangle\langle 6123\rangle\langle 2356\rangle\langle 1235\rangle} + \frac{\langle 1345\rangle^3}{\langle 1245\rangle\langle 2345\rangle\langle 1234\rangle\langle 1235\rangle}$$

$$\frac{\langle 3456\rangle^3}{\langle 2345\rangle\langle 2356\rangle\langle 2346\rangle\langle 2456\rangle} + \frac{\langle 1456\rangle^3}{\langle 1245\rangle\langle 1256\rangle\langle 2456\rangle\langle 1246\rangle} + \frac{\langle 1346\rangle^3}{\langle 1234\rangle\langle 6123\rangle\langle 1246\rangle\langle 2346\rangle}$$

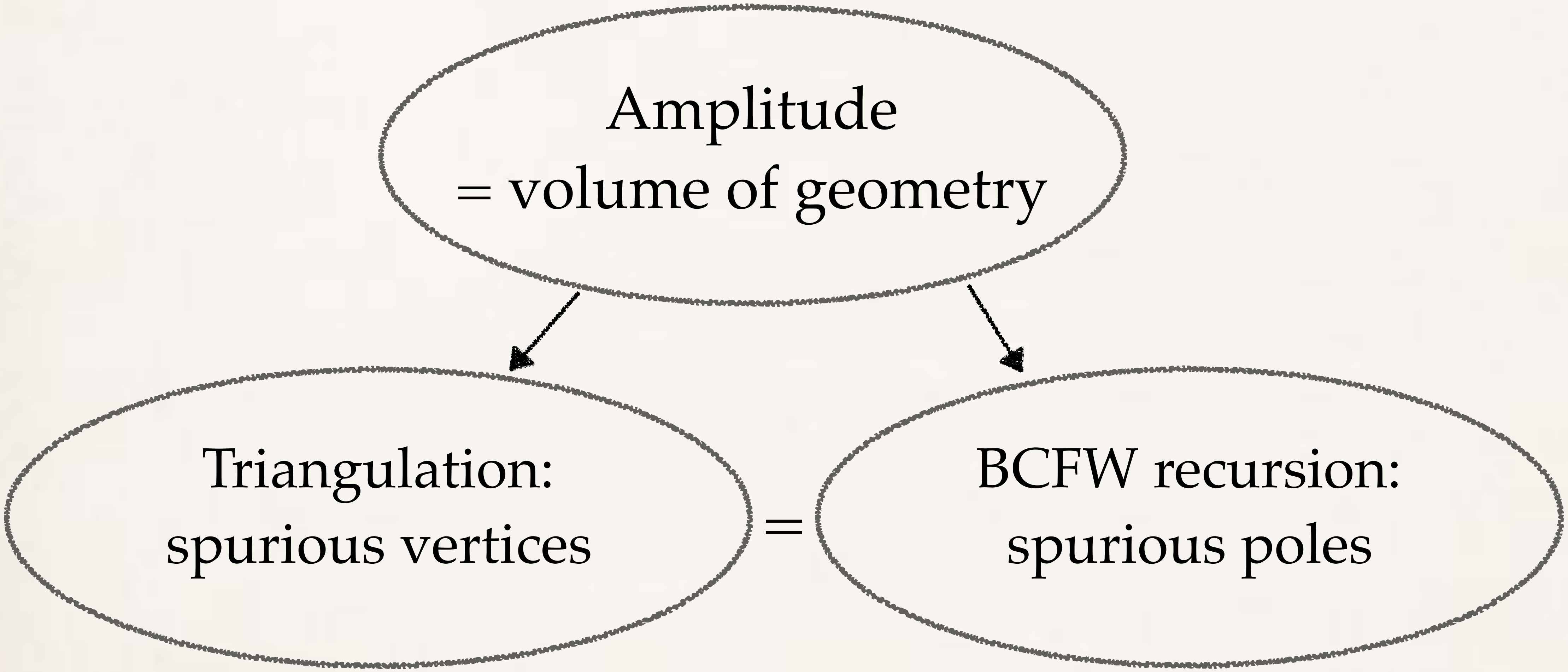
For helicity amplitude $1^-2^-3^-4^+5^+6^+$

Two different triangulations

Gluon recap

Amplitude
= volume of geometry

Gluon recap



rigid formulas = fixed by geometry

manifest dual conformal symmetry

Graviton amplitudes

Graviton amplitudes

- No momentum twistors, use spinor helicity variables
- MHV amplitude: non-trivial, does not factorize

$$A_n = \frac{\det H}{(abc)(def)}$$ (Hodges 2012)

Other formulas

(Berends, Giele, Kuijf 1988)
(Bern, Dixon, Perelstein, Rozowsky 1999
(Mason, Skinner 2008)
(Spradlin, Volovich, Wen 2009)

- Natural proposal: study BCFW recursion relations
  - Problem: good large-z behavior

$$A_n(z) \sim \frac{1}{z^2}$$

$$\oint \frac{dz\, A(z)(a+bz)}{z} = 0$$

  - Many different formulas, no rigidity, they all look different

Example of BCFW formula

- Explicit example: (34) shift

(Cachazo, Svrcek)

$$A(1^-,2^-,3^-,4^+,5^+,6^+) = D_1 + \overline{D}_1^{flip} + D_2 + D_3 + \overline{D}_3^{flip} + D_6.$$

$$D_1 = \frac{\langle 23\rangle\langle 1|2+3|4]^7\big(\langle 1|2+3|4]\langle 5|3+4|2][51] + [12][45]\langle 51\rangle p_{234}^2\big)}{\langle 15\rangle\langle 16\rangle[23][34]^2\langle 56\rangle p_{234}^2\langle 1|3+4|2]\langle 5|3+4|2]\langle 5|2+3|4]\langle 6|3+4|2]\langle 6|2+3|4]}$$
$$+ (1\leftrightarrow 2).$$

$$D_2 = -\frac{\langle 13\rangle^7\langle 25\rangle[45]^7[16]}{\langle 16\rangle[24][25]\langle 36\rangle p_{245}^2\langle 1|2+5|4]\langle 6|2+5|4]\langle 3|1+6|5]\langle 3|1+6|2]}$$
$$+ (1\leftrightarrow 2) + (5\leftrightarrow 6) + (1\leftrightarrow 2, 5\leftrightarrow 6).$$

$$D_3 = \frac{\langle 13\rangle^8[14][56]^7\big(\langle 23\rangle\langle 56\rangle[62]\langle 1|3+4|5] + \langle 35\rangle[56]\langle 62\rangle\langle 1|3+4|2]\big)}{\langle 14\rangle[25][26]\langle 34\rangle^2 p_{134}^2\langle 1|3+4|2]\langle 1|3+4|5]\langle 1|3+4|6]\langle 3|1+4|2]\langle 3|1+4|5]\langle 3|1+4|6]}$$
$$+ (1\leftrightarrow 2).$$

$$D_6 = \frac{\langle 12\rangle[56]\langle 3|1+2|4]^8}{[21][14][24]\langle 35\rangle\langle 36\rangle\langle 56\rangle p_{124}^2\langle 5|1+2|4]\langle 6|1+2|4]\langle 3|5+6|1]\langle 3|5+6|2]}.$$

Reminder: Yang-Mills formula

Rewrite: $$\frac{\langle 1|2+3|4]^3}{s_{234}[23][34]\langle 56\rangle\langle 61\rangle\langle 5|3+4|2]} + \frac{\langle 3|4+5|6]^3}{s_{345}[61][12]\langle 34\rangle\langle 45\rangle\langle 5|3+4|2]}$$

$$A_{6,3} = \sum_S \frac{\langle 1|2+3|4]^3}{s_{234}\cdot[23][34]\cdot\langle 56\rangle\langle 61\rangle\cdot\langle 5|3+4|2]}$$

relabeling sum
$(123,456) \to (321,654)$

multiparticle

anti-holomorphic

holomorphic

spurious

Only adjacent indices appear

New graviton formula

(JT, in progress)

- Formula for graviton amplitude $1^-2^-3^-4^+5^+6^+$

$$\sum_S \frac{\langle 1|2+3|4]^6 \cdot \langle 3|4+5|6]}{s_{234} \cdot [23]^2[34][24] \cdot \langle 56\rangle^2\langle 61\rangle\langle 15\rangle \cdot \langle 5|3+4|2]} + \frac{s_{123}^7}{[12]^2[23]^2[13]^2 \cdot \langle 45\rangle^2\langle 56\rangle^2\langle 64\rangle^2}$$

in comparison to YM:

$$\sum_S \frac{\langle 1|2+3|4]^3}{s_{234} \cdot [23][34] \cdot \langle 56\rangle\langle 61\rangle \cdot \langle 5|3+4|2]}$$

New graviton formula

(JT, in progress)

- Formula for graviton amplitude $1^- 2^- 3^- 4^+ 5^+ 6^+$

$$\sum_S \frac{\langle 1|2+3|4]^6 \cdot \langle 3|4+5|6]}{s_{234} \cdot [23]^2[34][24] \cdot \langle 56\rangle^2 \langle 61\rangle\langle 15\rangle \cdot \langle 5|3+4|2]} + \frac{s_{123}^7}{[12]^2[23]^2[13]^2 \cdot \langle 45\rangle^2\langle 56\rangle^2\langle 64\rangle^2}$$

$S^3 \times S^3$

relabeling sum

$$\sum_S \frac{\langle 1|2+3|4]^3}{s_{234} \cdot [23][34] \cdot \langle 56\rangle\langle 61\rangle \cdot \langle 5|3+4|2]}$$

$(123, 456) \to (321, 654)$

New graviton formula

(JT, in progress)

- Formula for graviton amplitude $1^-2^-3^-4^+5^+6^+$

$$\sum_S \frac{\langle 1|2+3|4]^6 \cdot \langle 3|4+5|6]}{s_{234}\,[23]^2[34][24] \cdot \langle 56\rangle^2\langle 61\rangle\langle 15\rangle \cdot \langle 5|3+4|2]} + \frac{s_{123}^7}{[12]^2[23]^2[13]^2 \cdot \langle 45\rangle^2\langle 56\rangle^2\langle 64\rangle^2}$$

multi-particle
factorization channel

$$\sum_S \frac{\langle 1|2+3|4]^3}{s_{234}\,[23][34] \cdot \langle 56\rangle\langle 61\rangle \cdot \langle 5|3+4|2]}$$

New graviton formula

(JT, in progress)

- Formula for graviton amplitude $1^-2^-3^-4^+5^+6^+$

$$\sum_S \frac{\langle 1|2+3|4]^6 \cdot \langle 3|4+5|6]}{s_{234} \cdot [23]^2[34][24] \cdot \langle 56\rangle^2\langle 61\rangle\langle 15\rangle \cdot \langle 5|3+4|2]} + \frac{s_{123}^7}{[12]^2[23]^2[13]^2 \cdot \langle 45\rangle^2\langle 56\rangle^2\langle 64\rangle^2}$$

anti-holomorphic
factorization channel

$$\sum_S \frac{\langle 1|2+3|4]^3}{s_{234} \cdot [23][34] \cdot \langle 56\rangle\langle 61\rangle \cdot \langle 5|3+4|2]}$$

New graviton formula

(JT, in progress)

- Formula for graviton amplitude $1^-2^-3^-4^+5^+6^+$

$$\sum_S \frac{\langle 1|2+3|4]^6 \cdot \langle 3|4+5|6]}{s_{234} \cdot [23]^2[34][24] \, \langle 56\rangle^2\langle 61\rangle\langle 15\rangle \cdot \langle 5|3+4|2]} + \frac{s_{123}^7}{[12]^2[23]^2[13]^2 \cdot \langle 45\rangle^2\langle 56\rangle^2\langle 64\rangle^2}$$

holomorphic
factorization channel

$$\sum_S \frac{\langle 1|2+3|4]^3}{s_{234} \cdot [23][34] \cdot \langle 56\rangle\langle 61\rangle \, \langle 5|3+4|2]}$$

New graviton formula

(JT, in progress)

- Formula for graviton amplitude $1^-2^-3^-4^+5^+6^+$

$$\sum_S \frac{\langle 1|2+3|4]^6 \cdot \langle 3|4+5|6]}{s_{234} \cdot [23]^2[34][24] \cdot \langle 56\rangle^2\langle 61\rangle\langle 15\rangle \cdot \langle 5|3+4|2]} + \frac{s_{123}^7}{[12]^2[23]^2[13]^2 \cdot \langle 45\rangle^2\langle 56\rangle^2\langle 64\rangle^2}$$

spurious pole

$$\sum_S \frac{\langle 1|2+3|4]^3}{s_{234} \cdot [23][34] \cdot \langle 56\rangle\langle 61\rangle \cdot \langle 5|3+4|2]}$$

New graviton formula

(JT, in progress)

- Formula for graviton amplitude $1^-2^-3^-4^+5^+6^+$

$$\sum_S \frac{\langle 1|2+3|4]^6 \cdot \langle 3|4+5|6]}{s_{234}\cdot [23]^2[34][24]\cdot \langle 56\rangle^2\langle 61\rangle\langle 15\rangle \cdot \langle 5|3+4|2]} + \frac{s_{123}^7}{[12]^2[23]^2[13]^2\cdot\langle 45\rangle^2\langle 56\rangle^2\langle 64\rangle^2}$$

extra term: respects complete label symmetry

$$\sum_S \frac{\langle 1|2+3|4]^3}{s_{234}\cdot [23][34]\cdot\langle 56\rangle\langle 61\rangle\cdot\langle 5|3+4|2]}$$

Second formula

(JT, in progress)

- The other Yang-Mills formula: (1) + (3) + (5)

$$A_{6,3} = \sum_S \frac{\langle 23\rangle^3 [56]^3}{s_{234} \cdot [61] \cdot \langle 34\rangle \cdot \langle 2|3+4|5] \cdot \langle 4|5+6|1]} + \frac{s_{123}^3}{[12][23] \cdot \langle 45\rangle\langle 56\rangle \cdot \langle 4|5+6|1] \cdot \langle 6|4+5|3]}$$

Structure of poles again analogous!

- Gravity:

$$A_{6,3} = \sum_S \frac{\langle 23\rangle^6 [56]^6 \cdot \langle 6|1+2|3]}{s_{234} \cdot [61][51] \cdot \langle 34\rangle\langle 24\rangle \cdot \langle 2|3+4|5] \cdot \langle 4|5+6|1]^2} + \sum_S \frac{s_{123}^5 \cdot \langle 12\rangle\langle 23\rangle [46][56]}{[12][23] \cdot \langle 45\rangle\langle 56\rangle \cdot \langle 4|5+6|1]\langle 4|5+6|3] \cdot \langle 6|4+5|3]\langle 6|4+5|1]}$$

Outlook

- Very early stage, so far only have some nice formulas reminiscent of Amplituhedron geometry
- Challenge 1: What is the singularity structure?

$$\frac{\dots}{\dots\langle 12\rangle\langle 23\rangle\langle 13\rangle} \qquad A_n \xrightarrow{\lambda_3=\alpha\lambda_2} F_n \xrightarrow{\alpha=0} \mathcal{O}\left(\frac{1}{\alpha^2}\right)$$

  - it can not be only logarithmic
  - gravity amplitudes have double poles
- Challenge 2: What is the positive space?
  - needs to be formulated in momentum space
  - must capture all properties of gravity amplitudes

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apart from geometry new symmetry in tree-level graviton amplitudes?

Thank you!