Dynamical Renormalization Group Method Including Spin-Orbit precession

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Work with Adam K. Leibovich [arXiv:1908.05688],
Based on the work of Galley & Rothestein [arXiv:1609.08268]

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Motivation

- The binary inspiral orbit motion can be described by the post-Newtonian equations of motion.
- Adiabatic solutions average over the orbit, causing ambiguity and detail loss.
- Numerical approaches provide great precision but are time-consuming.
- Need analytic solutions without any averaging process to efficiently generate binary evolution covering full parameter space.
Binary Orbital Motion

The binary acceleration in post-Newtonian Expansion:

\[ a = a_N + a_{1PN} + a_{2PN} + \ldots \]
\[ + a_{SO} + a_{SS} + \ldots \]
\[ + a_{RR2.5PN} + a_{RR3.5PN} + \ldots \]
\[ + \ldots, \]

where \( a_N = -\frac{M}{r^2} \hat{n}, \)

\[ a_{SO} = \frac{1}{r^3} \left\{ 6 \hat{n} \left[ (\hat{n} \times \mathbf{v}) \cdot (2S + \Delta \Sigma) \right] - \left[ \mathbf{v} \times (7S + 3\Delta \Sigma) \right] \right. \]
\[ + 3\hat{r} \left[ \hat{n} \times (3S + \Delta \Sigma) \right] \right\}, \]

\[ a_{RR2.5PN} = \frac{M^2 \nu}{15r^4} \hat{r} \left( \frac{136M}{r} + 72v^2 \right) \mathbf{r} - \frac{8M^2 \nu}{5r^3} \left( \frac{3M}{r} + v^2 \right) \mathbf{v}, \]

where \( S = S_1 + S_2 \) and \( \Delta \Sigma = (m_1 - m_2)(S_2/m_2 - S_1/m_1). \)
Spin Precession Equations

The equations that describe the time evolution of the spin vectors:

\[ \dot{S}_a = \frac{1}{r^3} \left\{ (L_N \times S_a) \left( 2 + \frac{3}{2} \frac{m_b}{m_a} \right) - S_b \times S_a + 3(\hat{n} \cdot S_b)\hat{n} \times S_a \right\} \]

where \( \{a, b\} \) are the binary labels \( \{1, 2\} \), and \( L_N = \nu M (r \times v) \) is the Newtonian orbital angular momentum.
Dynamical Renormalization Group
(Chen, Goldenfeld, Oono, 1995)

- Physically-motivated approach to ordinary differential equation problems involving multiple scales, boundary layers with technically difficult asymptotic matching, and WKB analysis.

Conservative background quasi-circular orbit perturbed by radiation reaction.

The perturbations that grow secularly with time can be reummed using the Dynamical Renormalization Group method.

The resummed results are closed-form real-time solutions without any orbital or precession averaging.
The Procedure for DRG method

1. Determine the perturbation around a background solution given by the “bare parameters”.

2. Substitute the full solutions into EoMs and solve for the perturbations relative to an initial time $t_0$.

3. Write the bare parameters as the renormalized parameters plus “counter-terms”, and introduce an arbitrary renormalization scale $\tau$ through $t - t_0 = (t - \tau) + (\tau - t_0)$.

4. Use the counter-terms to cancel the secular terms that grow as $(\tau - t_0)$ grows.

5. Find the Renormalization Group Equations of the renormalized parameters and solve the RG Es to obtain the resummed solutions.
The Moving Frame and the Euler Angle

The fixed \( \{\hat{x}, \hat{y}, \hat{z}\} \) frame:
\( \hat{z} \) — the direction of the total angular momentum \( J / |J| \).

The moving \( \{\hat{n}, \hat{\lambda}, \hat{l}\} \) frame:
\( \hat{n} \) — the direction of the binary separation \( r = x_1 - x_2 \).
\( \hat{l} \) — the direction of the orbital angular momentum \( L_N \).

\( \alpha, \iota \) are the spherical inclination and azimuthal angles of \( \hat{l} \). \( \Phi \) is the angle between \( \hat{n} \) and \( \hat{x}_l = \frac{\hat{z} \times \hat{l}}{|\hat{z} \times \hat{l}|} \).
DRG Step 1: Define the perturbations

The equations of motion and the spin precession equations with 1.5PN spin-orbit + 2.5PN radiation reaction effects, written in the moving frame:

\[
\begin{align*}
\dot{r} - r\omega^2 &= -\frac{M}{r^2} + \frac{64M^3\nu}{15r^4} \dot{r} + \frac{16M^2\nu}{5r^3} \dot{r}^3 + \frac{16M^2\nu}{5r} \dot{\omega}^2 + \frac{\omega}{r^2} (5\dot{S}_l + 3\Delta\Sigma_i), \\
\dot{\omega} + 2\dot{r} \omega &= -\frac{24M^3\nu}{5r^3} \dot{\omega} - \frac{8M^2\nu}{5r^2} \dot{r}^2 \omega - \frac{8M^2\nu}{5} \omega^3 - \frac{2\dot{r}}{r^3} \dot{S}_l, \\
\ddot{\omega} &= \frac{2\ddot{r}}{r^4} S_\lambda + \frac{7}{r^3} S_n + \frac{3\Delta}{r^3} \Sigma_n. \\
\end{align*}
\]

\[
\begin{align*}
\frac{dS_n^a}{dt} &= (\omega - \Omega_a) S_\lambda^a, \\
\frac{dS_\lambda^a}{dt} &= - (\omega - \Omega_a) S_n^a + \omega S_i^a, \\
\frac{dS_i^a}{dt} &= -\ddot{\omega} S_\lambda^a, \\
\end{align*}
\]

9 Degrees of freedom:

\[
r(t), \ \omega(t), \ \phi(t) = \int_0^t \omega(t), \ \quad S^a = S_n^a \hat{n} + S_\lambda^a \hat{\lambda} + S_i^a \hat{l}.
\]
Write the solutions in the form of perturbations around background solutions, for example:

\[ r(t) = R_B + \delta r(t) + \delta r_S(t), \]
\[ \omega(t) = \Omega_B + \delta \omega(t) + \delta \omega_S(t) \]

- Background solutions \( R_B \) and \( \Omega_B \) are called the bare parameters only dependent on the initial time \( t_0 \).
- \( \delta r(t) \) and \( \delta \omega(t) \) are the non-spinning 2.5PN perturbations with only radiation reaction present.
- \( \delta r_S(t) \) and \( \delta \omega_S(t) \) are the 4PN perturbations as the results of the interaction between spin-orbit and radiation reaction effects.
DRG Step 2: Perturbative solutions

Expanding the EoMs up to the linear order of the unknown perturbation

\[
\delta \ddot{r}_S(t) - 2R_B \Omega_B \delta \omega_S(t) - 3\Omega_B^2 \delta r_S(t) = \frac{\delta \omega(t)}{R_B^2} (5S_l + 3\Delta \Sigma_l),
\]

\[
R_B \delta \dot{\omega}_S(t) + 2\Omega_B \delta \dot{r}_S(t) = - \left( \frac{2S_l}{R_B^3} \delta \dot{r}(t) + \left( 88S_l + \frac{264}{5} \Delta \Sigma_l \nu R_B^3 \Omega_B^6 \right) \right).
\]

Solve for the perturbative terms, for instance:

\[
\delta r_S(t) = - \left( \frac{144}{5} S_l + 48 \Delta \Sigma_l \right) \nu R_B^3 \Omega_B^5 (t - t_0)
\]

\[
+ \frac{(7S_l + 3\Delta \Sigma_l)}{2\Omega_B R_B^3} A_B \left[ 2\Omega_B (t - t_0) \cos \left( \Omega_B (t - t_0) + \Phi_B \right) - \sin \left( \Omega_B (t - t_0) + \Phi_B \right) \right]
\]

\[
+ A_B^S \cos \left( \Omega_B (t - t_0) + \Phi_B \right),
\]

- The constant coefficients for the general solutions that only depend on initial conditions are written as bare parameters.

- The solutions contains secular terms that grow as \((t - t_0)\), which are the “divergences” to be renormalized.
DRG Step 3: Parameters redefinition & subtraction scale

Write the bare parameters as the renormalized parameters plus counter-terms:

\[ R_B(t_0) = R_R(\tau) + \delta^5_R(\tau, t_0) + \delta^S_R(\tau, t_0), \]
\[ \Omega_B(t_0) = \Omega_R(\tau) + \delta^5_\Omega(\tau, t_0) + \delta^S_\Omega(\tau, t_0), \]
\[ \Phi_B(t_0) = \Phi_R(\tau) + \delta^5_\Phi(\tau, t_0) + \delta^S_\Phi(\tau, t_0), \]
\[ A^S_B(t_0) = A^S_R(\tau) + \delta^S_A(\tau, t_0). \]

Isolating counter-terms removing divergences at different PN orders.

Introduce the renormalization scale \( \tau \) through replacing all \((t - t_0)\) by \((t - \tau) + (\tau - t_0)\).

\[
r(t) = R_R + \delta^S_R - \frac{64\nu}{5} R^6_R \Omega^6_R (t - \tau) + A_R \sin(\Omega_R(t - \tau) + \Phi_R) \\
- \left( \frac{144}{5} S_l + 48 \Delta \Sigma_l \right) \nu R^3_R \Omega^5_R (t - \tau) - \left( \frac{144}{5} S_l + 48 \Delta \Sigma_l \right) \nu R^3_R \Omega^5_R (\tau - t_0) \\
+ \frac{(7S_l + 3\Delta \Sigma_l)}{2\Omega_R R^3_R} A_R \left[ 2\Omega_R(t - \tau) \cos(\Omega_R(t - \tau) + \Phi_R) + 2\Omega_R(\tau - t_0) \cos(\Omega_R(t - \tau) + \Phi_R) \\
- \sin(\Omega_R(t - \tau) + \Phi_R) \right] \\
+ A^S_R \cos(\Omega_R(t - \tau) + \Phi_R) + \delta^S_A \cos(\Omega_R(t - \tau) + \Phi_R),
\]
DRG Step 4: Remove the divergences

Using the counter-terms to absorb all the secular terms, we can fix the value of counter-terms, for instance:

$$\delta^S_R(\tau, t_0) = \left( \frac{144}{5} S_l + 48\Delta \Sigma_l \right) \nu R^3_R \Omega^5_R (\tau - t_0),$$

After this step, the perturbative solutions are explicitly independent of $t_0$. Choose the arbitrary renormalization scale $\tau = t$ to minimize the secular terms

$$r(t) = R_R(t) + \left( 1 - \frac{(7S_l + 3\Delta \Sigma_l)}{2\Omega_R(t)R^3_R(t)} \right) A_R(t) \sin \Phi_R(t) + A^S_R(t) \cos \Phi_R(t),$$

The solutions become functions of renormalized parameters.
DRG Step 5: RG equations

The bare parameters are independent of the arbitrary scale \( \tau \), as in
\[
\frac{dR_B}{d\tau} = 0,
\]
thus we can obtain the running of the renormalized \( R_R(\tau) \) through the derivative of counter-term  \( d\delta_R/d\tau \). The RG flow of the renormalized parameters are described by

\[
\begin{align*}
\frac{dR_R}{d\tau} & = - \frac{64\nu}{5} R_R^6(\tau) \Omega_R^6(\tau) - \left( \frac{144}{5} S_l + 48\Delta\Sigma_l \right) \nu R_R^3(\tau) \Omega_R^5(\tau), \\
\frac{d\Omega_R}{d\tau} & = \frac{96\nu}{5} R_R^5(\tau) \Omega_R^7(\tau) - \left( \frac{24}{5} S_l - \frac{216}{5} \Delta\Sigma_l \right) \nu R_R^2(\tau) \Omega_R^6(\tau), \\
\frac{d\Phi_R}{d\tau} & = \Omega_R(\tau), \\
\frac{dA_R^S}{d\tau} & = \left( 7S_l + 3\Delta\Sigma_l \right) \frac{A_R}{R_R^3}.
\end{align*}
\]

The resummed solutions to the equations of motion are given by combining the perturbative solutions and the results of the RG equations.
Final result

\[ r(t) = R_R(t) + \left( 1 - \frac{(7S_l + 3\Delta \Sigma_l)}{2\Omega(t)R^3_R(t)} \right) A_R(t) \sin \Phi_R(t) + A^S_R(t) \cos \Phi_R(t), \]

\[ \omega(t) = \Omega_R(t) - \frac{2\Omega(t)A_R(t)}{R(t)} \left( 1 - \frac{(5S_l + 3\Delta \Sigma_l)}{2\Omega(t)R^3_R(t)} \right) \sin \Phi_R(t) - \frac{2A^S_R(t)\Omega_R(t)}{R(t)} \cos \Phi_R(t), \]

\[ \phi(t) = \Phi_R(t) + \frac{2A_R(t)}{R(t)} \left( 1 - \frac{(19S_l + 9\Delta \Sigma_l)}{2\Omega(t)R^3_R(t)} \right) \cos \Phi_R(t) - \frac{2A^S_R(t)}{R(t)} \sin \Phi_R(t), \]

\[ \begin{align*}
&\frac{64\nu M^3}{5} t + \frac{1}{4} R(t)^4 + \frac{2\mathcal{I}}{5M^{1/2}} R(t)^{5/2} + \frac{\mathcal{J}^2}{M} R(t) + \frac{2\mathcal{J}^{8/3}}{\sqrt{3}M^{4/3}} \tan^{-1} \left( \frac{1}{\sqrt{3}} + \frac{2M^{1/6}R(t)^{1/2}}{\sqrt{3}\mathcal{J}^{1/3}} \right) \\
&+ \frac{\mathcal{J}^{8/3}}{3M^{4/3}} \ln \left( \frac{(\mathcal{J}^{1/3} - M^{1/6}R(t)^{1/2})^2}{\mathcal{J}^{2/3} + \mathcal{J}^{1/3}M^{1/6}R(t)^{1/2} + M^{1/3}R(t)} \right) = \text{constant.} \\
&\Omega^2_R(t)R^3_R(t) + \Omega_R(t)(5S_l + 3\Delta \Sigma_l) = M \\
&\Phi_R(t) + \frac{1}{32M^{5/2}\nu} R^{5/2}_R(t) = \frac{5(41S_l + 15\Delta \Sigma_l)}{256\nu M^2 \mathcal{J}^2} \left( \frac{64\nu M^3}{5} t + \frac{1}{4} R(t)^4 + \frac{2\mathcal{J}}{5M^{1/2}} R^{5/2}_R(t) \right) = \text{constant} \\
&A_R(t) = \text{constant} \\
&A^S_R(t) \left( \frac{7S_l + 3\Delta \Sigma_l}{64\nu M^2 \mathcal{J}^2} \right) \left( \frac{64\nu M^3}{5} t + \frac{1}{4} R(t)^4 + \frac{2\mathcal{J}}{5M^{1/2}} R^{5/2}_R(t) \right) = \text{constant} \\
&i \ln S^a_{+R}(t) - \Phi_R(t) - \frac{5\nu_a R^{3/2}_R(t)}{96M^3/2\nu^2} - \frac{5(41S_l + 15\Delta \Sigma_l)\nu_a}{384M^2\nu^2} \ln \left( M^{1/2}R^{3/2}_R(t) - \mathcal{J} \right) = \text{constant.}
\end{align*} \]
Numerical Comparison: orbital radius $r(t)$

$m_1/m_2 = 1$

$m_1/m_2 = 4$
Numerical Comparison: orbital phase $\phi(t)$

![Graph of orbital phase comparison](image1)

![Graph of orbital phase comparison](image2)
Numerical Comparison: spin vector $\vec{S}(t)$

$m_1/m_2=1$

$\vec{S}_{\text{Resummed}}$ vs $\vec{S}_{\text{Numerical}}$

$m_1/m_2=4$

$\vec{S}_{\text{Resummed}}$ vs $\vec{S}_{\text{Numerical}}$
Binary motion and Spin Precession animation

\( m_1 = 0.5 \quad | \quad m = 1 \quad | \quad S_1 = 0.354704 \)
\( m_2 = 0.5 \quad | \quad \nu = 0.25 \quad | \quad S_2 = 0.199866 \)

\( m_1 = 0.8 \quad | \quad m = 1 \quad | \quad S_1 = 0.607157 \)
\( m_2 = 0.2 \quad | \quad \nu = 0.16 \quad | \quad S_2 = 0.030598 \)

↖ Equal-mass system with anti-aligned spins

← Unequal mass system with mis-aligned spins
Conclusion and Outlooks

- We obtain the analytic solutions to the spinning binary dynamics at leading spin-orbit order with 2.5PN radiation reaction using the DRG method.

- We are adding the other non-spinning PN order terms in the equations of motion for a more complete solution to the dynamics.

\[ \mathbf{a} = \mathbf{a}_N + \mathbf{a}_{1\text{PN}} + \mathbf{a}^{\text{SO}}_{\text{cons}} + \mathbf{a}_{2\text{PN}} + \mathbf{a}_{\text{RR}} + \mathbf{a}_{\text{RR1PN}} + \mathbf{a}_{\text{SO RR}} + \mathbf{a}_{\text{RR2PN}} \]

- More things to do: Large eccentricity and spin-spin order.