Dynamical Renormalization Group Method Including Spin-Orbit precession

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Work with Adam K. Leibovich [arXiv:1908.05688],
Based on the work of Galley & Rothestein [arXiv:1609.08268]

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Motivation

- The binary inspiral orbit motion can be described by the post-Newtonian equations of motion.
- Adiabatic solutions average over the orbit, causing ambiguity and detail loss.
- Numerical approaches provide great precision but are time-consuming.
- Need analytic solutions without any averaging process to efficiently generate binary evolution covering full parameter space.

Binary Orbital Motion

The binary acceleration in post-Newtonian Expansion:

$$\begin{aligned}\boldsymbol{a} = {} & \boldsymbol{a}_{\mathrm{N}} + \boldsymbol{a}_{\mathrm{1PN}} + \boldsymbol{a}_{\mathrm{2PN}} + \ldots \\ & + \boldsymbol{a}_{\mathrm{SO}} + \boldsymbol{a}_{\mathrm{SS}} + \ldots \\ & + \boldsymbol{a}_{\mathrm{RR2.5PN}} + \boldsymbol{a}_{\mathrm{RR3.5PN}} + \ldots \\ & + \ldots,\end{aligned}$$

where

$$\begin{aligned}\boldsymbol{a}_{\mathrm{N}} &= -\frac{M}{r^2}\hat{\boldsymbol{n}}, \\ \boldsymbol{a}_{\mathrm{SO}} &= \frac{1}{r^3}\Big\{6\hat{\boldsymbol{n}}\Big[(\hat{\boldsymbol{n}}\times\boldsymbol{v})\cdot\Big(2\boldsymbol{S}+\Delta\boldsymbol{\Sigma}\Big)\Big] - \Big[\boldsymbol{v}\times\Big(7\boldsymbol{S}+3\Delta\boldsymbol{\Sigma}\Big)\Big] \\ &\quad + 3\dot{r}\Big[\hat{\boldsymbol{n}}\times\Big(3\boldsymbol{S}+\Delta\boldsymbol{\Sigma}\Big)\Big]\Big\}, \\ \boldsymbol{a}_{\mathrm{RR2.5PN}} &= \frac{M^2\nu}{15r^4}\dot{r}\Big(\frac{136M}{r}+72\boldsymbol{v}^2\Big)\boldsymbol{r} - \frac{8M^2\nu}{5r^3}\Big(\frac{3M}{r}+\boldsymbol{v}^2\Big)\boldsymbol{v},\end{aligned}$$

where $S = S_1 + S_2$ and $\Delta\boldsymbol{\Sigma} = (m_1 - m_2)(S_2/m_2 - S_1/m_1)$.

Spin Precession Equations

The equations that describe the time evolution of the spin vectors:

$$\dot{\boldsymbol{S}}_a = \frac{1}{r^3}\left\{ \left(\boldsymbol{L}_N \times \boldsymbol{S}_a\right)\left(2 + \frac{3}{2}\frac{m_b}{m_a}\right) - \boldsymbol{S}_b \times \boldsymbol{S}_a + 3(\hat{\boldsymbol{n}} \cdot \boldsymbol{S}_b)\hat{\boldsymbol{n}} \times \boldsymbol{S}_a \right\}$$

where $\{a, b\}$ are the binary labels $\{1, 2\}$, and $\boldsymbol{L}_N = \nu M(\boldsymbol{r} \times \boldsymbol{v})$ is the Newtonian orbital angular momentum.

Dynamical Renormalization Group

(Chen, Goldenfeld, Oono, 1995)

- Physically-motivated approach to ordinary differential equation problems involving multiple scales, boundary layers with technically difficult asymptotic matching, and WKB analysis.

Conservative background quasi-circular orbit perturbed by radiation reaction.

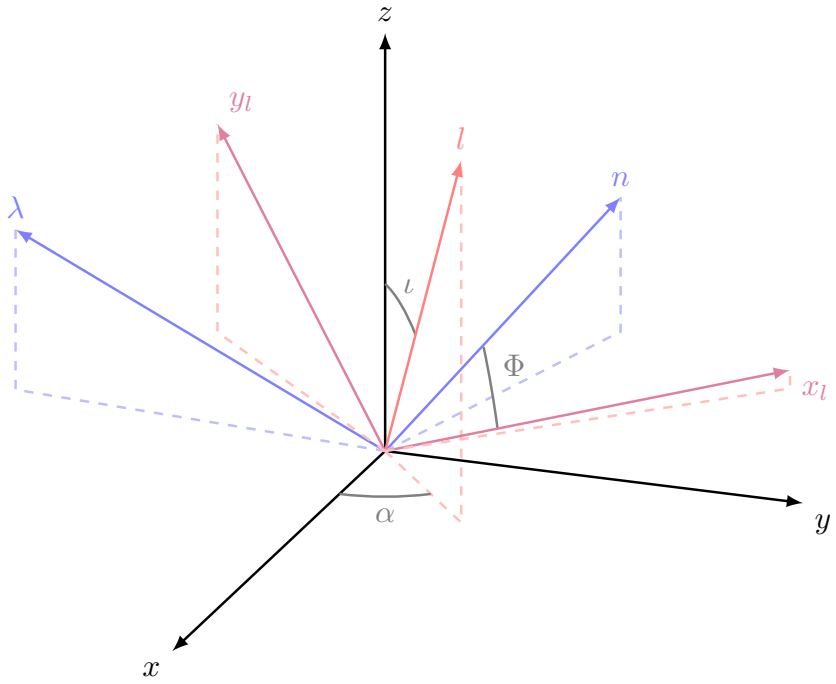
The perturbations that grow secularly with time can be reummed using the Dynamical Renormalization Group method.

The resummed results are closed-form real-time solutions without any orbital or precession averaging.

The Procedure for DRG method

1. Determine the perturbation around a background solution given by the "bare paramters".
2. Substitute the full solutions into EoMs and solve for the perturbations relative to an initial time t_0.
3. Write the bare parameters as the renormalized parameters plus "counter-terms", and introduce an arbitrary renormalization scale τ through $t - t_0 = (t - \tau) + (\tau - t_0)$.
4. Use the counter-terms to cancel the secular terms that grow as $(\tau - t_0)$ grows.
5. Find the Renormalization Group Equations of the renormalized parameters and solve the RGEs to obtain the resummed solutions.

The Moving Frame and the Euler Angle



The fixed $\{\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\hat{\boldsymbol{z}}\}$ frame:

$\hat{\boldsymbol{z}}$ — the direction of the total angular momentum $\boldsymbol{J}/J$.

The moving $\{\hat{\boldsymbol{n}},\hat{\boldsymbol{\lambda}},\hat{\boldsymbol{l}}\}$ frame:

$\hat{\boldsymbol{n}}$ — the direction of the binary separation $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$.

$\hat{\boldsymbol{l}}$ — the direction of the orbital angular momentum $\boldsymbol{L}_N$.

α, ι are the spherical inclination and azimuthal angles of $\hat{\boldsymbol{l}}$. Φ is the angle between $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{x}}_l = \frac{\hat{\boldsymbol{z}}\times\hat{\boldsymbol{l}}}{|\hat{\boldsymbol{z}}\times\hat{\boldsymbol{l}}|}$.

DRG Step 1: Define the perturbations

The equations of motion and the spin precession equations with 1.5PN spin-orbit + 2.5PN radiation reaction effects, written in the moving frame:

$$\ddot{r} - r\omega^2 = -\frac{M}{r^2} + \frac{64M^3\nu}{15r^4}\dot{r} + \frac{16M^2\nu}{5r^3}\dot{r}^3 + \frac{16M^2\nu}{5r}\dot{r}\omega^2 + \frac{\omega}{r^2}(5S_l + 3\Delta\Sigma_l),$$

$$r\dot{\omega} + 2\dot{r}\omega = -\frac{24M^3\nu}{5r^3}\omega - \frac{8M^2\nu}{5r^2}\dot{r}^2\omega - \frac{8M^2\nu}{5}\omega^3 - \frac{2\dot{r}}{r^3}S_l,$$

$$\varpi = \frac{2\dot{r}}{r^4\omega}S_\lambda + \frac{7}{r^3}S_n + \frac{3\Delta}{r^3}\Sigma_n.$$

$$\frac{\mathrm{d}S_n^a}{\mathrm{d}t} = \Big(\omega - \Omega_a\Big)S_\lambda^a,$$

$$\frac{\mathrm{d}S_\lambda^a}{\mathrm{d}t} = -\Big(\omega - \Omega_a\Big)S_n^a + \varpi S_l^a,$$

$$\frac{\mathrm{d}S_l^a}{\mathrm{d}t} = -\varpi S_\lambda^a,$$

9 Degrees of freedom:

$$r(t),\ \ \omega(t),\ \ \phi(t) = \int^t \omega(t), \quad \boldsymbol{S}^a = S_n^a\hat{\boldsymbol{n}} + S_\lambda^a\hat{\boldsymbol{\lambda}} + S_l^a\hat{\boldsymbol{l}}.$$

DRG Step 1: Define the perturbations

Write the solutions in the form of perturbations around background solutions, for example:

$$r(t) = R_B + \delta r(t) + \delta r_S(t),$$
$$\omega(t) = \Omega_B + \delta\omega(t) + \delta\omega_S(t)$$

- Background solutions R_B and Ω_B are called the bare parameters only dependent on the initial time t_0.
- $\delta r(t)$ and $\delta\omega(t)$ are the non-spinning 2.5PN perturbations with only radiation reaction present.
- $\delta r_S(t)$ and $\delta\omega_S(t)$ are the 4PN perturbations as the results of the interaction between spin-orbit and radiation reaction effects.

DRG Step 2: Perturbative solutions

Expanding the EoMs up to the linear order of the unknown perturbation

$$\delta\ddot{r}_S(t) - 2R_B\Omega_B\delta\omega_S(t) - 3\Omega_B^2\delta r_S(t) = \frac{\delta\omega(t)}{R_B^2}(5S_l + 3\Delta\Sigma_l),$$

$$R_B\delta\dot{\omega}_S(t) + 2\Omega_B\delta\dot{r}_S(t) = -\left(\frac{2S_l}{R_B^3}\delta\dot{r}(t) + \left(88S_l + \frac{264}{5}\Delta\Sigma_l\right)\nu R_B^3\Omega_B^6\right).$$

Solve for the perturbative terms, for instance:

$$\begin{aligned}\delta r_S(t) = &-\left(\frac{144}{5}S_l + 48\Delta\Sigma_l\right)\nu R_B^3\Omega_B^5(t-t_0)\\ &+\frac{(7S_l + 3\Delta\Sigma_l)}{2\Omega_B R_B^3}A_B\Big[2\Omega_B(t-t_0)\cos\left(\Omega_B(t-t_0)+\Phi_B\right) - \sin\left(\Omega_B(t-t_0)+\Phi_B\right)\Big]\\ &+A_B^S\cos\left(\Omega_B(t-t_0)+\Phi_B\right),\end{aligned}$$

- The constant coefficients for the general solutions that only depend on initial conditions are written as bare parameters.
- The solutions contains secular terms that grow as $(t - t_0)$, which are the "divergences" to be renormalized.

DRG Step 3: Parameters redefinition & subtraction scale

Write the bare parameters as the renormalized parameters plus counter-terms:

$$
\begin{aligned}
R_B(t_0) &= R_R(\tau) + \delta_R^{v^5}(\tau, t_0) + \delta_R^S(\tau, t_0), \\
\Omega_B(t_0) &= \Omega_R(\tau) + \delta_\Omega^{v^5}(\tau, t_0) + \delta_\Omega^S(\tau, t_0), \\
\Phi_B(t_0) &= \Phi_R(\tau) + \delta_\Phi^{v^5}(\tau, t_0) + \delta_\Phi^S(\tau, t_0), \\
A_B^S(t_0) &= A_R^S(\tau) + \delta_A^S(\tau, t_0).
\end{aligned}
$$

Isolating counter-terms removing divergences at different PN orders.

Introduce the renormalization scale τ through replacing all $(t - t_0)$ by $(t - \tau) + (\tau - t_0)$.

$$
\begin{aligned}
r(t) =& R_R + \delta_R^S - \frac{64\nu}{5} R_R^6 \Omega_R^6 (t-\tau) + A_R \sin\left(\Omega_R(t-\tau) + \Phi_R\right) \\
& - \left(\frac{144}{5} S_l + 48\Delta\Sigma_l\right) \nu R_R^3 \Omega_R^5 (t-\tau) - \left(\frac{144}{5} S_l + 48\Delta\Sigma_l\right) \nu R_R^3 \Omega_R^5 (\tau - t_0) \\
& + \frac{(7S_l + 3\Delta\Sigma_l)}{2\Omega_R R_R^3} A_R \Big[2\Omega_R(t-\tau)\cos\left(\Omega_R(t-\tau) + \Phi_R\right) + 2\Omega_R(\tau - t_0)\cos\left(\Omega_R(t-\tau) + \Phi_R\right) \\
& \qquad - \sin\left(\Omega_R(t-\tau) + \Phi_R\right) \Big] \\
& + A_R^S \cos\left(\Omega_R(t-\tau) + \Phi_R\right) + \delta_A^S \cos\left(\Omega_R(t-\tau) + \Phi_R\right),
\end{aligned}
$$

DRG Step 4: Remove the divergences

Using the counter-terms to absorb all the secular terms, we can fix the value of counter-terms, for instance:

$$\delta_R^S(\tau, t_0) = \left(\frac{144}{5} S_l + 48\Delta\Sigma_l\right) \nu R_R^3 \Omega_R^5 (\tau - t_0),$$

After this step, the perturbative solutions are explicitly independent of t_0. Choose the arbitrary renormalization scale $\tau = t$ to minimize the secular terms

$$r(t) = R_R(t) + \left(1 - \frac{(7S_l + 3\Delta\Sigma_l)}{2\Omega_R(t) R_R^3(t)}\right) A_R(t) \sin\Phi_R(t) + A_R^S(t) \cos\Phi_R(t),$$

The solutions become functions of renormalized parameters.

DRG Step 5: RG equations

The bare parameters are independent of the arbitrary scale τ, as in

$$\mathrm{d}R_B/\mathrm{d}\tau = 0,$$

thus we can obtain the running of the renormalized $R_R(\tau)$ though the derivative of counter-term $\mathrm{d}\delta_R/\mathrm{d}\tau$. The RG flow of the renormalized parameters are described by

$$\begin{aligned}
\frac{\mathrm{d}R_R}{\mathrm{d}\tau} =& -\frac{64\nu}{5}R_R^6(\tau)\Omega_R^6(\tau) - \left(\frac{144}{5}S_l + 48\Delta\Sigma_l\right)\nu R_R^3(\tau)\Omega_R^5(\tau),\\
\frac{\mathrm{d}\Omega_R}{\mathrm{d}\tau} =& \frac{96\nu}{5}R_R^5(\tau)\Omega_R^7(\tau) - \left(\frac{24}{5}S_l - \frac{216}{5}\Delta\Sigma_l\right)\nu R_R^2(\tau)\Omega_R^6(\tau),\\
\frac{\mathrm{d}\Phi_R}{\mathrm{d}\tau} =& \Omega_R(\tau),\\
\frac{\mathrm{d}A_R^S}{\mathrm{d}\tau} =& \left(7S_l + 3\Delta\Sigma_l\right)\frac{A_R}{R_R^3}.
\end{aligned}$$

The resummed solutions to the equations of motion are given by combining the perturbative solutions and the results of the RG equations .

Final result

$$r(t) = R_R(t) + \left(1 - \frac{(7S_l + 3\Delta\Sigma_l)}{2\Omega_R(t)R_R^3(t)}\right) A_R(t)\sin\Phi_R(t) + A_R^S(t)\cos\Phi_R(t),$$

$$\omega(t) = \Omega_R(t) - \frac{2\Omega_R(t)A_R(t)}{R_R(t)}\left(1 - \frac{(5S_l + 3\Delta\Sigma_l)}{2\Omega_R(t)R_R^3(t)}\right)\sin\Phi_R(t) - \frac{2A_R^S\Omega_R(t)}{R_R(t)}\cos\Phi_R(t),$$

$$\phi(t) = \Phi_R(t) + \frac{2A_R(t)}{R_R(t)}\left(1 - \frac{(19S_l + 9\Delta\Sigma_l)}{2\Omega_R(t)R_R^3(t)}\right)\cos\Phi_R(t) - \frac{2A_R^S(t)}{R_R(t)}\sin\Phi_R(t),$$

$$S_+^a(t) = \mathcal{S}_{+R}^a(t)\exp\left\{ i\left(\frac{2A_R^S(t)}{R_R(t)} - \frac{3\nu_a}{\nu}A_R^S(t)\Omega_R(t)^2 R_R(t)\right)\sin\Phi_R(t) - i\left(\frac{2A_R(t)}{R_R(t)} - \left(19S_l + 9\Delta\Sigma_l\right)\frac{A_R}{R_R^4\Omega_R} - \frac{3\nu_a}{\nu}\Omega_R^2 A_R R_R + \frac{\nu_a}{\nu}\left(\frac{29}{2}S_l + \frac{9}{2}\Delta\Sigma_l\right)\frac{A_R(t)\Omega_R(t)}{R_R(t)^2}\right)\cos\Phi_R(t)\right\}.$$

$$\frac{64\nu M^3}{5}t + \frac{1}{4}R_R(t)^4 + \frac{2\mathscr{S}}{5M^{1/2}}R_R(t)^{5/2} + \frac{\mathscr{S}^2}{M}R_R(t) + \frac{2\mathscr{S}^{8/3}}{\sqrt{3}M^{4/3}}\tan^{-1}\left(\frac{1}{\sqrt{3}} + \frac{2M^{1/6}R_R(t)^{1/2}}{\sqrt{3}\mathscr{S}^{1/3}}\right) + \frac{\mathscr{S}^{8/3}}{3M^{4/3}}\ln\left(\frac{\left(\mathscr{S}^{1/3} - M^{1/6}R_R(t)^{1/2}\right)^2}{\mathscr{S}^{2/3} + \mathscr{S}^{1/3}M^{1/6}R_R(t)^{1/2} + M^{1/3}R_R(t)}\right) = \text{constant}.$$

$$\Omega_R^2(t)R_R^3(t) + \Omega_R(t)\left(5S_l + 3\Delta\Sigma_l\right) = M$$

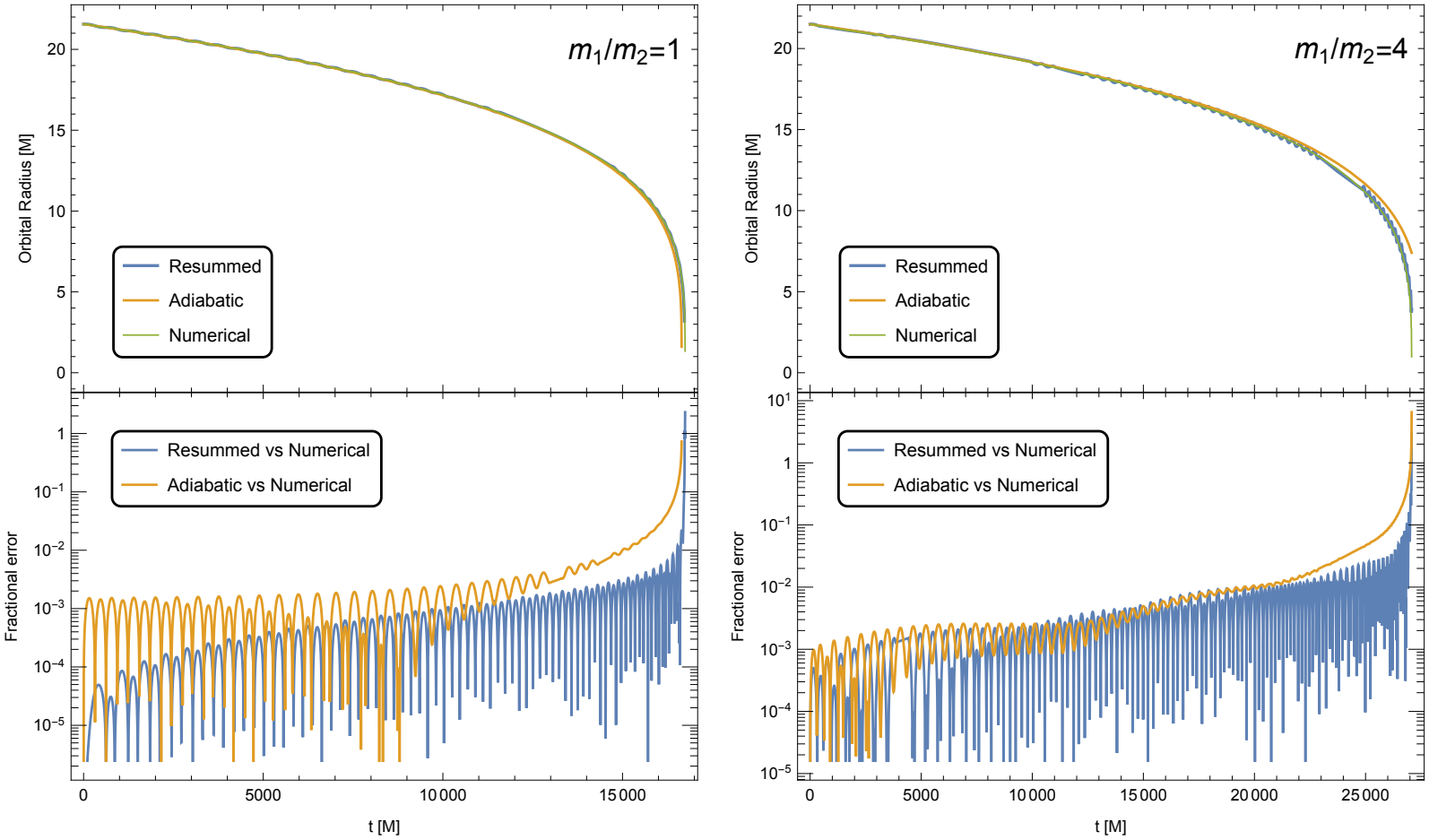
$$\Phi_R(t) + \frac{1}{32M^{5/2}\nu}R_R^{5/2}(t) - \frac{5\left(41S_l + 15\Delta\Sigma_l\right)}{256\nu M^2\mathscr{S}^2}\left(\frac{64\nu M^3}{5}t + \frac{1}{4}R_R^4(t) + \frac{2\mathscr{S}}{5M^{1/2}}R_R^{5/2}(t)\right) = \text{constant}$$

$$A_R(t) = \text{constant}$$

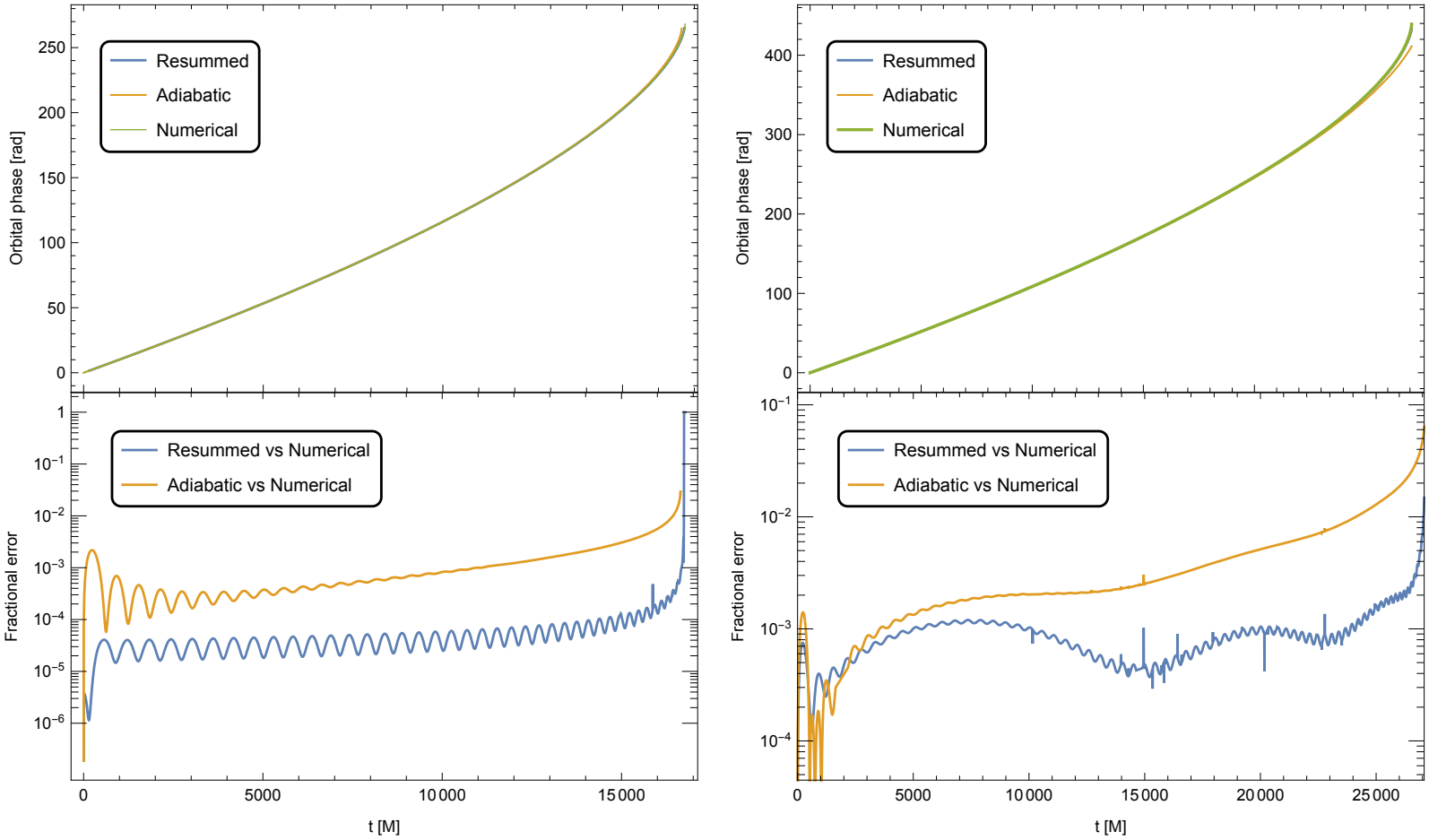
$$A_R^S(t) - \frac{5A_R\left(7S_l + 3\Delta\Sigma_l\right)}{64\nu M^2\mathscr{S}^2}\left(\frac{64\nu M^3}{5}t + \frac{1}{4}R_R^4(t) + \frac{2\mathscr{S}}{5M^{1/2}}R_R^{5/2}(t)\right) = \text{constant}$$

$$i\ln\mathcal{S}_{+R}^a(t) - \Phi_R(t) - \frac{5\nu_a R_R^{3/2}(t)}{96M^{3/2}\nu^2} - \frac{5\left(41S_l + 15\Delta\Sigma_l\right)\nu_a}{384M^2\nu^2}\ln\left(M^{1/2}R_R^{3/2}(t) - \mathscr{S}\right) = \text{constant}.$$

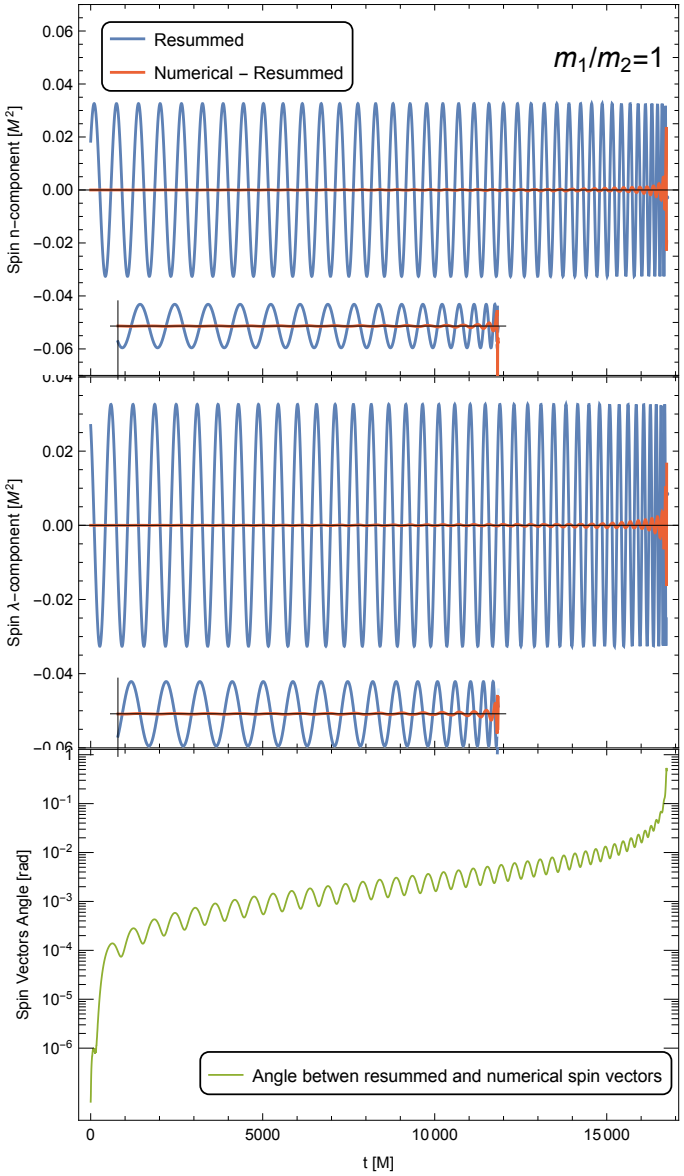
Numerical Comparison: orbital radius $r(t)$

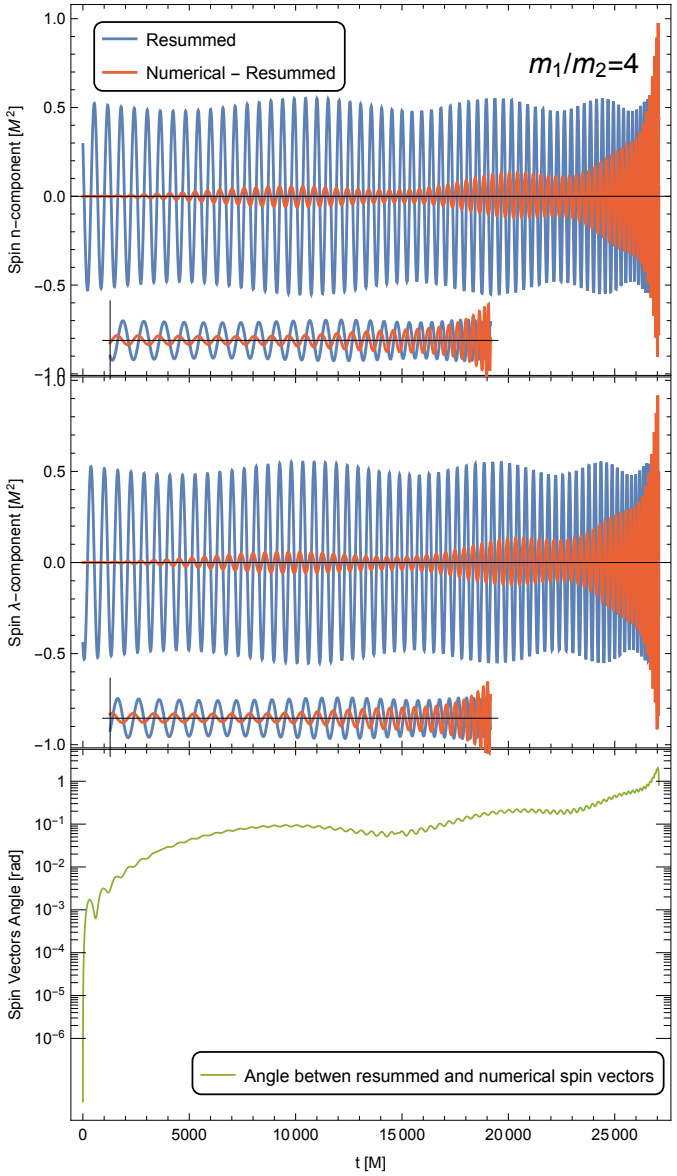


Numerical Comparison: orbital phase $\phi(t)$

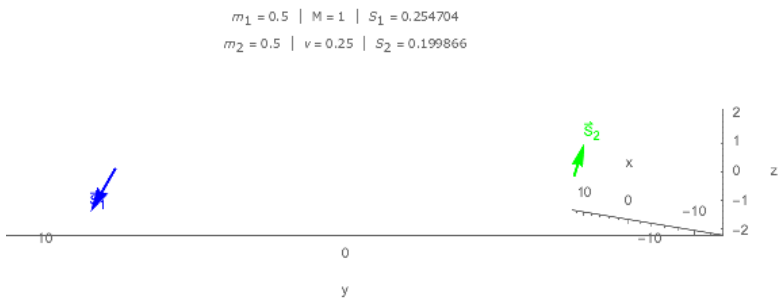


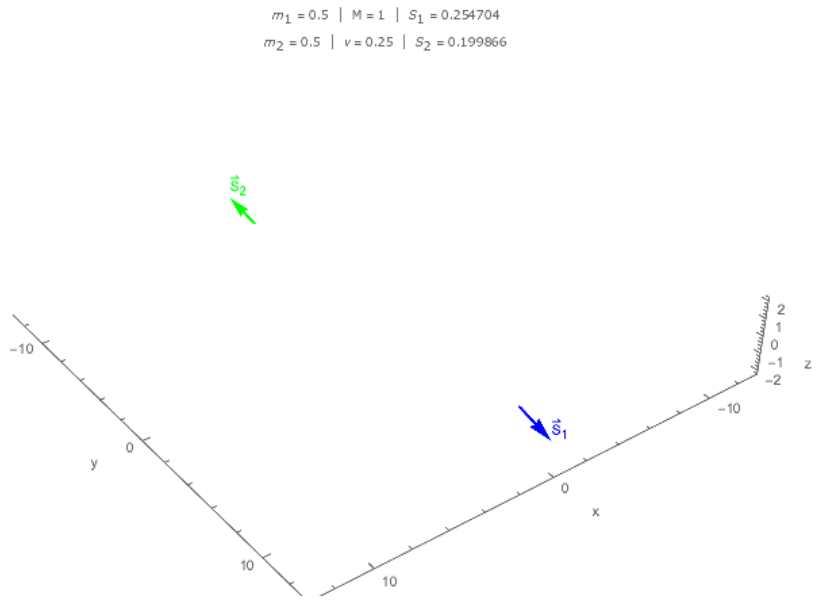
Numerical Comparison: spin vector $\vec{S}(t)$

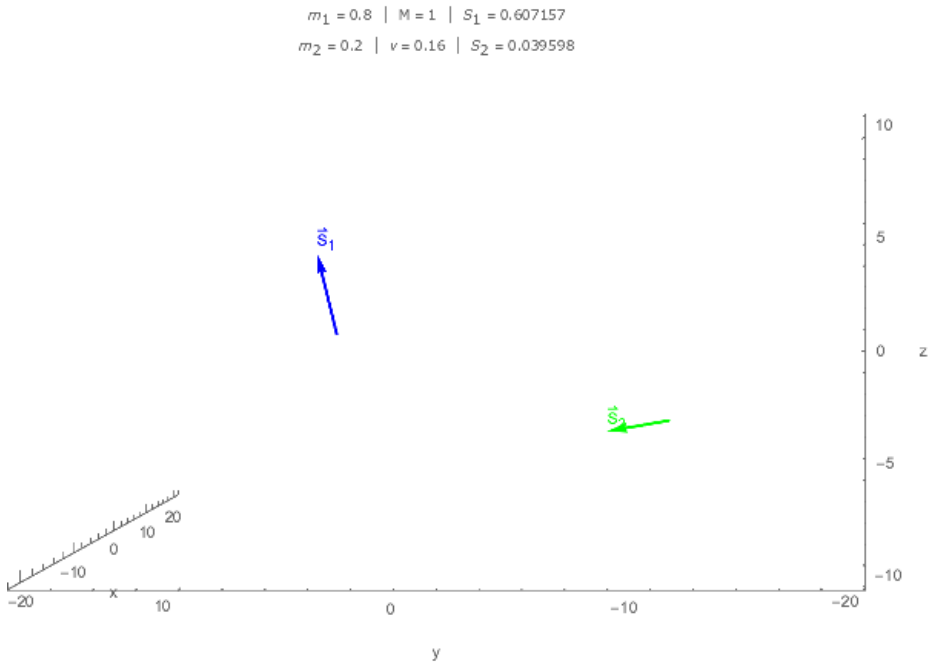




Binary motion and Spin Precession animation







↖ Equal-mass system with anti-aligned spins

← Unequal mass system with mis-aligned spins

Conclusion and Outlooks

- We obtain the analytic solutions to the spinnning binary dynamics at leading spin-orbit order with 2.5PN radiation reaction using the DRG method .

- We are adding the other non-spinning PN order terms in the equations of motion for a more complete solution to the dynamics.

$$\begin{aligned}\boldsymbol{a} =& \boldsymbol{a}_{\mathrm{N}} + \boldsymbol{a}_{\mathrm{1PN}} + \boldsymbol{a}^{\mathrm{SO}}_{\mathrm{cons}} + \boldsymbol{a}_{\mathrm{2PN}} \\ &+ \boldsymbol{a}_{\mathrm{RR}} + \boldsymbol{a}_{\mathrm{RR1PN}} + \boldsymbol{a}^{\mathrm{SO}}_{\mathrm{RR}} + \boldsymbol{a}_{\mathrm{RR2PN}} \qquad \Leftarrow \text{(Natalia's talk)}\end{aligned}$$

- More things to do: Large eccentricity and spin-spin order.