Gravity from BRST squared

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How general is the double copy?

- Can we construct the full space of the gravity theory without resorting to special gauge choices and coordinate choices ?
- We are seeking a dictionary that continues to hold when we perfom gauge/coordinate transformations.
- Can we construct a double copy dictionary in set-ups where we don't have guidance from amplitudes ?

Minimal YM^2

- Schematically:

$$A_\mu * \tilde{A}_\nu \equiv h_{\mu\nu} + B_{\mu\nu} + \eta_{\mu\nu}\phi$$

- Want to construct the full theory with arbitrary boundary conditions of graviton, dilaton and two-form from the double copy.
- If this is achievable, it should be possible to extract (pure) gravitational solutions by requiring $B_{\mu\nu} = \phi = 0$
- Start by constructing Lorenz-covariant dictionaries for all the fields compatible with symmetries and equations of motion.

Local symmetries

- We tensor left and right off-shell linearised gauge fields with arbitrary non-Abelian gauge groups G_L and G_R.
- At linear level, want to reproduce

$$\begin{aligned} \text{graviton:} \quad & \delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \\ \text{two-form:} \quad & \delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \\ \text{dilaton:} \quad & \delta \varphi = 0 \end{aligned}$$

from the (linearised) YM gauge field

$$\delta A^i_\mu = \partial_\mu \alpha^i + f^i{}_{jk} A^j_\mu \theta^k$$

Local symmetries

- We define[Anastasiou, Borsten, Duff, Hughes, SN 2014]:

$$Z_{\mu\nu}(x) = [A_\mu{}^i \star \Phi^{-1}_{ii'} \star \tilde{A}_\nu{}^{i'}](x)$$

where $\Phi_{ii'}$ is the "spectator" bi-adjoint scalar field introduced by [Hodges 2013] and [Cachazo 2014]

- The convolution is defined as

$$[f \star g](x) = \int d^4 y f(y) g(x-y).$$

and is a consequence of the momentum-space origin of squaring: product in momentum space is convolution in position space!

- Importantly, is **doesn't** obey the Leibnitz rule:

$$\partial_\mu (f \star g) = (\partial_\mu f) \star g = f \star (\partial_\mu g)$$

Field dictionary[Cardoso,Inverso,SN,Nampuri'18]

Write down most general dictionary for the product of two YM fields:

$$h_{\mu\nu} = A_\mu \circ \tilde{A}_\nu + A_\nu \circ \tilde{A}_\mu + q\eta_{\mu\nu} \left[A_\rho \circ \tilde{A}^\rho - \frac{1}{\Box}(\partial \cdot A) \circ (\partial \cdot \tilde{A}) \right]$$

$$B_{\mu\nu} = A_\mu \circ \tilde{A}_\nu - A_\nu \circ \tilde{A}_\mu$$

$$\phi = A_\rho \circ \tilde{A}^\rho - \frac{1}{\Box}(\partial \cdot A) \circ (\partial \cdot \tilde{A})$$

Field dictionary [Cardoso,Inverso,SN,Nampuri'18]

- Reproduces the correct local transformations at linear level:

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$
$$\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$$
$$\delta \phi = 0$$

- Allows us to read off parameter dictionaries:

$$\xi_\mu = \alpha \circ \tilde{A}_\mu + A_\mu \circ \tilde{\alpha},$$
$$\Lambda_\mu = \alpha \circ \tilde{A}_\mu - A_\mu \circ \tilde{\alpha},$$

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Problem 1: graviton - dilaton

Couple arbitrary external sources to all equations:

- Yang-Mills

$$\partial^\mu F^i_{\mu\nu} = j^i_\nu, \quad \partial^\mu ({}^*F^i_{\mu\nu}) = 0, \quad \partial^\mu j^i_\mu = 0$$

- gravity

$$R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R = j^{(h)}_{\mu\nu}$$
$$\partial^\rho H_{\rho\mu\nu} = j^{(B)}_{\mu\nu}$$
$$\Box\phi = j^{(\phi)}$$

- spectator

$$\Box\Phi_{ii'} = j^{(\Phi)}_{ii'}$$

Sources have a dual role : they ensure proper fall-off for the fields to be convoluted and they will illuminate a fundamental issue of the double copy.

Problem 1: graviton - dilaton

Making use of the field dictionaries and the eom, we can read off the surce dictionaries:

$$j^{(h)}_{\mu\nu} = -j_{(\mu} \circ \tilde{j}_{\nu)} + (q+1)\left[\eta_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{\Box}\right] j_\rho \circ \tilde{j}^\rho$$

$$j^{(B)}_{\mu\nu} = 2 j_{[\mu} \circ \tilde{j}_{\nu]}$$

$$j^{(\phi)} = j_\rho \circ \tilde{j}^\rho$$

Problem 1: graviton - dilaton

$$j^{(h)}_{\mu\nu} = -j_{(\mu} \circ \tilde{j}_{\nu)} + (q+1) \left[\eta_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{\Box} \right] j_\rho \circ \tilde{j}^\rho$$

$$j^{(B)}_{\mu\nu} = 2 j_{[\mu} \circ \tilde{j}_{\nu]}$$

$$j^{(\phi)} = j_\rho \circ \tilde{j}^\rho$$

Note that out theory is constrained:

$$\boxed{j^\phi \propto -T^{(h)\rho}_{\ \ \rho}}$$

Obstruction to obtaining pure gravity - even for the most general dictionary !!

Related issue: d.o.f. counting

- On-shell, the counting is $2 * 2 = 4$:

$$A^+ \otimes \tilde{A}^+ = g^{++} \qquad A^- \otimes \tilde{A}^- = g^{--}$$
$$A^+ \otimes \tilde{A}^- = \phi \qquad A^- \otimes \tilde{A}^+ = B$$

- Looking at the off-shell counting, $3 * 3 \neq 10$:

$$[A_\mu] = [\tilde{A}_\mu] = 4 - 1 = 3 \qquad (A_\mu \to A_\mu + \partial_\mu \alpha)$$
$$[h_{\mu\nu}] = 10 - 4 = 6 \qquad (h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)})$$
$$[B_{\mu\nu}] = 6 - (4 - 1) = 3 \qquad (B_{\mu\nu} \to B_{\mu\nu} + \partial_{[\mu}\Lambda_{\nu]}, \; \Lambda_\mu \to \Lambda_\mu + \partial_\mu \Lambda)$$
$$[\phi] = 1$$

- Similar issues for double copy of SUSY multiplets.

Problem 2: gauge mapping

- The double-copy is usually formulated with some specific gauge fixing on both the YM and the gravity side.
- There is no general procedure determining a mapping between these corresponding gauge choices.
- This can lead to issues, particularly when studying off-shell or gauge-dependent objects [Plefka,Shi,Steinhoff,Wang'19].

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BRST

- Introduced to avoid issues caused by gauge symmetry in path integrals.
- Schematically

$$S_{BRST} = \int d^D x \Big(\mathcal{L}_0[f] + b\left(G[f] - \tfrac{\xi}{2} b\right) - \bar{c} Q\left(G[f]\right) - f j^{(f)} + \bar{j} c + \bar{c} j \Big) ,$$

where $\mathcal{L}_0[f]$ is the classical action for the field f, $G[f]$ is the gauge-fixing functional and b is the Lautrup-Nakanishi Lagrange multiplier field.

- Note that, *unlike in the standard treatment, we have coupled sources to the ghost and anti-ghost*.

BRST-symmetries

As before, we require that the BRST symmetries of YM system:

$$QA_\mu = \partial_\mu c, \quad Qc = 0, \quad Q\bar{c} = \frac{1}{\xi} G(A)$$

induce the correct symmetries for the gravitational fields:

$$\begin{aligned} Qh_{\mu\nu} &= 2\partial_{(\mu}c_{\nu)}, & Qc_\mu &= 0, & Q\bar{c}_\mu &= \frac{1}{\xi^{(h)}} G_\mu[h,\varphi], \\ QB_{\mu\nu} &= 2\partial_{[\mu}d_{\nu]}, & Qd_\mu &= \partial_\mu d, & Q\bar{d}_\mu &= \frac{1}{\xi^{(B)}} G_\mu[B,\eta], \\ Q\varphi &= 0. \end{aligned}$$

Let us now make a choice of gauge fixing functional on the YM side, and set

$$G[A] \equiv \partial^\mu A_\mu, \quad G[\tilde{A}] \equiv \partial^\mu \tilde{A}_\mu.$$

BRST dictionary[Anastasiou,Borsten,Duff,SN,Zoccali'18]

The most general dictionary in the absence of $\Box^{-1}$ terms and compatible with symmetries is:

$$
\begin{aligned}
h_{\mu\nu} =& A_\mu \circ \tilde{A}_\nu + A_\nu \circ \tilde{A}_\mu + a\eta_{\mu\nu}\left(A^\rho \circ \tilde{A}_\rho + \xi c^\alpha \circ \tilde{c}_\alpha\right),\\
B_{\mu\nu} =& A_\mu \circ \tilde{A}_\nu - A_\nu \circ \tilde{A}_\mu,\\
\varphi =& A^\rho \circ \tilde{A}_\rho + \xi c^\alpha \circ \tilde{c}_\alpha.
\end{aligned}
$$

where we have introduced the OSp(2) ghost singlet

$$c^\alpha \circ \tilde{c}_\alpha = c \circ \tilde{\bar{c}} - \bar{c} \circ \tilde{c}.$$

Source issue - problem 1

- On the YM side, the e.o.m. are

$$\Box A_\mu - \tfrac{\xi+1}{\xi}\partial_\mu(\partial A) = j_\mu$$
$$\Box c = j^{(c)}, \quad \Box\bar{c} = j^{(\bar{c})}$$

- On the gravity side, we have

$$\Box h_{\mu\nu} - \tfrac{\xi_{(h)}+2}{\xi_{(h)}}\left(2\partial^\rho\partial_{(\mu}h_{\nu)\rho} - \partial_\mu\partial_\nu h\right) = j_{\mu\nu}$$
$$\Box\varphi = j^{(\varphi)}$$

Source issue - problem 1

- Making use of the field dictionaries and eom, we now read off source dictionaries (for simplicity, set $a = \xi = -1$, $\xi_{(h)} = -2$):

$$
\begin{aligned}
j^{(h)}_{\mu\nu} =& 2\frac{1}{\Box} j_{(\mu} \circ \tilde{j}_{\nu)} - \eta_{\mu\nu} j^{\alpha} \circ \tilde{j}_{\alpha} \\
j^{(\varphi)} =& \frac{1}{\Box} j^{\rho} \circ \tilde{j}_{\rho} - \frac{1}{\Box} j^{\alpha} \circ \tilde{j}_{\alpha}
\end{aligned}
$$

with

$$
j^{\alpha} \circ \tilde{j}_{\alpha} = j^{(c)} \circ \tilde{j}^{(\bar{c})} - j^{(\bar{c})} \circ \tilde{j}^{(c)}
$$

- Note that we can now set $j^{(\varphi)} = 0$ without affecting $j^{(h)}_{\mu\nu}$.
- Reminiscent of removal of unwanted dilaton in [Luna,Nicholson,O'Connell,White'17,Johansson,Ochirov'15]

Off-shell d.o.f. counting

	$\tilde{A}_\mu\ 4^{(0)}$	$\tilde{c}\ 1^{(1)}$	$\tilde{\bar{c}}\ 1^{(-1)}$
$A_\mu\ 4^{(0)}$	$16^{(0)}$	$4^{(1)}$	$4^{(-1)}$
$c\ 1^{(1)}$	$4^{(1)}$	$1^{(2)}$	$1^{(0)}$
$\bar{c}\ 1^{(-1)}$	$4^{(-1)}$	$1^{(0)}$	$1^{(-2)}$

Table: Degree of freedom counting, graded by ghost number

Gauge mapping

- The BRST gives an algorithm for mapping YM gauge choice to gravity gauge choice. Illustrate with simple, covariant example.
- Choose YM gauge fixing functional

$$G[A] \equiv \partial^\mu A_\mu, \quad G[\tilde{A}] \equiv \partial^\mu \tilde{A}_\mu$$

- From the graviton dictionary

$$h_{\mu\nu} = A_\mu \circ \tilde{A}_\nu + A_\nu \circ \tilde{A}_\mu + a\eta_{\mu\nu} \left(A^\rho \circ \tilde{A}_\rho + \xi c^\alpha \circ \tilde{c}_\alpha \right),$$

we read off the gravitational ghost dictionary

$$c_\mu = c \circ \tilde{A}_\mu + A_\mu \circ \tilde{c},$$

Gauge mapping

- Conjugation immediately gives us the anti-ghost

$$\bar{c}_\mu = \bar{c} \circ \tilde{A}_\mu + A_\mu \circ \tilde{\bar{c}},$$

- We know that the BRST transformation of this should be

$$Q\bar{c}_\mu = \frac{1}{\xi} G_\mu[h, \varphi],$$

BUT we can compute $Q\bar{c}_\mu$ directly, using the the YM transformation rules:

$$Q\bar{c}_\mu = \frac{1}{\xi}\left[\partial^\rho A_\rho \circ \tilde{A}_\mu + A_\mu \circ \partial^\rho \tilde{A}_\rho\right] + \partial_\mu c^\alpha \circ \tilde{c}_\alpha$$

Then, inverting the graviton and dilaton dictionaries, we read off

$$G_\mu[h, \varphi] = \partial^\nu h_{\nu\mu} - \frac{1}{2}\partial_\mu h + \left(1 + \frac{D-2}{2}a\right)\partial_\mu \varphi$$

Direct gravity Lagrangian construction

- Start with BRST YM Lagrangian:

$$\mathcal{L} = \delta_{ab}\frac{1}{2}\Big[A^{\mu a}\Box A^{b}_{\mu} + \frac{\xi+1}{\xi}\partial A^{a}\partial A^{b} - \varepsilon^{\alpha\beta}c^{a}_{\alpha}\Box c^{b}_{\beta} - A^{\mu a}j^{b}_{\mu} + \varepsilon^{\alpha\beta}c^{a}_{\alpha}\Box j^{b}_{\beta}\Big]$$

- Double replacement rule to account for space-time indices and OSP(2) indices labelling the ghosts:

$$A^{\mu a} \to \begin{cases} A^{\mu\nu} = A^{\mu}\circ\tilde{A}^{\nu} \\ A^{\mu\alpha} = A^{\mu}\circ c^{\alpha} \end{cases}, \quad c^{\alpha a} \to \begin{cases} c^{\alpha\nu} = c^{\alpha}\circ\tilde{A}^{\nu} \\ c^{\alpha\beta} = c^{\alpha}\circ c^{\beta} \end{cases}$$

- Get BRST gravity Lagrangian, with gauge fixing functional as determined by the symmetries.

Next order

- Cannot use the convolution to directly construct the fields.
- Construct the action at higher orders in perturbation theory by employing a result connecting perturbative expansions of classical fields to amplitudes of increasing order [Boulware,Brown'68,Luna,Monteiro,Nicholson,Ochirov, O'Connell,Westerberg,White'17].
- Apply BCJ replacement rules to get to gravity action.

Next order

Expand

$$A_\mu = \sum_i g^{i-1} A_\mu^{(i)}, \quad c_\alpha = \sum_i g^{i-1} c_\alpha^{(i)}.$$

Pick $\xi = -1$ for simplicity, then we can rearrange the action at order g^1 as:

$$\mathcal{L} = \delta_{ab}\frac{1}{2}\Big[A^{\mu a(1)}\Box A_\mu^{b(2)} - \varepsilon^{\alpha\beta} c_\alpha^{a(1)}\Box c_\beta^{b(2)}\Big] + if^a{}_{bc}\int \bar{d}p\bar{d}k\bar{d}q\hat{\delta}\Big[n^{\mu\nu\sigma}A_\mu^{(1)a}(p)A_\nu^{(1)b}(k)A_\sigma^{(1)c}(q) + n^\mu A_\mu^{(1)a}(p)c^{(1)\alpha b}(k)c^{(1)}{}_\alpha{}^c(q)\Big]$$

Next order

- Use BCJ replacement rules to send colour to kinematics.
- Replacement rules for $A_\mu^{a(1)}$ and $c^{a(1)}$ as derived at linear order.
- Replacement rule for $A_\mu^{a(2)}$ and $c^{a(2)}$ take into account terms in the action which fail to display LR factorisation, e.g.

$$A^{(2)}_{\mu\nu} \propto h_\mu{}^{\alpha(1)} h^{(1)}_{\alpha\nu}$$

  similar to field redefinitions in [Bern,Grant'99].
- Finally, read off second order gauge fixing functional for gravity by comparison with stadard action; note that it appears in the BRST action in the form $G_\mu[h]^{(1)} G^{\mu(2)}[h]$.

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Double-copy in a non-flat background

- Can we construct a double copy dictionary in set-ups where we don't have guidance from amplitudes ?
- Further develop the maths needed for a general double copy.
- Applications of double copy to realistic cosmological backgrounds.

Double copy on S^2

- Remember in flat space the convolution was crucial for writing a dictionary compatible with symmetries:

$$[f \star g](x) = \int d^4y f(y) g(x-y).$$

Double copy on S^2

- Remember in flat space the convolution was crucial for writing a dictionary compatible with symmetries:

$$[f \star g](x) = \int d^4y f(0 + y) g(x - y).$$

- Notice we are integrating over the goup of translations, which act transitively on flat space.
- Another crucial property is factorisation when we Fourier transform to the dual space (in this case momentum space):

$$\mathcal{F}[f \star g](p) = \mathcal{F}[f](p) \cdot \mathcal{F}[g](p)$$

Double copy on S^2

- On S^2, one can define, for scalars [Driscoll,Healy'94]:

$$k \star f(\theta,\phi) = \left(\int_{g\in \mathrm{SO}(3)} dg \;\; k(g\eta)\Lambda(g) \right) f(\theta,\phi)$$
$$= \int_{g\in \mathrm{SO}(3)} dg \;\; k(g\eta) f\left(g^{-1}(\theta,\phi)\right) dg$$

  using the fact that $SO(3)$ acts transitively on the sphere.

- The convolution factorises in the dual space

$$(k \star f)_l^m = 2\pi \sqrt{\tfrac{4\pi}{2l+1}} \; k_l^m \cdot f_l^0$$

  and satisfies the derivative rule

$$\partial_\mu(f \star g) = (\partial_\mu f) \star g = f \star (\partial_\mu g)$$

Double copy on S^2

- We would now like to extend the spherical convolution to tensor fields.
- We find it useful to first recast tensors on S^2 as scalars on SO(3), as described in [Gelfand,Minlos,Shapiro'63,Burridge'69]. Let $M_{\alpha_1\ldots\alpha_p}(\mathbf{x})$ be a tensor field in $\mathbb{R}^3$ and $g \in SO(3)$. We define

$$\mathcal{M}_{\alpha_1\ldots\alpha_p}(r,g) = g_{\alpha_1\beta_1}\ldots g_{\alpha_p\beta_p} M_{\beta_1\ldots\beta_p}\left(rg^{-1}\mathbf{e_3}\right)$$

- We also find it useful to perform a change of basis

$$\mathcal{M}_{a_1\ldots a_p}(r,g) = C_{a_1\alpha_1}\ldots C_{a_p\alpha_p}\mathcal{M}^{\alpha_1\ldots\alpha_p}(r,g),$$

where

$$C_{\alpha a} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}.$$

This can be thought of as going to a helicity basis; here α runs over $-1, 0, +1$.

Double copy on S^2

- Finally, we restrict to S^2 by setting

$$r = R_0$$

witha R_0 a constant, and setting all tensor components in the $\hat{\mathbf{r}}$ direction to vanish.

- We can now expand into generalised spherical harmonics

$$\mathcal{M}_{a_1 \ldots a_p}(g) = \sum_{l \geq A, |m| \leq l} \left(\mathcal{M}_{a_1 \ldots a_p}\right)_l^m T^l_{Am}(g)$$

with $A = a_1 + \ldots + a_p$, and T^l_{ab} is related to the usual Wigner matrices through:

$$T^l_{ab}(g) = (-1)^{b-a} \bar{D}^l_{ba}(g)$$

Double copy on S^2

- Finally, we introduce the tensor convolution [Borsten,Jubb,Makwana,SN'19]:

$$k_{a_1\dots a_m} \star f_{b_1\dots b_n}(\omega) = \left(\int dg\ k_{a_1\dots a_m}(g)\Lambda^A(g)\right) f_{b_1\dots b_n}(\omega)$$
$$= \int dg\ k_{a_1\dots a_m}(g)[X^A f]_{b_1\dots b_n}\left(g^{-1}\omega\right).$$

Here, $\omega \in \mathrm{SO}(3)$ and $\Lambda^A(g)$ is an operator induced by the action of SO(3) on the sphere, and weighted by $A = a_1 + \cdots + a_m$.

Double copy on S^2

The operator X^A is defined through its action on the generalised spherical harmonics, $[X^A f]_{b_1 \ldots b_n} = (f_{b_1 \ldots b_n})_l^m [X^A T]^l_{Bm}(\omega)$, where

$$[X^A T]^l_{Bm}(\omega) := \Omega^l_{(A,B)} \; T^l_{B+A,m+A}(\omega)$$

Here the prefactor

$$\Omega^l_{(A,B)} = \frac{\Omega^l_{-AB}}{\Omega^l_0},$$

with

$$\Omega^l_N = \sqrt{\frac{(l+N)(l-N+1)}{2}},$$

is introduced for convenience, as it will allow for the correct matching of symmetries between YM and the gravity theory

Double copy on S^2

- The tensor convolution factorises in dual space as required:

$$\left(k_{a_1\ldots a_m} \star f_{b_1\ldots b_n}\right)_l^m = \frac{8\pi^2}{2l+1}\Omega^l_{(A,B)}\left(k_{a_1\ldots a_m}\right)_l^m \cdot \left(f_{b_1\ldots b_n}\right)_l^0$$

- It also satisfies the necessary derivative rule

$$V_a \circ \partial_b s = \nabla_b\left(V_a \circ s\right) = \left(\nabla_b V_a\right) \circ s$$

thus allowing us to reproduce the diffeos

$$Qh_{ab} = 2\nabla_{(a}c_{b)}$$

from gauge transformations of the factors.

Double copy on S^2 and Einstein-static universe

- Dictionary is now defined similarly to flat space, and we similarly work out symmetries, gauge mapping in the BRST formalism and source dictionaries.
- Have also extended to an Einstein-static universe by combining the spherical convolution with a flat standard convolution over the time dimension.
- Extension to other background manifolds; higher orders ?

Thank You !