g-factor of free and bound leptons

SchwingerFest2018: g-2

Mani L. Bhaumik Institute for Theoretical Physics

Andrzej Czarnecki 🍁 University of Alberta

December 3, 2018

Outline

Electroweak corrections to the anomalous magnetic moments

Another anniversary (95th) related to g-2 in QED

g-factor of a bound electron: an alternative route to New Physics
- new two (three) loop corrections

25 years ago: SLAC Summer Institute, 26 July - 3 August 1993

SPIN AND PRECISION ELECTROWEAK PHYSICS*

William J. Marciano
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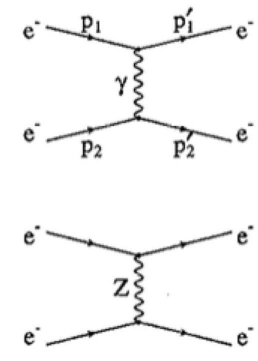
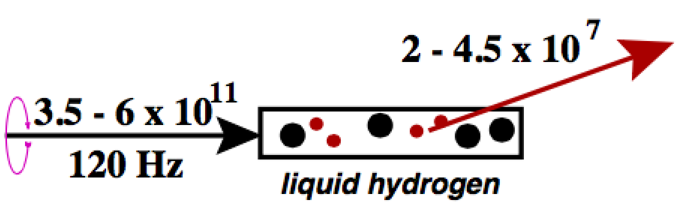
Brookhaven National Laboratory
Upton, New York 11973

ABSTRACT

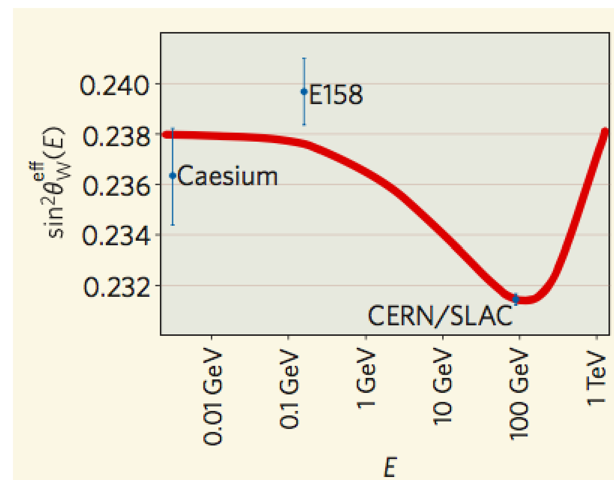
A perspective on fundamental parameters and precision tests of the Standard Model is given. Weak neutral current reactions are discussed with emphasis on those processes involving (polarized) electrons. The role of electroweak radiative corrections in determining the top quark mass and probing for "new physics" is described.



(Received 26 July 1995)



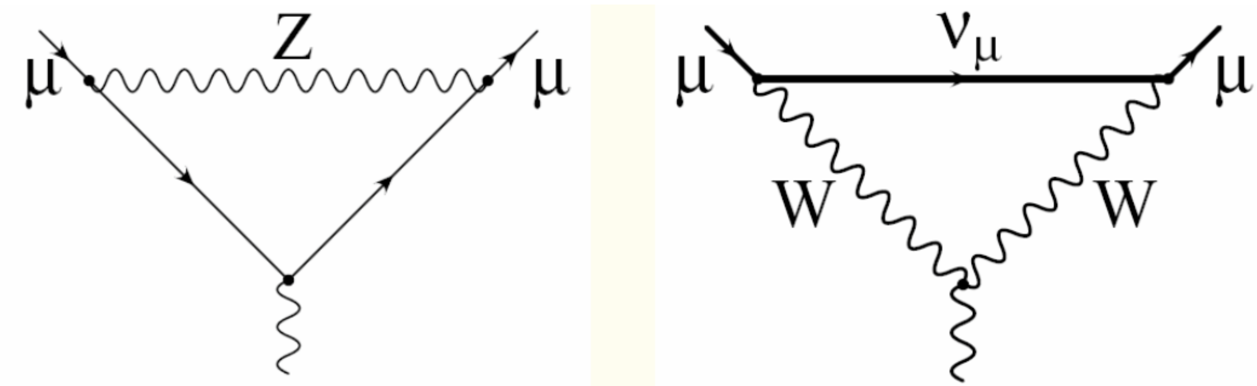
E158 @ SLAC, 2005



NATURE|Vol 435|26 May 2005

Electroweak corrections to g-2 of the muon

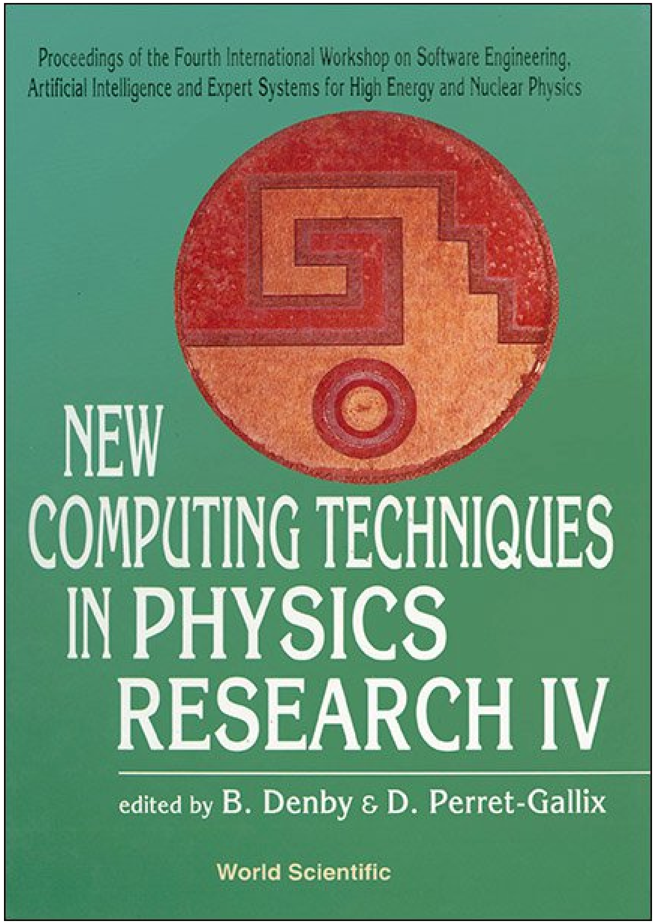
The original motivation for the Brookhaven experiment;
one-loop: contribute 195 / 10^{11} to the total g-2



$$(-1 \quad +2) \cdot \frac{5G_\mu m_\mu^2}{24\sqrt{2}\pi^2} \simeq 195\cdot 10^{-11}$$

How about two-loop electroweak corrections?
Then (1995) none had been calculated.
g-2 was the right observable to start with (no tree-level effect)

1995 Pisa Conference



Automatic Calculation of 2-loop Weak Corrections to Muon Anomalous Magnetic Moment *

Toshiaki KANEKO †
Faculty of General Education, Meijigakuin University
Totsuka, Yokohama 244 , JAPAN

and

Nobuya NAKAZAWA
Department of Physics, Kogakuin University
Nishi-Shinjuku 1-24, Shinjuku, Tokyo 160, Japan

Received (April 6, 1995)

arXiv:hep-ph/9505278

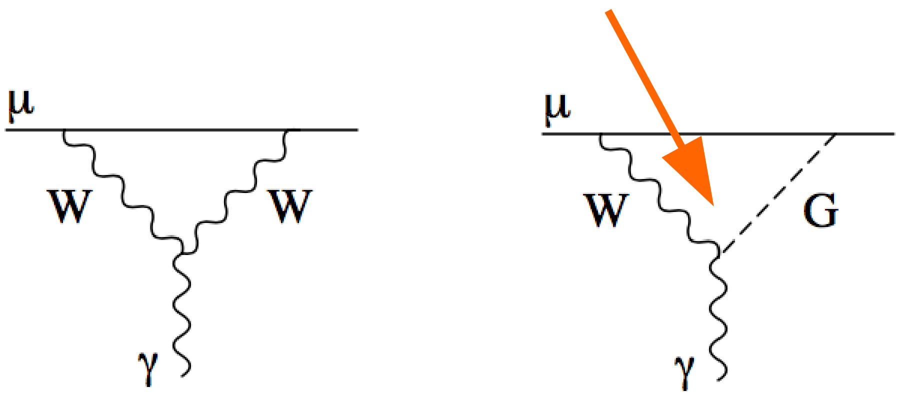
1678 diagrams

Also at Pisa 1995:
Ettore Remiddi reported his result with Stefano Laporta: analytical 3-loop QED g-2.

Our approach:

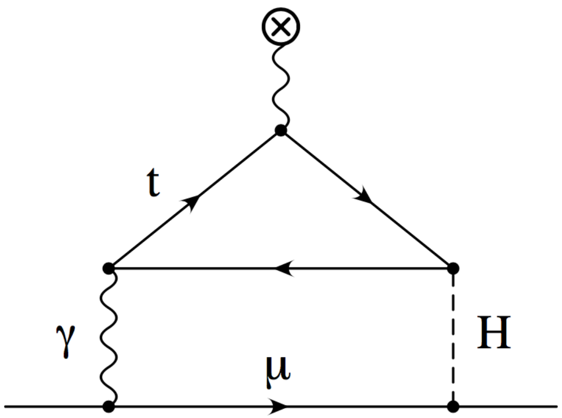
Neglect 92 per cent of diagrams
- drop all multiple scalar couplings (factor of ~6 decrease)
- adopt a special gauge: non-linear (factor of ~2)

Can eliminate WGγ vertex:



- evaluate the remaining ~140 diagrams analytically
  (asymptotic expansions,
  Karlsruhe QCD expertise: Chetyrkin, Steinhauser)

Asymptotic expansions: an example



There are three mass scales in this amplitude: top, Higgs, muon.

We are interested in very small Higgs masses (possibly even below the muon).

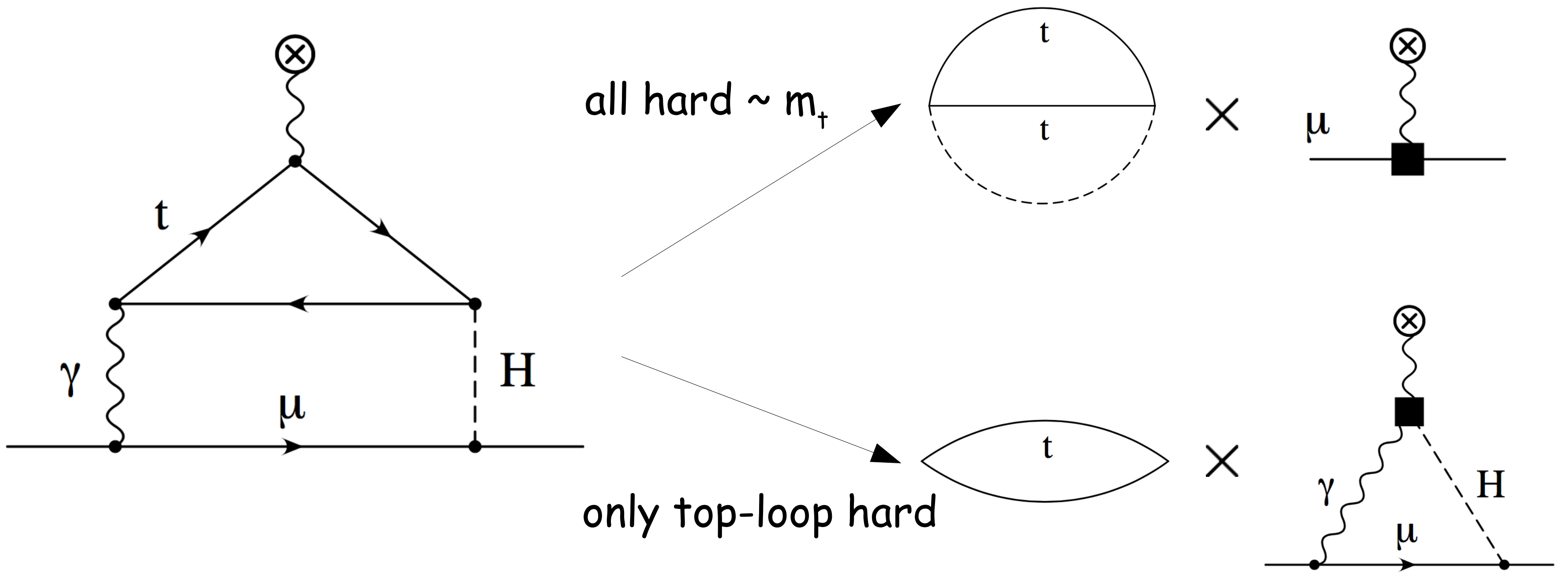
(Work in progress with W. J. Marciano)

$$\Delta g = \sum_n \left(\frac{m_H^2}{m_t^2}\right)^n \left[f_n + g_n \cdot \ln \frac{m_H^2}{m_t^2} \right]$$

functions of $\dfrac{m_H^2}{m_\mu^2}$

How are the coefficient functions computed?

Consider various scaling regimes of loop momenta; “regions”:



One scale eliminated

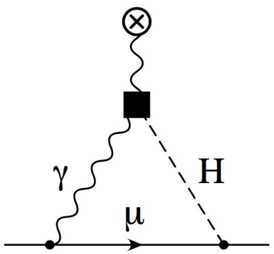
The remaining two scales are separated similarly. Consider two cases,

$$z_h \ll 1 \quad \text{and} \quad z_h \gg 1, \qquad z_h \equiv \frac{m_H}{m_\mu}$$

Results for the coefficient function f_0 (leading order in inverse top mass expansion)

$z_h \ll 1$ (Higgs lighter than muon)

$$-\frac{176}{45}-\frac{16}{15}\ln\frac{m_t^2}{m_\mu^2}+z_h\left(\frac{64}{45}\pi\right)+z_h^2\left(-\frac{16}{15}+\frac{32}{15}\ln z_h\right)+\ldots+z_h^{26}\left(-\frac{1}{8365982625}\right)$$



$z_h \gg 1$ (Higgs heavier than muon)

$$-\frac{104}{45}-\frac{16}{15}\ln\frac{m_t^2}{m_H^2}+z_h^{-2}\left(\frac{64}{45}\ln z_h-\frac{16}{135}\right)+z_h^{-4}\left(\frac{32}{15}\ln z_h-\frac{28}{45}\right)$$

$$+\ldots+z_h^{-36}\left(101896256\ln z_h-\frac{181409946447178}{2834325}\right)$$

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Odd powers present: non-analyticity

The same coefficient of the top-mass log

$z_h \gg 1$ (Higgs heavier than muon)

$$-\frac{104}{45}-\frac{16}{15}\ln\frac{m_t^2}{m_H^2}+z_h^{-2}\left(\frac{64}{45}\ln z_h-\frac{16}{135}\right)+z_h^{-4}\left(\frac{32}{15}\ln z_h-\frac{28}{45}\right)$$

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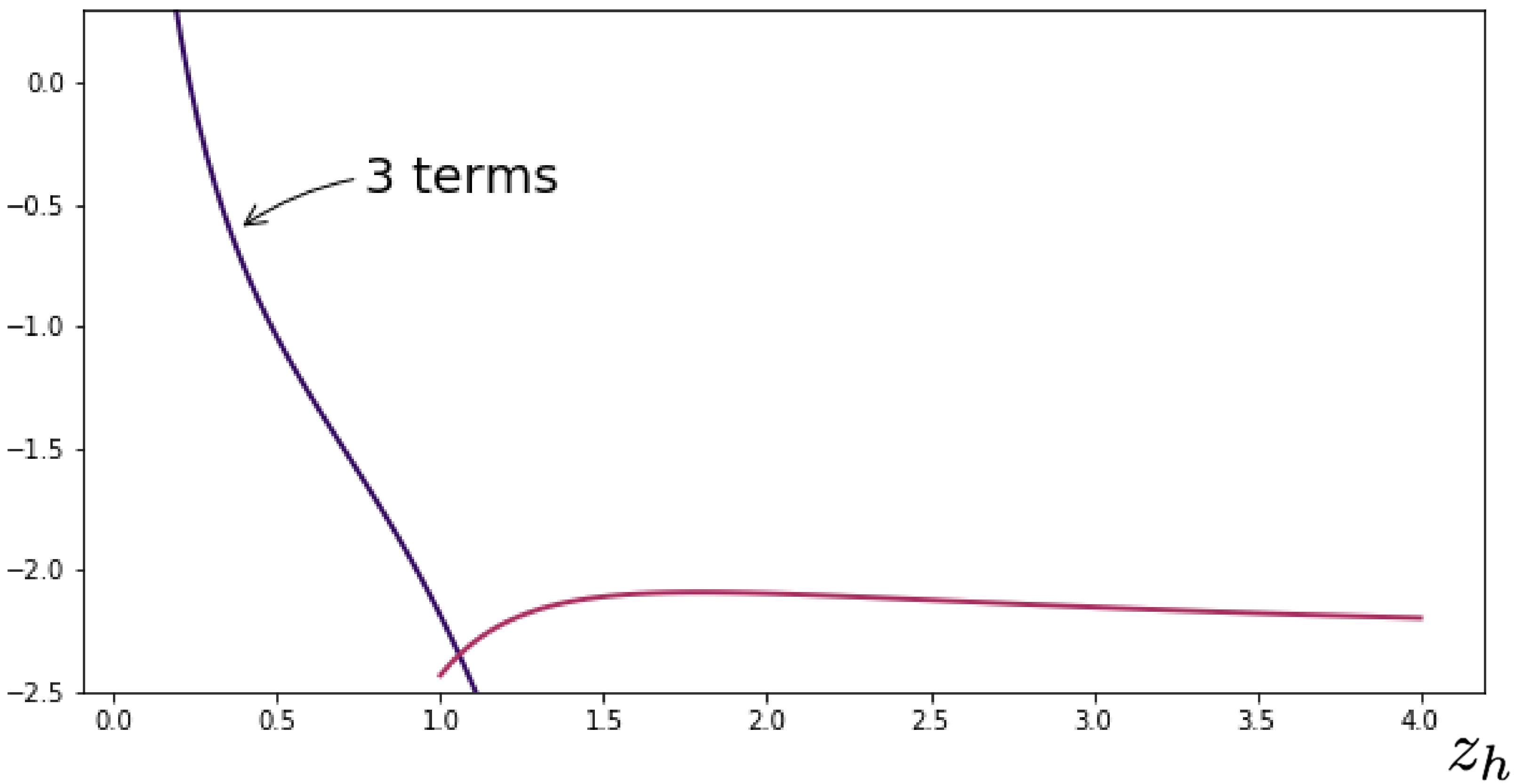
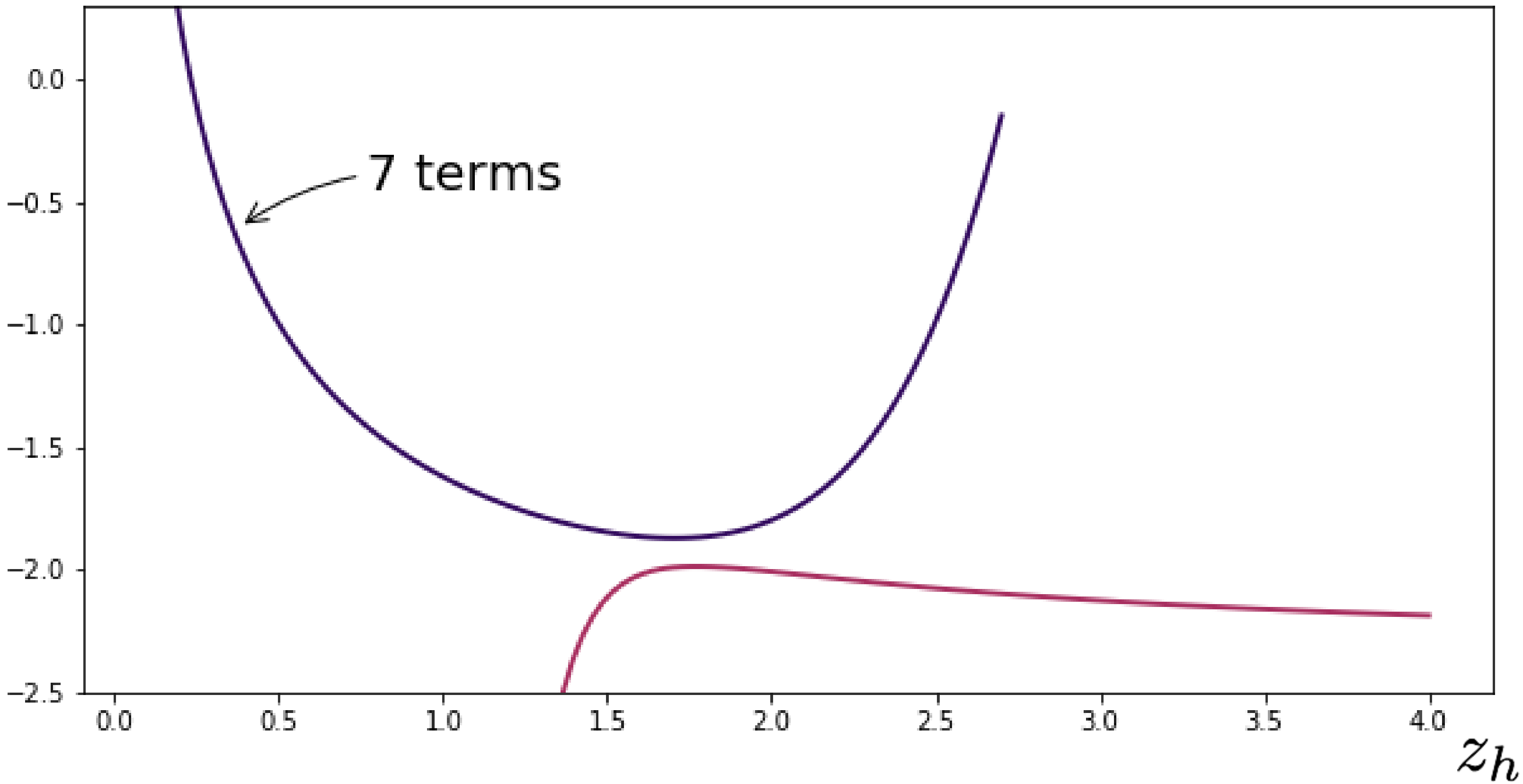
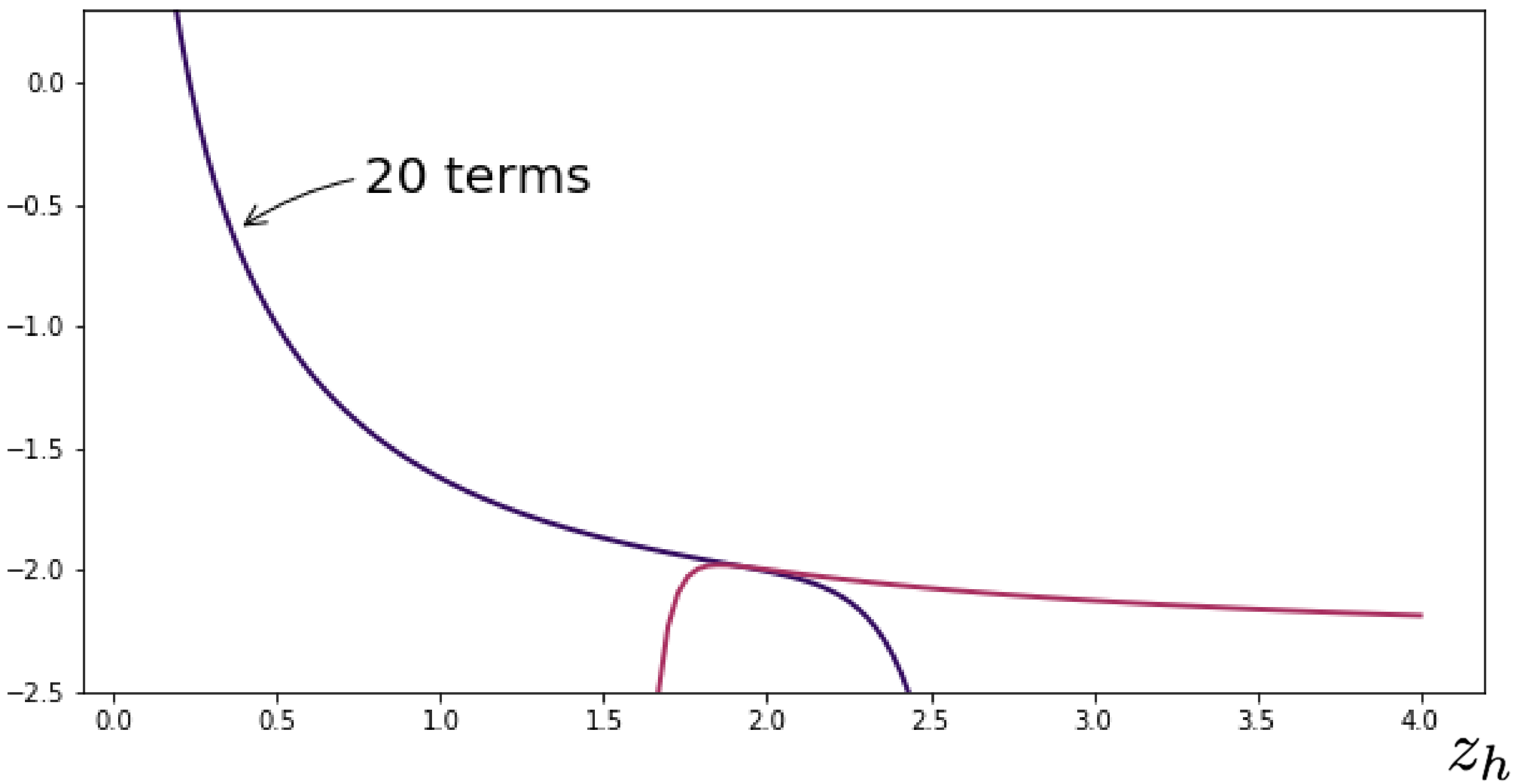
Behavior of the expansions



Behavior of the expansions



Behavior of the expansions



With so much data: the exact coefficients can be guessed

For example, odd powers:

$$a_{2n+1} = \frac{\pi}{15}\frac{(2n-1)!!}{8^n(n+2)!}$$

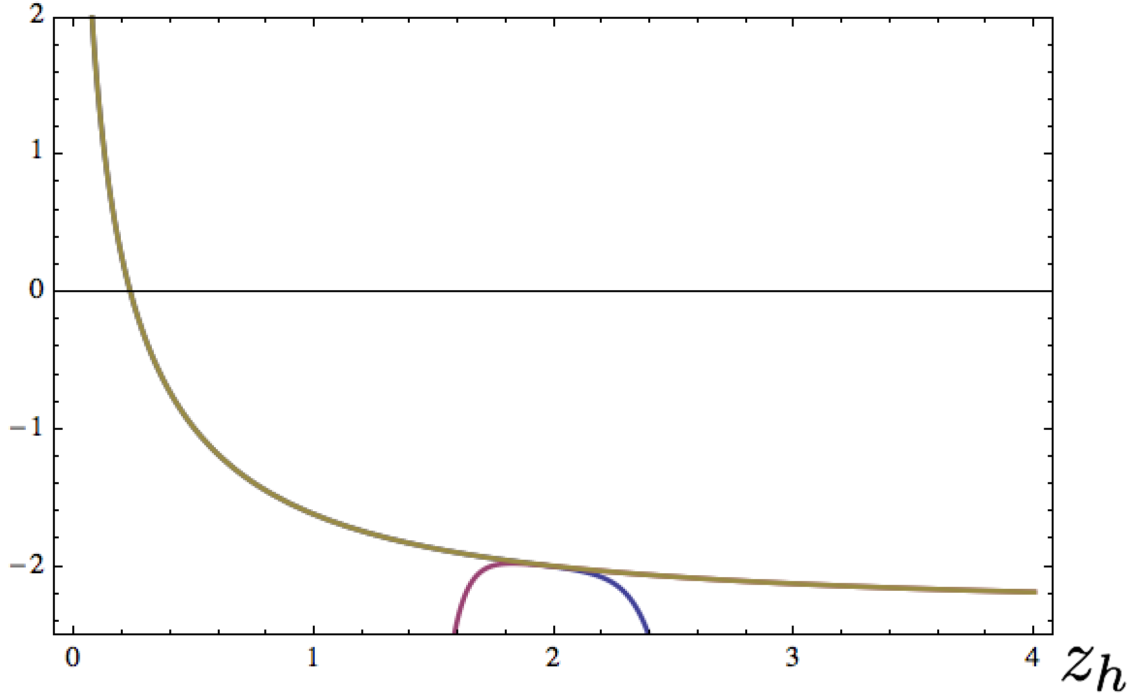
What is the convergence radius? (d'Alembert criterion)

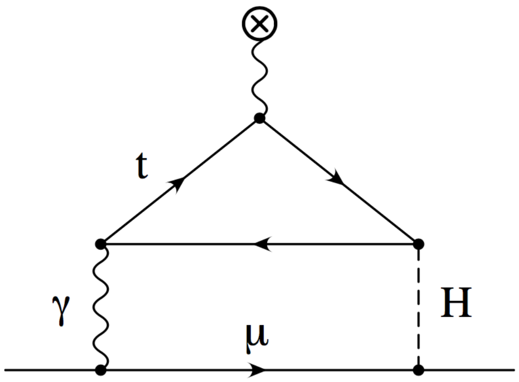
$$\frac{a_{2n+1}z_h^{2n+1}}{a_{2n-1}z_h^{2n-1}} \sim \frac{(2n-1)!!}{(2n-3)!!}\frac{8^{n-1}}{8^n}\frac{(n+1)!}{(n+2)!}z_h^2 \sim \frac{z_h^2}{4}$$

Convergent for $z_h < 2$

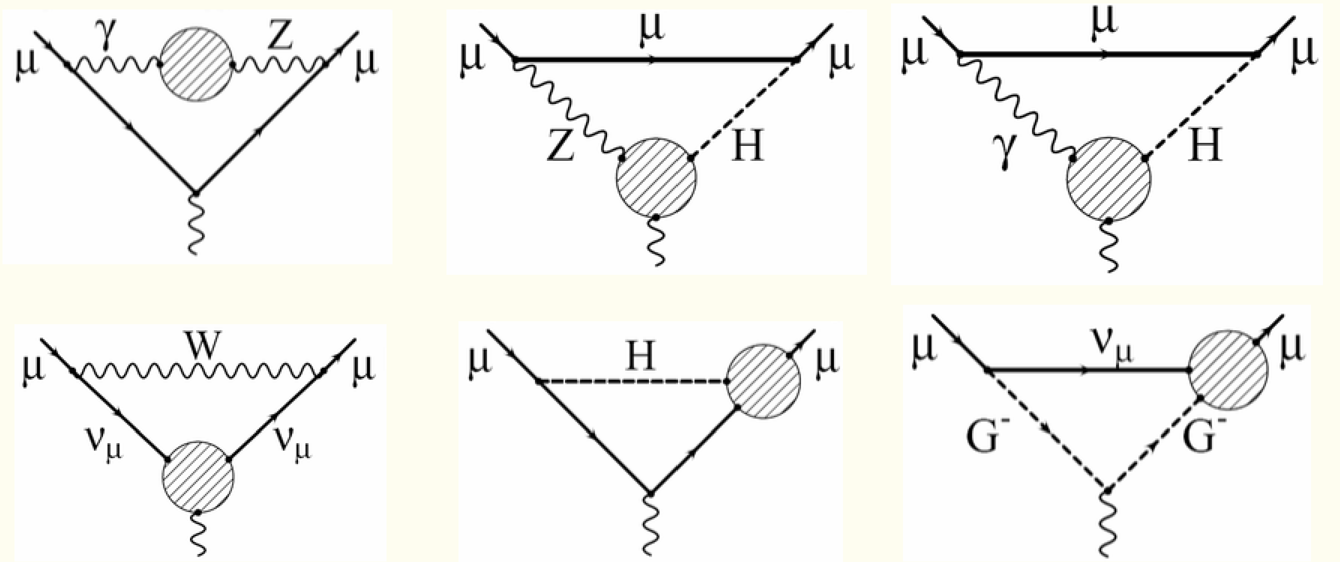
Now sum the series to all orders

$$\frac{16}{45}\left[-11-3\ln\frac{m_t^2}{m_\mu^2}+z_h^2\left(1+6\ln z_h\right)-z_h^4\ln z_h-z_h\left(z_h^2-4\right)\begin{cases}\sqrt{4-z_h^2}\,\arccos\frac{z_h}{2} & z_h<2\\ -\sqrt{z_h^2-4}\,\ln\left(\sqrt{\frac{z_h^2}{4}-1}+\frac{z_h}{2}\right) & z_h>2\end{cases}\right]$$





All two-loop electroweak effects



Most important: photonic corrections $\rightarrow$ large logs

$$\frac{\alpha}{\pi} G_\mu m_\mu^2 \ln \frac{M_W^2}{m_\mu^2} \sim -23\% \text{ of one-loop}$$

Kukhto et al.
AC, Krause, Marciano
Heinemaier, Stockinger, Weiglein
AC, Marciano, Vainshtein

Total two-loop correction: $\boxed{41(1) \cdot 10^{-11}}$

$$195 \rightarrow 154(1) \cdot 10^{-11}$$

Recent numerical calculation

Numerical calculation of the full two-loop electroweak corrections to muon (g-2)

Tadashi Ishikawa[*]
High Energy Accelerator Organization(KEK), 1-1 OHO Tsukuba Ibaraki 305-0801,Japan

Nobuya Nakazawa[†]
Department of Physics, Kogakuin University, Shinjuku,Tokyo 163-8677,Japan

Yoshiaki Yasui[‡]
Department of Management, Tokyo Management College,Ichikawa, Chiba 272-0001, Japan
(Dated: November 1, 2018)

1780 diagrams; $\Delta(g\text{-}2)_{Weak-2loop} = -38.6(\pm 0.1) \times 10^{-11}$

$\sim 6\%$ smaller than the widely accepted value

Difference seems to come mainly from fermion loops.
Unclear treatment of Higgs couplings to fermions.
(Constituent quark masses?)

Another anniversary (yesterday, 95th)

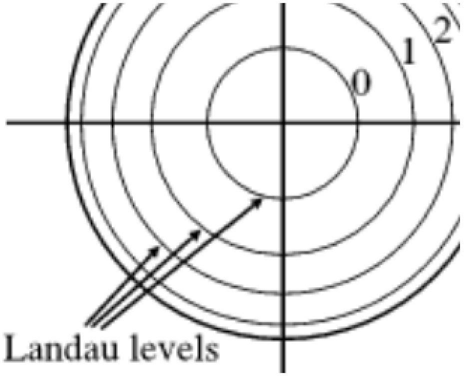
Joaquin Mazdak Luttinger

Born 2 December 1923
New York City, United States

PHYSICAL REVIEW VOLUME 74, NUMBER 8

A Note on the Magnetic Moment of the Electron

J. M. Luttinger*
Swiss Federal Institute of Technology, Zurich, Switzerland
(Received May 19, 1948)

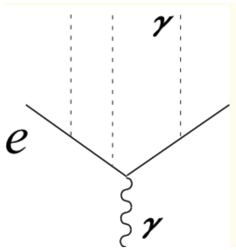


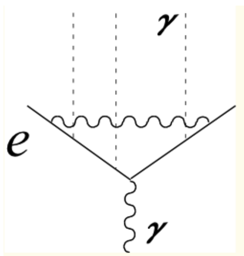
$$g=2(1+(1/2\pi)(e^2/\hbar c)). \qquad (12)$$

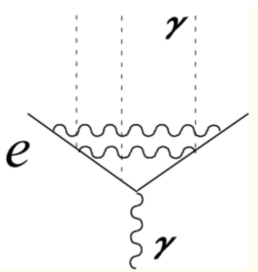
This corresponds exactly to the result of Schwinger.

Bound-electron *g*-factor

Bound-electron g-2: binding and loops







$$g = 2 - \frac{2\,(Z\alpha)^2}{3} - \frac{(Z\alpha)^4}{6} + \ldots$$

Breit 1928

$$+\frac{\alpha}{\pi}\left[1 + \frac{(Z\alpha)^2}{6} + (Z\alpha)^4\,(a_{41}\ln Z\alpha + a_{40}) + \ldots\right]$$

Pachucki, Jentschura, Yerokhin (2004)

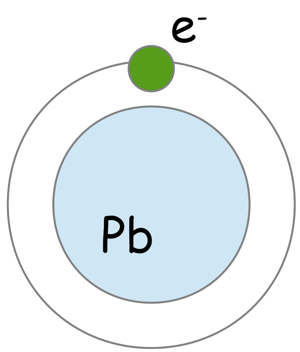
$$+\underbrace{\left(\frac{\alpha}{\pi}\right)^2\left[-0.65..\left(1 + \frac{(Z\alpha)^2}{6}\right) + (Z\alpha)^4\,(b_{41}\ln Z\alpha + b_{40}) + \ldots\right]}_{\text{two-loop corrections}}$$

Pachucki, AC Jentschura, Yerokhin (2005)

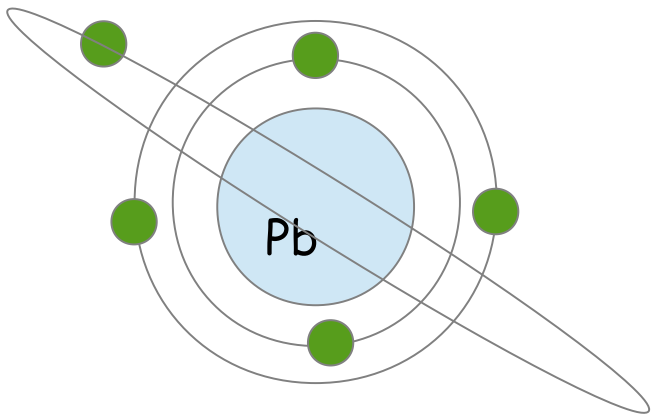
A new source of alpha: highly-charged ions

$$g \simeq 2 - \frac{2\,(Z\alpha)^2}{3} \longrightarrow \frac{\delta\alpha}{\alpha} \sim \frac{1}{(\alpha Z)^2}\sqrt{(\delta g_{\rm exp})^2 + (\delta g_{\rm th})^2}$$

large Z favorable



Hydrogen-like lead



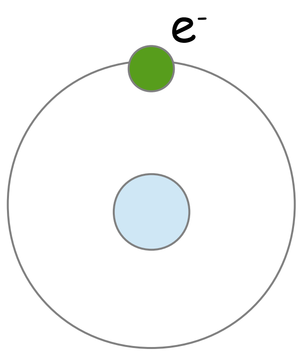
Boron-like lead

There is a combination of *g*-factors in both ions where the sensitivity to the nuclear structure largely cancels, but the sensitivity to alpha remains.

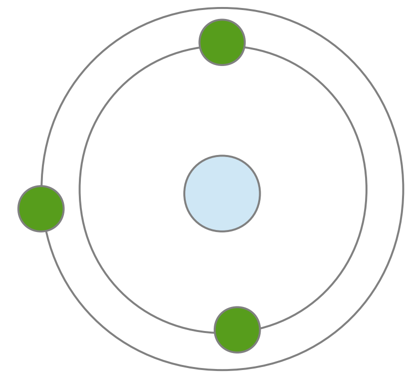
Shabaev, Glazov, Oreshkina, Volotka, Plunien, Kluge, Quint

New idea: medium-charged ions

$$g \simeq 2 - \frac{2\,(Z\alpha)^2}{3}$$



Hydrogen-like ion



Lithium-like ion

Combine H-like and Li-like to remove nuclear dependence;
then combine with a different nucleus, to remove free-g dependence!

Much interesting theoretical work remains to be done.

Yerokhin, Berseneva, Harman, Tupitsyn, Keitel: PRL (2016)

One- and two-loop binding corrections at (Zα)^5

One-loop: Pachucki+Puchalski, PRA96 (2017) 032503
conceptual breakthrough

$$\Delta g \sim \alpha\,(Z\alpha)^5$$

$$\Delta g \sim \alpha^2\,(Z\alpha)^5$$

Sources of $\alpha^2 (Z\alpha)^5$ effects:
- additional short-distance potential generated by Lamb-shift diagrams
- energy-dependence of the Lamb shift (and B-field change of energy)
- modification of the electron response to the B-field
- modification of the B-field by the Coulomb field

PHYSICAL REVIEW LETTERS **120,** 043203 (2018)

Two-Loop Binding Corrections to the Electron Gyromagnetic Factor

Andrzej Czarnecki,[1] Matthew Dowling,[1] Jan Piclum,[1,2] and Robert Szafron[1,3]

Summary

* Electroweak corrections to the muon *g*-2: 2018 complete numerical calculation confirms previous approximations

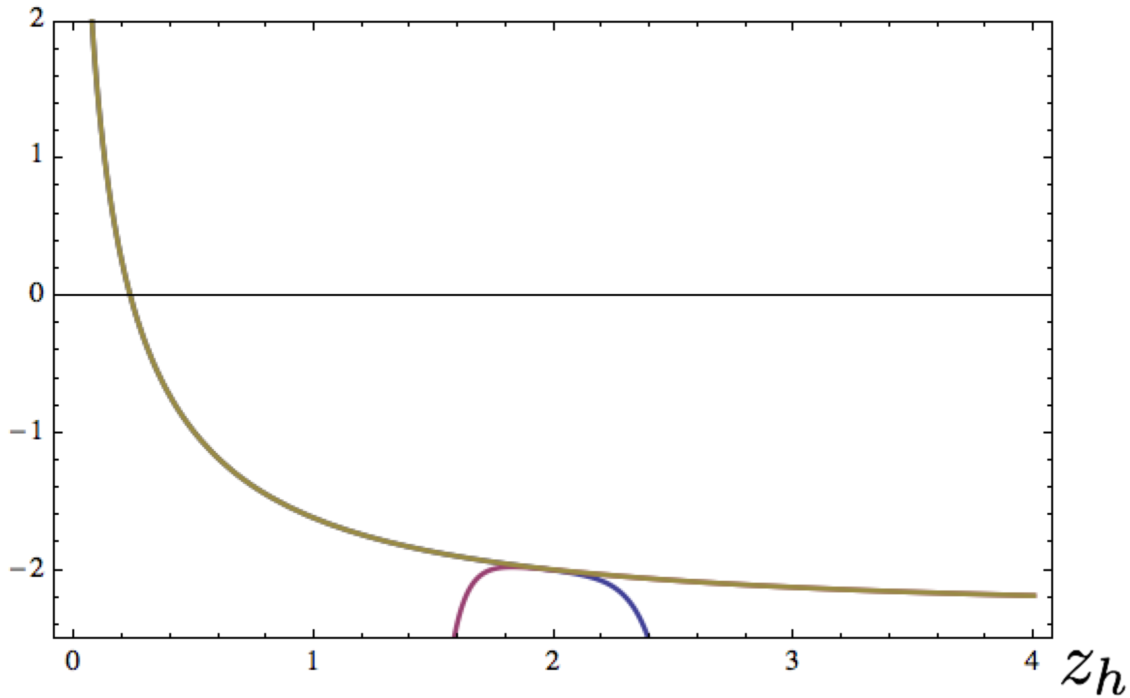
* Bound-electron *g*-factor: potential new source of alpha; Theory: richer than for free particles (and less developed)

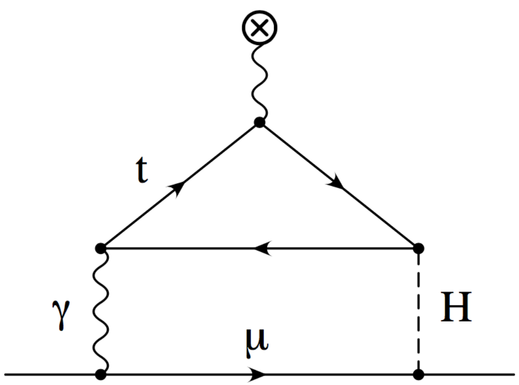
* Synergy with beautiful experiments: mass of the electron and, in future, the fine structure constant.

* For *g*: $\alpha(Z\alpha)^5$ and $\alpha^2(Z\alpha)^5$ newly-finished

Now sum the series to all orders

$$\frac{16}{45}\left[-11-3\ln\frac{m_t^2}{m_\mu^2}+z_h^2\left(1+6\ln z_h\right)-z_h^4\ln z_h-z_h\left(z_h^2-4\right)\begin{cases}\sqrt{4-z_h^2}\arccos\frac{z_h}{2} & z_h<2\\ -\sqrt{z_h^2-4}\ln\left(\sqrt{\frac{z_h^2}{4}-1}+\frac{z_h}{2}\right) & z_h>2\end{cases}\right]$$

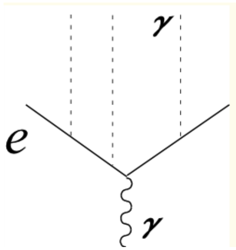


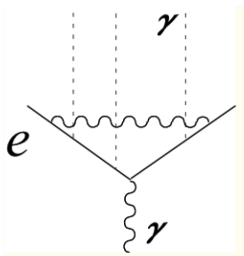


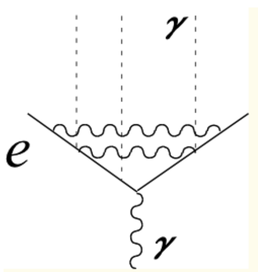
Reconcile logarithms:

$$\frac{16}{45}\left(-3\ln\frac{m_t^2}{m_\mu^2}+6\ln z_h\right)=\frac{16}{15}\left(-\ln\frac{m_t^2}{m_\mu^2}+\ln\frac{m_H^2}{m_\mu^2}\right)=-\frac{16}{15}\ln\frac{m_t^2}{m_H^2}.$$

Recent experimental improvement







$$g = 2 - \frac{2\,(Z\alpha)^2}{3} - \frac{(Z\alpha)^4}{6} + \dots$$
$$+\frac{\alpha}{\pi}\left[1 + \frac{(Z\alpha)^2}{6} + (Z\alpha)^4\,(a_{41}\ln Z\alpha + a_{40}) + \dots\right]$$
$$+\left(\frac{\alpha}{\pi}\right)^2\left[-0.65..\left(1+\frac{(Z\alpha)^2}{6}\right) + (Z\alpha)^4\,(b_{41}\ln Z\alpha + b_{40}) + \dots\right]$$

$$b_{41} = \frac{28}{9}$$
$$b_{40} = -\cancel{16.4}$$

-18.0

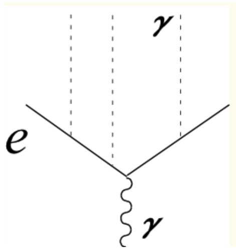
Together, new experiments in Mainz and this theory improved the accuracy of m_e by about a factor 3,

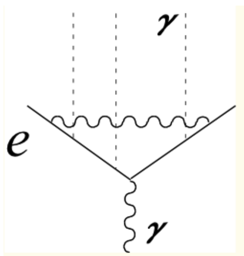
$$\frac{m_e}{u} = 0.000\;548\;579\;909\;32\,(29)\,(1)$$

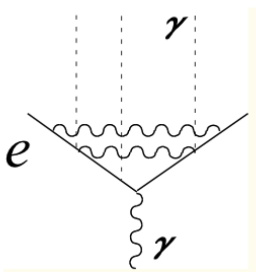
$$\frac{m_e}{u} = 0.000\;548\;579\;909\;067\,(17)$$

Nature 2014
Sturm et al

Recent experimental improvement







$$g = 2 - \frac{2\,(Z\alpha)^2}{3} - \frac{(Z\alpha)^4}{6} + \ldots$$
$$+\frac{\alpha}{\pi}\left[1 + \frac{(Z\alpha)^2}{6} + (Z\alpha)^4\,(a_{41}\ln Z\alpha + a_{40}) + \ldots\right]$$
$$+\left(\frac{\alpha}{\pi}\right)^2\left[-0.65..\left(1+\frac{(Z\alpha)^2}{6}\right) + (Z\alpha)^4\,(b_{41}\ln Z\alpha + b_{40}) + \ldots\right]$$

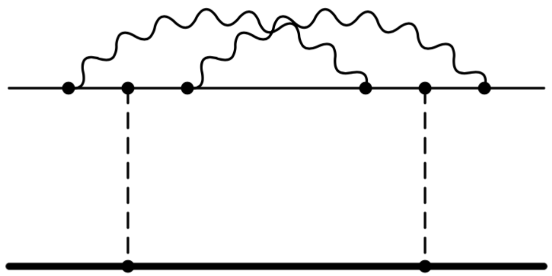
$$\frac{m_e}{u} = 0.000\ 548\ 579\ 909\ 32\,(29)\,(1)$$

$$\frac{m_e}{u} = 0.000\ 548\ 579\ 909\ 067\,(17)$$

Nature 2014
Sturm et al

Next theory challenge:
$(Z\alpha)^5$ effects.

Short-distance potential at $O(\alpha^2 (Z\alpha)^2)$



+ 18 other diagrams with two-loop self-energy

First computed for Lamb shift in hydrogen:
Pachucki, PRL 72, 3154 (1994);
Eides and Shelyuto, PRA 52, 954 (1995).

Improved precision by reduction to master integrals:
Dowling, Mondejar, Piclum, *AC*, PRA 81, 022509 (2010).

$$a_e = -1.912245764926445574152647167439830054060873390658725345\ldots\left(\frac{\alpha}{\pi}\right)^4$$

Crucial tool: Laporta algorithm

Integration by parts

↓

Identities among loop integrals
with various powers of propagators

↓

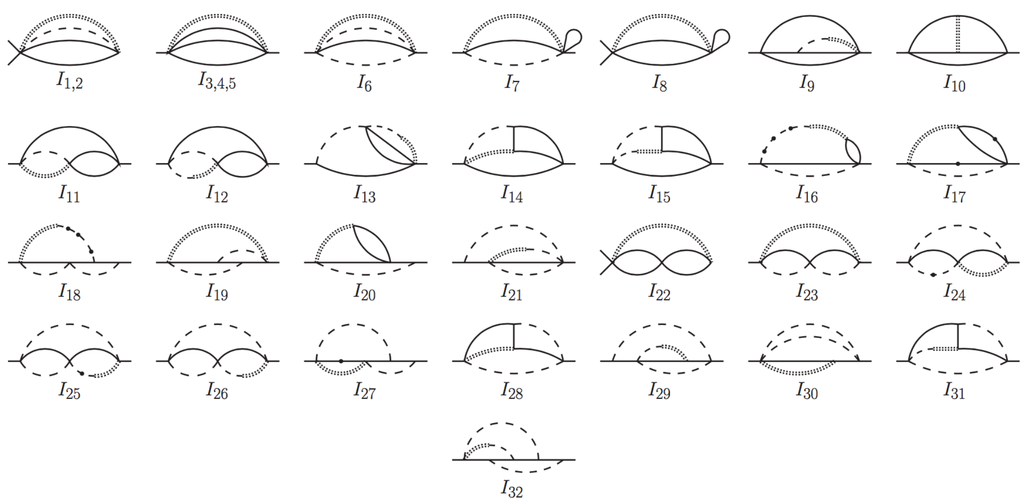
System of difference equations

↓

Solution in terms of a basis of master integrals

Milestone papers:

Tkachev PLB 100 (1981) 65
Chetyrkin and Tkachev NPB 192 (1981) 159
Laporta IJMP 15 (2000) 5087
Smirnov JHEP 0810 (2008) 107



Result: short-distance potential

$$\Delta V = c\,\alpha^2\,(Z\alpha)^2\,\delta^3(\boldsymbol{r})$$

Lamb shift

$$\Delta E = -\frac{7.72381(4)}{\pi}\alpha^2\,(Z\alpha)^5\,m$$

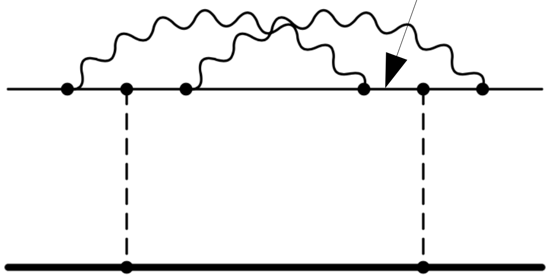
Correction to g

$$\Delta g = \frac{4\Delta E}{m}$$

Karshenboim, PLA 266 (2000) 380

Next effect: energy dependence of Lamb

Propagators depend on the energy
of the electron

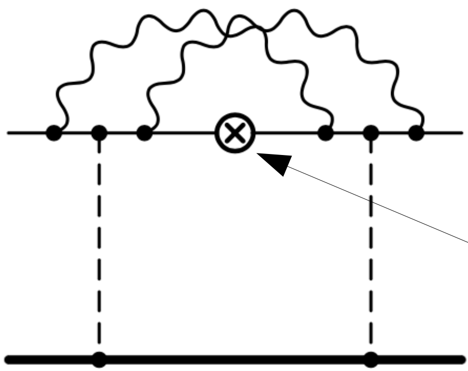


The energy is shifted in the magnetic field,

$$\Delta E = -\boldsymbol{\mu} \cdot \boldsymbol{B} = \frac{e}{2m} g \boldsymbol{s} \cdot \boldsymbol{B}$$

$$g_3 = g \frac{\mathrm{d}\,\mathrm{Lamb}}{\mathrm{d}E} = \left(2 + \frac{\alpha}{\pi}\right) \frac{\mathrm{d}\,\mathrm{Lamb}}{\mathrm{d}E}$$

Modified electron response to the B-field



External magnetic field

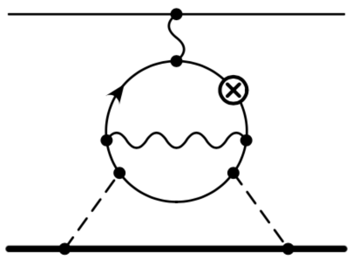
A group of about a hundred 3-loop diagrams: automatically generated from Lamb. Reduce to the same master integrals.

Gauge dependent: need g_3 to cancel the ξ-dependence.

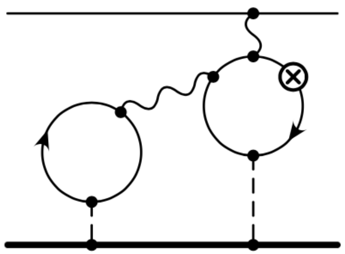
Coulomb field modifies the magnetic field

Karshenboim, Milstein, PLB 549 (2002) 321

"Magnetic loop": a vacuum-polarization effect,



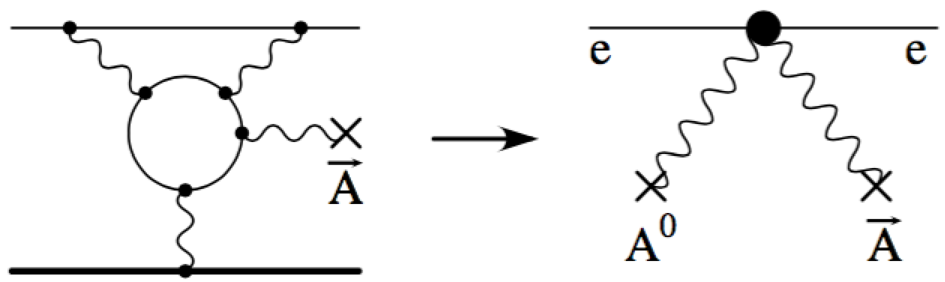
$$g^{\mathrm{MLPH}} = \left(-\frac{7543}{16200} - \frac{303587}{10125\pi} + \frac{92368}{2025\pi}\ln 2\right)\alpha^2\alpha_Z^5$$



$$g^{\mathrm{MLVP}} = \left(\frac{628}{8505\pi} - \frac{1}{54}\right)\alpha^2\alpha_Z^5$$

Note: technically simpler --> analytical result.
Numerically small.

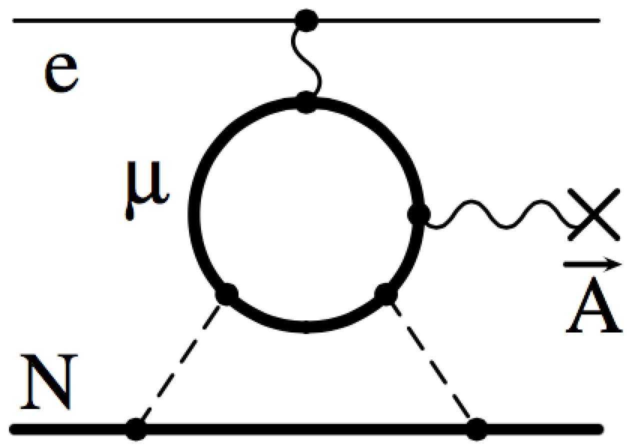
Light-by-light contributions



$$\delta V = \frac{e^2}{2m}\left(2\eta\sigma^{ij}B^{ik}\nabla^j E^k + \xi\sigma^{ij}B^{ij}\nabla^k E^k\right)$$

$$g^{(2)} = \left(\frac{\alpha}{\pi}\right)^2 \frac{(Z\,\alpha)^4}{n^3}\left\{\frac{28}{9}\,\ln[(Z\,\alpha)^{-2}] + \frac{258917}{19440} - \frac{4}{9}\,\ln k_0 - \frac{8}{3}\,\ln k_3 + \frac{113}{810}\,\pi^2 - \frac{379}{90}\,\pi^2\,\ln 2 + \frac{379}{60}\,\zeta(3)\right.$$
$$\left. + \left(\frac{16-19\pi^2}{108}\right)_{\rm LBL} + \frac{1}{n}\left[-\frac{985}{1728} - \frac{5}{144}\,\pi^2 + \frac{5}{24}\,\pi^2\,\ln 2 - \frac{5}{16}\,\zeta(3)\right]\right\},$$

Light-by-light: virtual-muon "magnetic loop"



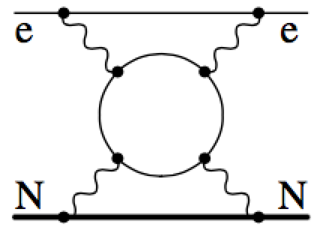
$$g_{\mathrm{ML}}(\mathrm{muon}) = \frac{7}{216}\alpha\,(Z\alpha)^5\left(\frac{m_e}{m_\mu}\right)^3$$

Recently considered numerically
in Belov et al, 1610.01340.

we find that this is the only difference
with the virtual-electron loop.
(Karshenboim & Milstein 2002)

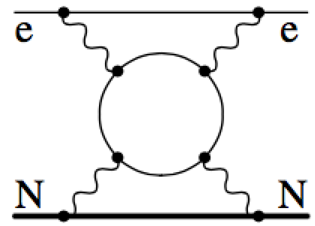
Light-by-light contribution to the Lamb shift

We consider two momentum regions in

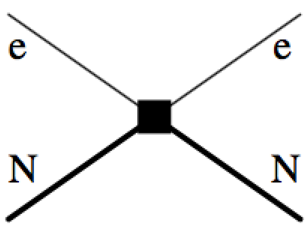


Light-by-light contribution to the Lamb shift

We consider two momentum regions in



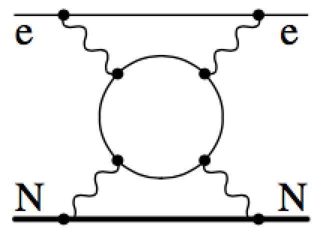
If all loops are
short-distance (hard)



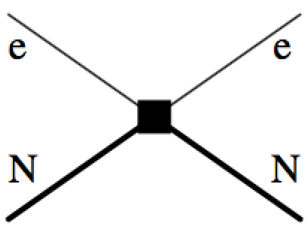
$$\alpha^2 \left(Z\alpha\right)^2 \cdot \psi_0^2 \to \alpha^2 \left(Z\alpha\right)^5$$

Light-by-light contribution to the Lamb shift

We consider two momentum regions in



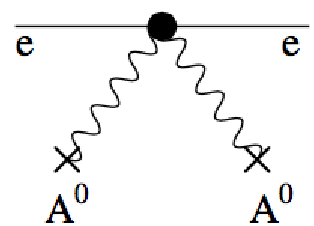
If all loops are short-distance (hard)



$$\alpha^2 (Z\alpha)^2 \cdot \psi_0^2 \to \alpha^2 (Z\alpha)^5$$

Pachucki; Eides+Shelyuto
Dowling, Mondejar, Piclum, AC

If the lowest loop is soft



$$\alpha^2 \langle e^2 \boldsymbol{E}^2 \rangle \sim \alpha^2 \left\langle \frac{(Z\alpha)^2}{r^4} \right\rangle$$

$$\to \alpha^2 (Z\alpha)^6 \ln Z\alpha$$

AC, Szafron
(Jentschura, AC, Pachucki)

How large is the resulting Lamb shift?

Decreases 1S-2S splitting by about 280 Hz.

For comparison, the experimental error is 10 Hz (~ 10^{-15} relative error)