

# *Twin supergravities from (Yang-Mills)<sup>2</sup>*

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based on

[arXiv:1301.4176 arXiv:1309.0546 arXiv:1312.6523  
arXiv:1402.4649 arXiv:1408.4434  
arXiv:1602.08267 arXiv:1610.07192  
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A. Marrani, S. Nagy and M. Zoccali]

## 1.0 Basic idea

- Strong nuclear, Weak nuclear and Electromagnetic forces described by Yang-Mills gauge theory (non-abelian generalisation of Maxwell). Gluons, W, Z and photons have spin 1.
- Gravitational force described by Einstein's general relativity. Gravitons have spin 2.
- But maybe  $(spin\ 2) = (spin\ 1)^2$ . If so:
  - 1) Do global gravitational symmetries follow from flat-space Yang-Mills symmetries?
  - 2) Do local gravitational symmetries and Bianchi identities follow from flat-space Yang-Mills symmetries?
  - 3) What about twin supergravities with same bosonic lagrangian but different fermions?

## 1.1 Gravity as square of Yang-Mills

- A recurring theme in attempts to understand the quantum theory of gravity and appears in several different forms:
- Closed states from products of open states and KLT relations in string theory [Kawai, Lewellen, Tye:1985, Siegel:1988],
- On-shell  $D = 10$  Type IIA and IIB supergravity representations from on-shell  $D = 10$  super Yang-Mills representations [Green, Schwarz and Witten:1987],
- Vector theory of gravity [Svidzinsky 2009]
- Supergravity scattering amplitudes from those of super Yang-Mills in various dimensions, [Bern, Carrasco, Johanson:2008, 2010; Bern, Huang, Kiermaier, 2010, 2012, Montiero, O'Connell, White 2011, 2014, Bianchi:2008, Elvang, Huang:2012, Cachazo:2013, Dolan:2013]
- Ambitwistor strings [Hodges:2011, Mason:2013, Geyer:2014]

## 1.2 Local and global symmetries from Yang-Mills

- LOCAL SYMMETRIES: general covariance, local lorentz invariance, local supersymmetry, local p-form gauge invariance  
[ [arXiv:1408.4434](#), [Physica Scripta 90 \(2015\)](#)]
- GLOBAL SYMMETRIES eg  $G = E_7$  in  $D = 4, \mathcal{N} = 8$  supergravity  
[[arXiv:1301.4176](#) [arXiv:1312.6523](#) [arXiv:1402.4649](#) [arXiv:1502.05359](#)]
- TWIN SUPERGRAVITIES FROM (YANG-MILLS)<sup>2</sup>  
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- LOCAL SYMMETRIES

## 2.1. Product?

- Most of the literature is concerned with products of momentum-space scattering amplitudes, but we are interested in products of off-shell left and right Yang-Mills field in coordinate-space

$$A_\mu(x)(L) \otimes A_\nu(x)(R)$$

so it is hard to find a conventional field theory definition of the product.

- Where do the gauge indices go?
- Does it obey the Leibnitz rule

$$\partial_\mu(f \otimes g) = (\partial_\mu f) \otimes g + f \otimes (\partial_\mu g)$$

If not, why not?

## 2.2 Convolution

- Here we present a  $G_L \times G_R$  product rule :

$$[A_\mu^i(L) \star \Phi_{ii'} \star A_\nu^{i'}(R)](x)$$

where  $\Phi_{ii'}$  is the “spectator” bi-adjoint scalar field introduced by Hodges [Hodges:2011] and Cachazo *et al* [Cachazo:2013] and where  $\star$  denotes a convolution

$$[f \star g](x) = \int d^4y f(y)g(x-y).$$

Note  $f \star g = g \star f$ ,  $(f \star g) \star h = f \star (g \star h)$ , and, importantly obeys

$$\partial_\mu(f \star g) = (\partial_\mu f) \star g = f \star (\partial_\mu g)$$

and not Leibnitz

$$\partial_\mu(f \otimes g) = (\partial_\mu f) \otimes g + f \otimes (\partial_\mu g)$$

## 2.3 Gravity/Yang-Mills dictionary

For concreteness we focus on

- $\mathcal{N} = 1$  supergravity in  $D = 4$ , obtained by tensoring the  $(4 + 4)$  off-shell  $\mathcal{N}_L = 1$  Yang-Mills multiplet  $(A_\mu(L), \chi(L), D(L))$  with the  $(3 + 0)$  off-shell  $\mathcal{N}_R = 0$  multiplet  $A_\mu(R)$ .
- Interestingly enough, this yields the new-minimal formulation of  $\mathcal{N} = 1$  supergravity [Sohnius,West:1981] with its 12+12 multiplet  $(h_{\mu\nu}, \psi_\mu, V_\mu, B_{\mu\nu})$
- The dictionary is,

$$\begin{aligned} Z_{\mu\nu} \equiv h_{\mu\nu} + B_{\mu\nu} &= A_\mu{}^i(L) \star \Phi_{ii'} \star A_\nu{}^{i'}(R) \\ \psi_\nu &= \chi^i(L) \star \Phi_{ii'} \star A_\nu{}^{i'}(R) \\ V_\nu &= D^i(L) \star \Phi_{ii'} \star A_\nu{}^{i'}(R), \end{aligned}$$

## 2.4 Yang-Mills symmetries

- The left supermultiplet is described by a vector superfield  $V^i(L)$  transforming as

$$\begin{aligned}\delta V^i(L) &= \Lambda^i(L) + \bar{\Lambda}^i(L) + f^i_{jk} V^j(L) \theta^k(L) \\ &\quad + \delta_{(a,\lambda,\epsilon)} V^i(L).\end{aligned}$$

Similarly the right Yang-Mills field  $A_\nu{}^{i'}(R)$  transforms as

$$\begin{aligned}\delta A_\nu{}^{i'}(R) &= \partial_\nu \sigma^{i'}(R) + f^{i'}_{j'k'} A_\nu{}^{j'}(R) \theta^{k'}(R) \\ &\quad + \delta_{(a,\lambda)} A_\nu{}^{i'}(R).\end{aligned}$$

and the spectator as

$$\delta \Phi_{ii'} = -f^j_{ik} \Phi_{jj'} \theta^k(L) - f^{j'}_{i'k'} \Phi_{ij'} \theta^{k'}(R) + \delta_a \Phi_{ii'}.$$

Plugging these into the dictionary gives the gravity transformation rules.

## 2.5 Gravitational symmetries

$$\begin{aligned}\delta Z_{\mu\nu} &= \partial_\nu \alpha_\mu(L) + \partial_\mu \alpha_\nu(R), \\ \delta \psi_\mu &= \partial_\mu \eta, \\ \delta V_\mu &= \partial_\mu \Lambda,\end{aligned}$$

where

$$\begin{aligned}\alpha_\mu(L) &= A_\mu{}^i(L) \star \Phi_{i\bar{i}'} \star \sigma^{i'}(R), \\ \alpha_\nu(R) &= \sigma^i(L) \star \Phi_{i\bar{i}'} \star A_\nu{}^{i'}(R), \\ \eta &= \chi^i(L) \star \Phi_{i\bar{i}'} \star \sigma^{i'}(R), \\ \Lambda &= D^i(L) \star \Phi_{i\bar{i}'} \star \sigma^{i'}(R),\end{aligned}$$

illustrating how the local gravitational symmetries of general covariance, 2-form gauge invariance, local supersymmetry and local chiral symmetry follow from those of Yang-Mills.

## 2.6 Lorentz multiplet

New minimal supergravity also admits an off-shell Lorentz multiplet  $(\Omega_{\mu ab}^-, \psi_{ab}, -2V_{ab}^+)$  transforming as

$$\delta \mathcal{V}^{ab} = \Lambda^{ab} + \bar{\Lambda}^{ab} + \delta_{(a, \lambda, \epsilon)} \mathcal{V}^{ab}. \quad (1)$$

This may also be derived by tensoring the left Yang-Mills superfield  $V^i(L)$  with the right Yang-Mills field strength  $F^{abi'}(R)$  using the dictionary

$$\mathcal{V}^{ab} = V^i(L) \star \Phi_{ii'} \star F^{abi'}(R),$$

$$\Lambda^{ab} = \Lambda^i(L) \star \Phi_{ii'} \star F^{abi'}(R).$$

## 2.7 Bianchi identities

- The corresponding Riemann and Torsion tensors are given by

$$R_{\mu\nu\rho\sigma}^+ = -F_{\mu\nu}{}^i(L) \star \Phi_{ii'} \star F_{\rho\sigma}{}^{i'}(R) = R_{\rho\sigma\mu\nu}^-.$$

$$T_{\mu\nu\rho}^+ = -F_{[\mu\nu}{}^i(L) \star \Phi_{ii'} \star A_{\rho]}{}^{i'}(R) = -A_{[\rho}{}^i(L) \star \Phi_{ii'} \star F_{\mu\nu]}{}^{i'}(R) = -T_{\mu\nu\rho}^-$$

- One can show that (to linearised order) both the gravitational Bianchi identities

$$DT = R \wedge e \tag{2}$$

$$DR = 0 \tag{3}$$

follow from those of Yang-Mills

$$D_{[\mu}(L)F_{\nu\rho]}{}^i(L) = 0 = D_{[\mu}(R)F_{\nu\rho]}{}^{i'}(R)$$

## 2.9 To do

- Convoluting the off-shell Yang-Mills multiplets  $(4 + 4, \mathcal{N}_L = 1)$  and  $(3 + 0, \mathcal{N}_R = 0)$  yields the  $12 + 12$  new-minimal off-shell  $\mathcal{N} = 1$  supergravity.
- Clearly two important improvements would be to generalise our results to the full non-linear transformation rules and to address the issue of dynamics as well as symmetries.
- What is the Yang-Mills origin of the Equivalence Principle?

- GLOBAL SYMMETRIES

## 3.1 Triality Algebra

- Second, the triality algebra  $\text{tri}(\mathbb{A})$

$$\text{tri}(\mathbb{A}) \equiv \{(A, B, C) \mid A(xy) = B(x)y + xC(y)\}, \quad A, B, C \in \mathfrak{so}(n), \quad x, y \in \mathbb{A}.$$

$$\text{tri}(\mathbb{R}) = 0$$

$$\text{tri}(\mathbb{C}) = \mathfrak{so}(2) + \mathfrak{so}(2)$$

$$\text{tri}(\mathbb{H}) = \mathfrak{so}(3) + \mathfrak{so}(3) + \mathfrak{so}(3)$$

$$\text{tri}(\mathbb{O}) = \mathfrak{so}(8)$$

[Barton and Sudbery:2003]:

## 3.2 Global symmetries of supergravity in D=3

- MATHEMATICS: Division algebras:  $R, C, H, O$

*(DIVISION ALGEBRAS)<sup>2</sup> = MAGIC SQUARE OF LIE ALGEBRAS*

- PHYSICS:  $N = 1, 2, 4, 8$   $D = 3$  Yang – Mills

*(YANG – MILLS)<sup>2</sup> = MAGIC SQUARE OF SUPERGRAVITIES*

- CONNECTION:  $N = 1, 2, 4, 8 \sim R, C, H, O$

*MATHEMATICS MAGIC SQUARE = PHYSICS MAGIC SQUARE*

- The  $D = 3$   $G/H$  grav symmetries are given by  $ym$  symmetries

$$G(\text{grav}) = \text{tri } ym(L) + \text{tri } ym(R) + 3[ym(L) \times ym(R)].$$

eg

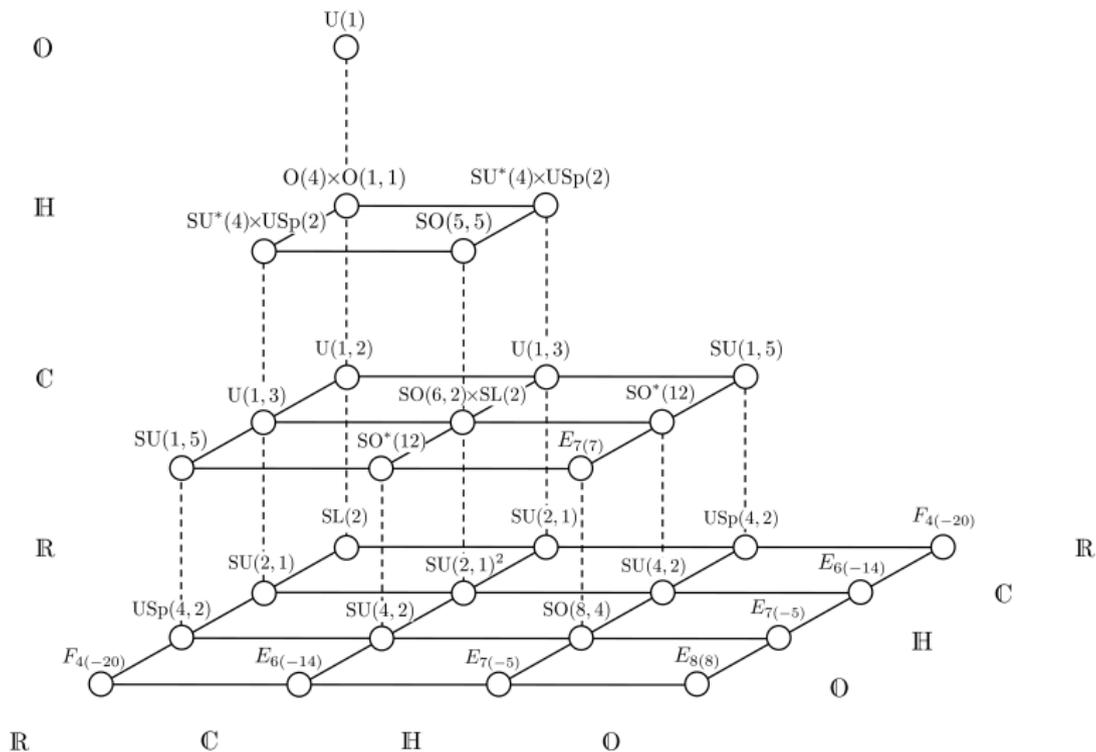
$$E_{8(8)} = SO(8) + SO(8) + 3(\mathbb{O} \times \mathbb{O})$$
$$248 = 28 + 28 + (8_v, 8_v) + (8_s, 8_s) + (8_c, 8_c)$$

## 3.3 Final result

	R	C	H	O
R	$\mathcal{N} = 2, f = 4$ $G = \text{SL}(2, \mathbb{R}), \text{dim } 3$ $H = \text{SO}(2), \text{dim } 1$	$\mathcal{N} = 3, f = 8$ $G = \text{SU}(2, 1), \text{dim } 8$ $H = \text{SU}(2) \times \text{SO}(2), \text{dim } 4$	$\mathcal{N} = 5, f = 16$ $G = \text{USp}(4, 2), \text{dim } 21$ $H = \text{USp}(4) \times \text{USp}(2), \text{dim } 13$	$\mathcal{N} = 9, f = 32$ $G = F_{4(-20)}, \text{dim } 52$ $H = \text{SO}(9), \text{dim } 36$
C	$\mathcal{N} = 3, f = 8$ $G = \text{SU}(2, 1), \text{dim } 8$ $H = \text{SU}(2) \times \text{SO}(2), \text{dim } 4$	$\mathcal{N} = 4, f = 16$ $G = \text{SU}(2, 1)^2, \text{dim } 16$ $H = \text{SU}(2)^2 \times \text{SO}(2)^2, \text{dim } 8$	$\mathcal{N} = 6, f = 32$ $G = \text{SU}(4, 2), \text{dim } 35$ $H = \text{SU}(4) \times \text{SU}(2) \times \text{SO}(2), \text{dim } 19$	$\mathcal{N} = 10, f = 64$ $G = E_{6(-14)}, \text{dim } 78$ $H = \text{SO}(10) \times \text{SO}(2), \text{dim } 46$
H	$\mathcal{N} = 5, f = 16$ $G = \text{USp}(4, 2), \text{dim } 21$ $H = \text{USp}(4) \times \text{USp}(2), \text{dim } 13$	$\mathcal{N} = 6, f = 32$ $G = \text{SU}(4, 2), \text{dim } 35$ $H = \text{SU}(4) \times \text{SU}(2) \times \text{SO}(2), \text{dim } 19$	$\mathcal{N} = 8, f = 64$ $G = \text{SO}(8, 4), \text{dim } 66$ $H = \text{SO}(8) \times \text{SO}(4), \text{dim } 34$	$\mathcal{N} = 12, f = 128$ $G = E_{7(-5)}, \text{dim } 133$ $H = \text{SO}(12) \times \text{SO}(3), \text{dim } 69$
O	$\mathcal{N} = 9, f = 32$ $G = F_{4(-20)}, \text{dim } 52$ $H = \text{SO}(9), \text{dim } 36$	$\mathcal{N} = 10, f = 64$ $G = E_{6(-14)}, \text{dim } 78$ $H = \text{SO}(10) \times \text{SO}(2), \text{dim } 46$	$\mathcal{N} = 12, f = 128$ $G = E_{7(-5)}, \text{dim } 133$ $H = \text{SO}(12) \times \text{SO}(3), \text{dim } 69$	$\mathcal{N} = 16, f = 256$ $G = E_{8(8)}, \text{dim } 248$ $H = \text{SO}(16), \text{dim } 120$

- The  $\mathcal{N} > 8$  supergravities in  $D = 3$  are unique, all fields belonging to the gravity multiplet, while those with  $\mathcal{N} \leq 8$  may be coupled to  $k$  additional matter multiplets [Marcus and Schwarz:1983; deWit, Tollsten and Nicolai:1992]. The real miracle is that tensoring left and right YM multiplets yields the field content of  $\mathcal{N} = 2, 3, 4, 5, 6, 8$  supergravity with  $k = 1, 1, 2, 1, 2, 4$ : just the right matter content to produce the U-duality groups appearing in the magic square.

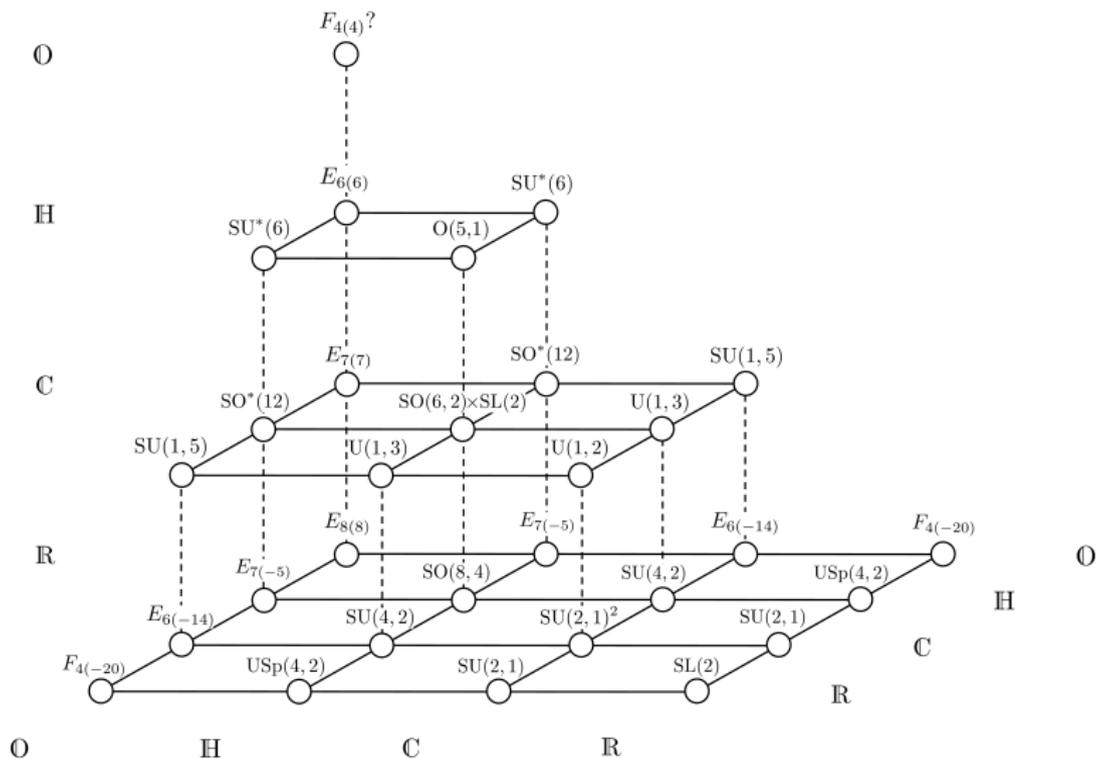
# 3.4 Magic Pyramid: G symmetries



## 4.7 Summary Gravity: Conformal Magic Pyramid

- We also construct a *conformal* magic pyramid by tensoring conformal supermultiplets in  $D = 3, 4, 6$ .
- The missing entry in  $D = 10$  is suggestive of an exotic theory with G/H duality structure  $F_{4(4)}/Sp(3) \times Sp(1)$ .

# 3.5 Conformal Magic Pyramid: G symmetries



- TWIN SUPERGRAVITIES

## 4.1. Twins?

- We consider so-called ‘twin supergravities’ - pairs of supergravities with  $\mathcal{N}_+$  and  $\mathcal{N}_-$  supersymmetries,  $\mathcal{N}_+ > \mathcal{N}_-$ , with identical bosonic sectors - in the context of tensoring super Yang-Mills multiplets.

[Gunaydin, Sierra and Townsend Dolivet, Julia and Kounnas Bianchi and Ferrara]

- Classified in [Roest and Samtleben Duff and Ferrara]
- Related work in [Chiodaroli, Gunaydin, Johansson, Roiban, 2015]



## 4.3 Example: $\mathcal{N}_+ = 6$ and $\mathcal{N}_- = 2$ twin supergravities

- The  $D = 4, \mathcal{N} = 6$  supergravity theory is unique and determined by supersymmetry. The multiplet consists of

$$\mathbf{G}_6 = \{g_{\mu\nu}, 16A_\mu, 30\phi; 6\Psi_\mu, 26\chi\}$$

- Its twin theory is the magic  $\mathcal{N} = 2$  supergravity coupled to 15 vector multiplets based on the Jordan algebra of  $3 \times 3$  Hermitian quaternionic matrices  $\mathfrak{J}_3(\mathbb{H})$ . The multiplet consists of

$$\mathbf{G}_2 \oplus 15\mathbf{V}_2 = \{g_{\mu\nu}, 2\Psi_\mu, A_\mu\} \oplus 15\{A_\mu, 2\chi, 2\phi\}$$

- In both cases the 30 scalars parametrise the coset manifold

$$\frac{\mathrm{SO}^*(12)}{\mathrm{U}(6)}$$

and the 16 Maxwell field strengths and their duals transform as the **32** of  $\mathrm{SO}^*(12)$  where  $\mathrm{SO}^*(2n) = O(n, H)$

## 4.4 Yang-Mills origin twin supergravities

- Key idea: reduce the degree of supersymmetry by using 'fundamental' matter multiplets

$$\chi^{\text{adj}} \longrightarrow \chi^{\text{fund}}$$

- Twin supergravities are systematically related through this process
- Generates new from old (supergravities that previously did not have a Yang-Mills origin)

## 5.7 Yang-Mills origin of (6, 2) twin supergravities

$$\mathcal{N} = 6$$

- The  $\mathcal{N} = 6$  multiplet is the product of  $\mathcal{N} = 2$  and  $\mathcal{N} = 4$  vector multiplets,

$$[\mathbf{V}_2 \oplus \mathbf{C}_2^\rho] \otimes \tilde{\mathbf{V}}_4 = \mathbf{G}_6,$$

- $\mathbf{G}_\mathcal{N}$ ,  $\mathbf{V}_\mathcal{N}$  and  $\mathbf{C}_\mathcal{N}$  denote the  $\mathcal{N}$ -extended gravity, vector, and spinor multiplets
- The hypermultiplet  $\mathbf{C}_2^\rho$  carries a non-adjoint representation  $\rho$  of  $G$
- $\mathbf{C}_2^\rho$  does not 'talk' to the right adjoint valued multiplet  $\tilde{\mathbf{V}}_4$

## 5.8. Yang-Mills origin of (6, 2) twin supergravities

To generate the twin  $\mathcal{N}_- = 2$  theory:

- Replace the right  $\mathcal{N} = 4$  Yang-Mills by an  $\mathcal{N} = 0$  multiplet

$$[\mathbf{V}_2 \oplus \mathbf{C}_2^\rho] \otimes \tilde{\mathbf{V}}_4 \longrightarrow [\mathbf{V}_2 \oplus \mathbf{C}_2^\rho] \otimes [\tilde{\mathbf{A}} \oplus \tilde{\chi}^{\rho\alpha} \oplus \tilde{\phi}^{[\alpha\beta]}]$$

- Here  $\tilde{\chi}^\alpha$  in the adjoint of  $\tilde{\mathbf{G}}$  and  $\mathbf{4}$  of  $SU(4)$  is replaced by  $\tilde{\chi}^{\rho\alpha}$  in a non-adjoint representation of  $\tilde{\mathbf{G}}$
- $\tilde{\chi}^{\rho\alpha}$  does not ‘talk’ to the right adjoint valued multiplet  $\mathbf{V}_2$ , but does with  $\mathbf{C}_2^\rho$
- Gives a “sum of squares”

$$[\mathbf{V}_2 \oplus \mathbf{C}_2^\rho] \otimes [\tilde{\mathbf{A}} \oplus \tilde{\chi}^{\rho\alpha} \oplus \tilde{\phi}^{[\alpha\beta]}] = \mathbf{V}_2 \otimes [\tilde{\mathbf{A}} \oplus \tilde{\phi}^{[\alpha\beta]}] \oplus [\mathbf{C}_2^\rho \otimes \tilde{\chi}^{\rho\alpha}] = \mathbf{G}_2 \oplus 15\mathbf{V}_2$$

## 5.8. Sum of squares

Introduce bi-fundamental scalar  $\Phi^{a\tilde{a}}$  to obtain sum of squares off-shell:

- Block-diagonal spectator field  $\Phi$  with bi-adjoint and bi-fundamental sectors

$$\Phi = \begin{pmatrix} \Phi^{i\tilde{i}} & 0 \\ 0 & \Phi^{a\tilde{a}} \end{pmatrix}.$$

- The off-shell dictionary correctly captures the sum-of-squares rule:

$$[\mathbf{V}_{\mathcal{N}_L} \oplus \mathbf{C}_{\mathcal{N}_L}^\rho] \circ \Phi \circ [\tilde{\mathbf{V}}_{\mathcal{N}_R} \oplus \tilde{\mathbf{C}}_{\mathcal{N}_R}^{\tilde{\rho}}] = \mathbf{V}_{\mathcal{N}_L}^i \circ \Phi_{i\tilde{i}} \circ \tilde{\mathbf{V}}_{\mathcal{N}_R}^{\tilde{i}} \oplus \mathbf{C}_{\mathcal{N}_L}^a \circ \Phi_{a\tilde{a}} \circ \tilde{\mathbf{C}}_{\mathcal{N}_R}^{\tilde{a}}$$

- Crucially, the gravitational symmetries are correctly generated by those of the Yang-Mills-matter factors via  $\star$  and  $\Phi$ .

## 5.10 Universal rule

- This construction generalises: all pairs of twin supergravity theories in the pyramid are related in this way

Super Yang-Mills factors

$$[\mathbf{V}_{\mathcal{N}_L} \oplus \mathbf{C}_{\mathcal{N}_L}] \otimes \tilde{\mathbf{V}}_{\mathcal{N}_R} \longrightarrow [\mathbf{V}_{\mathcal{N}_L} \oplus \mathbf{C}_{\mathcal{N}_L}] \otimes [\tilde{\mathbf{A}} \oplus \tilde{\chi} \oplus \tilde{\phi}]$$

↓

↓

$\mathbf{G}_{\mathcal{N}_+} + \textit{matter}$

$\xrightarrow{\textit{twin}}$

$\mathbf{G}_{\mathcal{N}_-} + \textit{matter}$

Twin supergravities

## 5.12. Remarks

- Twin relations relations gives new from old
- Raises the question: what class of gravitational theories are double-copy constructible?
- What about supergravity coupled to the MSSM: is it a double-copy?

- “Supergravity is very compelling but it has yet to prove its worth by experiment”

MJD “What’s up with gravity?”  
New Scientist 1977

- “...a remark still unfortunately true at Supergravity@25. Let us hope that by the Supergravity@50 conference we can say something different.”

MJD “M-theory on manifolds of  $G_2$  holonomy”  
Supergravity@25 2001

- “I’m glad I said 50 and not 40”  
MJD “Twin supergravities from Yang-Mills squared”  
Supergravity@40 2016