The Galois structure of $g-2$

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Framework:

perturbative Quantum Field Theory (pQFT).

Message:

There exists a mathematical structure—motivic Galois Theory (motGT)—that can be used to derive and/or analyze results in pQFT.

There seems to exists a deep connection between pQFT and motGT.

Result:

$g - 2_{QED}$ in motGT (Laporta's result translated by `HyperlogProcedures`).

motGT

Motivic GT is an advanced mathematical theory. It was envisioned by A. Grothendieck and developed by many contemporary mathematicians (P. Deligne, A. Goncharov, F. Brown, ...). Although motGT has roots in abstract mathematics, it comes with two very concrete structures: A Galois coaction

$$\Delta : P \longrightarrow P \otimes P^{\mathrm{d}R}$$

and an f alphabet.

$$\psi : P \longrightarrow P^{\mathrm{f}} = \sum_{\text{words } w} c_w w, \quad c_w \in \overline{\mathbb{Q}}[2\pi \mathrm{i}, \ldots].$$

Every algebraic integral (amplitude) can be translated into a suitable f alphabet.

In the f alphabet there exist no integral identities (it is the solution of an integral).

The f alphabet is a shuffle algebra. The coaction is deconcatenation.

The translation map ψ can be obtained by the decomposition algorithm (F. Brown).

Caveat:

The f alphabet needs (and depends on) an algebra basis of the numbers and functions considered.

In a sector where one has full control over the Galois coaction and the f alphabet, these mathematical structures revolutionize integration.

Example:

Multiple zeta values (MZV) with extensions by some roots of unity.

The MZV datamine (J. Blümlein, D.J. Broadhurst, J.A.M. Vermaseren) is superseeded by motGT.

In particular, it is not necessary to solve systems of equations gained by integral transformations.

Regretfully, in pQFT we do not yet have enough knowledge to use motGT alone to solve integrals.

The polylogarithmic part of Laporta's fourth order result of $g-2_{QED}$ contains extensions of MZVs by sixth and fourth roots of unity.
We need two alphabets (which do not mix): MZV(6) and MZV(4).
Every letter has a weight 1,2,3,.... By coincidence, $g-2_{QED}$ uses (up to Laporta's result) one letter per weight in each alphabet. We use this coincidence to code the weight of the letter by lexicographic order. So, a is the weight one letter in the alphabet for MZV(6). The letter c is the weight 3 letter in MZV(4).
We obtain [O.S.: *The Galois coaction on the electron anomalous magnetic moment*, Communications in Number Theory and Physics, **12**, no. 2 (2017)]:

$$a_e=\frac{1}{2}\left(\frac{\alpha}{\pi}\right)+\left(\frac{197}{144}+\frac{1}{12}\pi^2+\frac{27}{32}c-\frac{1}{4}a\pi^2\right)\left(\frac{\alpha}{\pi}\right)^2$$
$$+\left(\frac{28259}{5184}+\frac{17101}{810}\pi^2+\frac{139}{16}c-\frac{149}{9}a\pi^2-\frac{525}{32}ca+\frac{1969}{8640}\pi^4-\frac{1161}{128}e+\frac{83}{64}c\pi^2\right)\left(\frac{\alpha}{\pi}\right)^3$$
$$+\left(\frac{1243127611}{130636800}+\frac{30180451}{155520}\pi^2-\frac{255842141}{2419200}c-\frac{8873}{36}a\pi^2+\frac{126909}{2560}\frac{d}{\mathrm{i}\sqrt{3}}\right.$$
$$-\frac{84679}{1280}ca+\frac{169703}{3840}\frac{b\pi^2}{\mathrm{i}\sqrt{3}}+\frac{779}{108}aa\pi^2+\frac{112537679}{3110400}\pi^4-\frac{2284263}{25600}e$$
$$+\frac{8449}{96}caa-\frac{12720907}{345600}c\pi^2-\frac{231919}{97200}a\pi^4+\frac{150371}{256}\frac{f}{\mathrm{i}\sqrt{3}}+\frac{313131}{1280}ea$$
$$-\frac{121383}{1280}db-\frac{14662107}{51200}cc+\frac{8645}{128}\frac{cab}{\mathrm{i}\sqrt{3}}-\frac{231}{4}caaa-\frac{16025}{48}\frac{d\pi^2}{\mathrm{i}\sqrt{3}}$$
$$+\frac{4403}{384}ca\pi^2-\frac{136781}{1920}bb\pi^2+\frac{7069}{75}bb\pi^2-\frac{1061123}{14400}ac\pi^2+\frac{1115}{72}\frac{aab\pi^2}{\mathrm{i}\sqrt{3}}$$
$$+\frac{781181}{20736}\frac{b\pi^4}{\mathrm{i}\sqrt{3}}-\frac{4049}{1080}aa\pi^4+\frac{90514741}{54432000}\pi^6-\frac{95624828289}{2050048}g-\frac{29295}{512}dba$$
$$+\frac{107919}{512}cca+\frac{337365}{256}cac-\frac{55618247}{409600}e\pi^2-\frac{1055}{256}bba\pi^2+\frac{26}{3}bba\pi^2$$
$$+\frac{553}{4}aca\pi^2-\frac{35189}{1024}aac\pi^2+\frac{79147091}{2211840}c\pi^4-\frac{3678803}{4354560}a\pi^6$$
$$\left.+\sqrt{3}(E_{4a}+E_{5a}+E_{6a}+E_{7a})+E_{6b}+E_{7b}+U\right)\left(\frac{\alpha}{\pi}\right)^4+\ldots.$$

Result 1: Sparsity

- The result for a_e uses only two alphabets.
- The two alphabets do not mix.
- The alphabet MZV(6) has two weight 1 letters $a = \log 2$ and $\log 3$. The letter $\log 3$ is absent in a_e.
- Many words are absent in a_e (e.g. ac at order $(\alpha/\pi)^3$, see Galois conjugates).

Note that sparsity may become important when one tries to use the f alphabet for integration.

Result 2: Universality

Consider residues of primitive log-divergent graphs in massless ϕ^4 theory. We have information on the used alphabets up to loop order 11 by

- explicit calculations and
- the c_2 invariant.

We conjecture that to all orders the only polylogarithmic alphabets in ϕ^4 are MZV(6) and MZV(4).
Until now, no letter $\log 3$ was found in ϕ^4.

Result 3: Galois conjugates

Galois conjugates are words (with powers of $2\pi\mathrm{i}$) which one gets from clipping off right letters. Up to weight 4 we get the following short list of Galois conjugates:

wt 0 1

wt 2 π^2

wt 3 c, $\pi^2 a$

wt 4 d, ca, $\pi^2 b$, $\pi^2 b$, $\pi^2 aa$, π^4

We conjecture that this list is complete to all orders in the polylogarithmic sector of $g-2_{QED}$ (see F. Brown).
This (coaction) conjecture has predictive power.