Soft Graviton Theorem in Generic Quantum Theory of Gravity

Mritunjay Verma

Harish-Chandra Research Institute, Allahabad, India

QCD Meets Gravity

University of California, Los Angeles, December 2017

What is Soft Graviton Theorem ?

- We consider an S matrix involving some soft gravitons and some finite energy particles in a theory of gravity coupled with some arbitrary set of fields.

- The amplitude can be expanded in a power series in the soft momenta.

- The soft graviton theorem is the statement that the first three terms of the expansion involve the amplitude without the soft graviton multiplied by a soft factor.

- Our goal will be to prove this statement for the leading and the subleading terms (i.e. first two terms) of the expansion.

(Sen; Laddha, Sen; Chakrabarti, Kashyap, Sahoo, Sen, MV)

General Strategy

- We consider a theory of gravity coupled with matter fields which possesses general coordinate invariance.

- We shall assume that the general coordinate invariance is not broken by the quantum effects.

- Instead of the original action of the theory, we shall work with the general coordinate invariant 1PI effective action.

- We shall further assume that the S-matrix of the theory is infrared finite. If this is not the case (e.g., in $d \le 4$), then our derivation will be valid only for the tree level amplitudes.

General Strategy

- The full quantum corrected S-matrix of the theory can be obtained by the tree diagrams computed using the 1PI action.

- To derive the Feynman rules for the 1PI action, we first need to gauge fix it. This requires adding a gauge fixing term.

- We shall choose a gauge fixing term which breaks the general coordinate invariance but preserves the Lorentz invariance (e.g., it could be $(\partial_\mu h^{\mu\nu})^2 = 0$, where $h_{\mu\nu}$ describes the gravitons).

General Strategy

- Using this gauge fixed action, one can, in principle, now derive the Feynman rules of the theory and compute the amplitudes involving the soft gravitons.
- However, this requires the knowledge of 1PI action explicitly which we do not know in general. Hence, we shall proceed in an alternative way.
- Since we only need to work with tree diagrams, any line in a diagram (which contains some soft and some finite energy particles) can be identified with a soft or finite energy line.
- This allows us to treat soft and finite energy gravitons as two different particles.

General Strategy

- This means we can pretend that the gauge fixed action does not describe the soft gravitons but only the finite energy particles.

- The soft graviton field $S_{\mu\nu}$ can now be introduced by covariantizing the gauge fixed action with respect to the metric $\eta_{\mu\nu} + 2S_{\mu\nu}$.

- This will mean promoting the ordinary derivatives in the action to covariant derivatives (w.r.t. soft graviton metric) and possibly introducing the "non-minimal" terms involving the coupling of Riemann tensor with the finite energy fields.

General Strategy

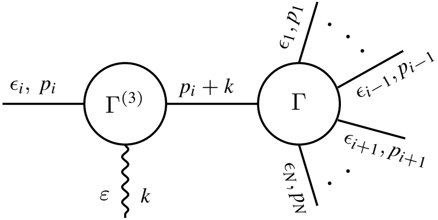
- This procedure will generate the coupling between the soft gravitons and the finite energy fields in a specific manner dictated by the covariantization procedure.

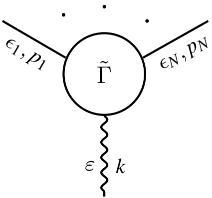
- As we shall see, even though the 1PI action depends upon the theory, the effect of covariantization (and hence the coupling between soft and hard particles) can be worked out in a universal manner.

- We shall now describe all the diagrams involving the soft gravitons which can be drawn using the vertices generated by the covariantization procedure.

Diagrams
(Single Soft Graviton)

- A single external graviton can attach itself either to an external hard line of the diagram or somewhere in the interior of the diagram. Thus, the class of diagrams which contribute to the amplitude with one external soft graviton are

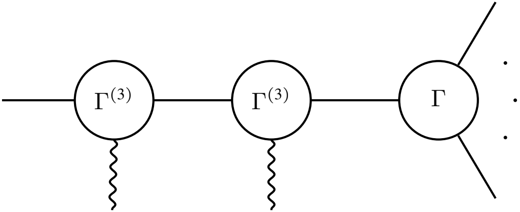




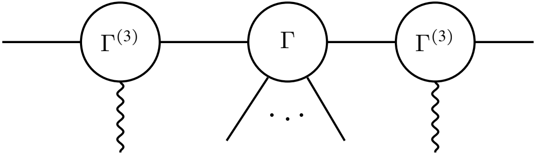
Diagrams
(Two Soft Gravitons)

- Two soft gravitons can appear in 6 different classes of diagrams (suppressing the momenta and polarizations of external lines)

1.

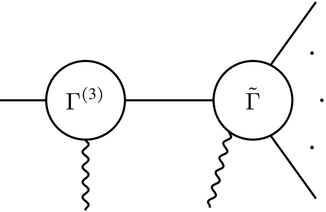


2.

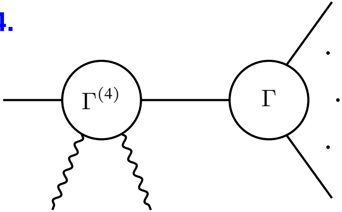


Diagrams
(Two Soft Gravitons)

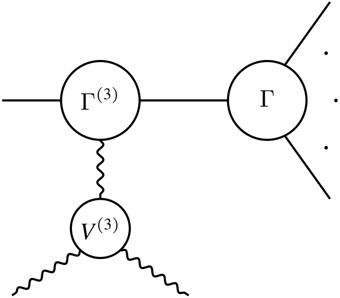
3.



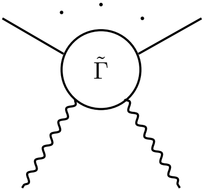
4.



5.



6.



Leading vs. Subleading Contributions

- By looking at diagrams, we find that the pole in the soft momenta arises when a soft graviton gets attached to an external line.
- This happens as follows: suppose the soft graviton carrying momentum k gets attached to the i^{th} external line. The diagram will then contain a factor of

$$\frac{1}{(p_i + k)^2 + M^2} = \frac{1}{(p_i^2 + M^2) + k^2 + 2p_i \cdot k}$$

Since graviton is on-shell, we have $k^2 = 0$. Now, the internal line carrying momentum $p_i + k$ can represent the same particle as the line carrying the external momentum p_i. In such a case, $p_i^2 + M^2 = 0$ and we get a pole in the soft momenta.

Leading vs. Subleading Contributions

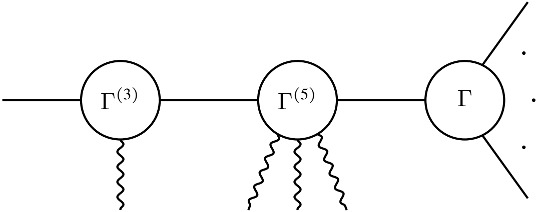
- No such pole arises when the soft graviton gets attached to an internal vertex or internal line.
- Due to this reason, the first diagram in the case of single soft graviton and the first two diagrams in the case of two soft gravitons start contributing at leading order.
- The leading contribution is proportional to $\frac{1}{k_\mu}$ for the single soft graviton and is proportional to $\frac{1}{k_\mu k_\nu}$ for the two soft gravitons.
- The contribution of the other diagrams starts at the subleading order.
- In the case of the two soft gravitons, the contribution of the 6th diagram starts at the subsubleading order.

Multiple Gravitons

- One might naively expect that as the number of soft gravitons increase, we would encounter more and more vertices.

- However, it turns out that for the case of multiple soft graviton theorem upto subleading order, we don't need any new vertex other than $\Gamma^{(3)}, \Gamma^{(4)}$ and $V^{(3)}$.

- E.g., consider the case of 4 soft gravitons. One might expect a 5-point vertex involving 3 soft gravitons and two hard particles as shown below



Multiple Gravitons

- However, we note that for the case of 4 soft gravitons, the leading and subleading contributions will go as

$$\sim \quad \frac{f_1}{k_\mu k_\nu k_\rho k_\sigma} + \frac{f_2}{k_\mu k_\nu k_\rho}$$

But, the above diagram involving $\Gamma^{(5)}$ will go as $\frac{1}{k_\mu k_\nu}$ and hence is subsubleading.

- We can do a detailed analysis of this and find it to be true for the case of an arbitrary number of soft gravitons. Thus, upto subleading order, we don't need to worry about any new coupling between the soft and hard particles.

(Chakrabarti, Kashyap, Sahoo, Sen, MV)

Determining $\Gamma^{(3)}$ and $\Gamma^{(4)}$ (Covariantization Procedure)

- We now turn to the evaluation of the diagrams.

- To evaluate the diagrams, we need to first determine the Feynman rules of the theory. In particular, we need the expression of propagators and the vertex factors $\Gamma^{(3)}, \Gamma^{(4)}$ and $V^{(3)}$.

- The $\Gamma^{(3)}$ and $\Gamma^{(4)}$ can be obtained by covariantizing the gauge fixed 1PI effective action with respect to one and two soft gravitons respectively as we describe now.

Determining $\Gamma^{(3)}$ and $\Gamma^{(4)}$ (Covariantization Procedure)

- The most general form for the quadratic part of the gauge fixed 1PI effective action can be written as

$$S^{(2)} = \frac{1}{2}\int \frac{d^d q_1}{(2\pi)^d}\frac{d^d q_2}{(2\pi)^d}(2\pi)^d \delta^d(q_1+q_2)\Phi_\alpha(q_1)\mathcal{K}^{\alpha\beta}(q_2)\Phi_\beta(q_2)$$

  where we can choose $\mathcal{K}^{\alpha\beta}(q) = \mathcal{K}^{\beta\alpha}(-q)$.

- In position space, $\mathcal{K}^{\alpha\beta}$ will correspond to some differential operator acting on the finite energy fields Φ_β.

- The covariantization essentially amounts to replacing the ordinary derivatives present in this operator into the covariant derivatives.

Determining $\Gamma^{(3)}$ and $\Gamma^{(4)}$ (Covariantization Procedure)

- We shall use the convention that all the finite energy fields carry flat tangent space indices.
- Hence, a generic term invoving the differential operator in position space will be of the form

$$\partial_{a_1} \cdots \partial_{a_n} \Phi_\alpha$$

  where, a_i are flat space vector indices and α denotes flat space spinor or tensor index.
- The covariantization amounts to the replacement

$$\partial_{a_1} \cdots \partial_{a_n} \Phi_\alpha \to E_{a_1}^{\ \mu_1} \cdots E_{a_n}^{\ \mu_n} D_{\mu_1} \cdots D_{\mu_n} \Phi_\alpha$$

Determining $\Gamma^{(3)}$ and $\Gamma^{(4)}$ (Covariantization Procedure)

- We shall parametrize the soft background metric as

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + 2S_{\mu\nu} + 2S_{\mu\rho}S^{\rho}_{\ \nu} + \cdots ; \quad S_{\mu\nu} = S_{\nu\mu} \ , \quad S_{\mu}^{\ \mu} = 0$$

- This parametrization implies for the inverse vielbein

$$E_a^{\ \mu} = \delta_a^{\ \mu} - S_a^{\ \mu} + \frac{1}{2} S_a^{\ b} S_b^{\ \mu} + \cdots$$

- The covariant derivative is given by

$$D_\mu \Phi_\alpha = \partial_\mu \Phi_\alpha + \frac{1}{2}\omega_\mu^{ab} (J_{ab})_\alpha^{\ \beta} \Phi_\beta$$

  where $(J_{ab})_\alpha^{\ \beta}$ is the generator of the spin angular momentum operator and the spin connection is given by

$$\omega_\mu^{ab} = \partial^b S_\mu^{\ a} - \partial^a S_\mu^{\ b} + \cdots$$

Determining $\Gamma^{(3)}$ and $\Gamma^{(4)}$ (Covariantization Procedure)

Parametrizing soft graviton as $S_{\mu\nu} = \varepsilon_{\mu\nu} e^{ik\cdot x}$ and working in momentum space, the effect of covariantization w.r.t. a single soft graviton produces the following coupling between the two hard particles and one soft graviton

$$
\begin{aligned}
&\frac{1}{2}\int \frac{d^d q_1}{(2\pi)^d}\frac{d^d q_2}{(2\pi)^d}(2\pi)^d \delta^d(q_1+q_2+k) \\
&\Phi_\alpha(q_1)\Big[-\varepsilon_{\mu\nu} q_2^\nu \frac{\partial}{\partial q_{2\mu}}\mathcal{K}^{\alpha\beta}(q_2) + \frac{1}{2}(k_b \varepsilon_{a\mu} - k_a \varepsilon_{b\mu})\frac{\partial}{\partial q_{2\mu}}\mathcal{K}^{\alpha\gamma}(q_2)(J_{ab})_\gamma^{\ \beta} \\
&-\frac{1}{2}\frac{\partial^2 \mathcal{K}^{\alpha\beta}(q_2)}{\partial q_{2\mu}\partial q_{2\nu}} q_{2\rho}(k_\mu \varepsilon_\nu^{\ \rho} + k_\nu \varepsilon_\mu^{\ \rho} - k^\rho \varepsilon_{\mu\nu}) + \mathcal{O}(k_\mu k_\nu)\Big]\Phi_\beta(q_2)
\end{aligned}
$$

Determining $\Gamma^{(3)}$ and $\Gamma^{(4)}$ (Covariantization Procedure)

Using this action, we can now read off the vertex factor $\Gamma^{(3)}$ to be

$$
\begin{aligned}
&\Gamma^{(3)\alpha\beta}(\varepsilon, k; q_1, q_2) \\
&= \frac{i}{2}\Bigg[-\varepsilon_{\mu\nu} q_2^{\nu} \frac{\partial \mathcal{K}^{\alpha\beta}(q_2)}{\partial q_{2\mu}} - \varepsilon_{\mu\nu} q_1^{\nu} \frac{\partial \mathcal{K}^{\beta\alpha}(q_1)}{\partial q_{1\mu}} \\
&-\frac{1}{2}\Big(\frac{\partial \mathcal{K}^{\alpha\gamma}(q_2)}{\partial q_{2\mu}} \left(J^{ab}\right)_{\gamma}^{\ \beta} + \frac{\partial \mathcal{K}^{\beta\gamma}(q_1)}{\partial q_{1\mu}} \left(J^{ab}\right)_{\gamma}^{\ \alpha} \Big) (k_a\, \varepsilon_{b\mu} - k_b\, \varepsilon_{a\mu}) \\
&-\frac{1}{2}\Big(\frac{\partial^2 \mathcal{K}^{\alpha\beta}(q_2)}{\partial q_{2\mu} \partial q_{2\nu}} q_{2\rho} + \frac{\partial^2 \mathcal{K}^{\beta\alpha}(q_1)}{\partial q_{1\mu} \partial q_{1\nu}} q_{1\rho} \Big) \left(k_\mu \varepsilon_{\nu}^{\ \rho} + k_\nu \varepsilon_{\mu}^{\ \rho} - k^{\rho} \varepsilon_{\mu\nu} \right) \Bigg]
\end{aligned}
$$

Determining $\Gamma^{(3)}$ and $\Gamma^{(4)}$ (Covariantization Procedure)

Following the similar procedure, the result of keeping terms upto quadratic in the soft graviton field gives the quartic vertex

$$
\begin{aligned}
&\Gamma^{(4)\alpha\beta}(\varepsilon_1, k_1; \varepsilon_2, k_2; q_1, q_2) \\
= \quad & i\left[\frac{1}{4}\varepsilon_{1\rho}^{\ \ \nu}\varepsilon_{2\mu\nu}q_2^{\mu}\frac{\partial\mathcal{K}^{\alpha\beta}(q_2)}{\partial q_{2\rho}} + \frac{1}{4}\varepsilon_{1\rho}^{\ \ \nu}\varepsilon_{2\mu\nu}q_1^{\mu}\frac{\partial\mathcal{K}^{\beta\alpha}(q_1)}{\partial q_{1\rho}}\right. \\
& +\frac{1}{4}\varepsilon_{1\mu}^{\ \ \nu}\varepsilon_{2\nu\rho}q_2^{\rho}\frac{\partial\mathcal{K}^{\alpha\beta}(q_2)}{\partial q_{2\mu}} + \frac{1}{4}\varepsilon_{1\mu}^{\ \ \nu}\varepsilon_{2\nu\rho}q_1^{\rho}\frac{\partial\mathcal{K}^{\beta\alpha}(q_1)}{\partial q_{1\mu}} \\
& \left.+\frac{1}{2}\varepsilon_{1\rho\mu}\varepsilon_{2\sigma\nu}q_2^{\mu}q_2^{\nu}\frac{\partial^2\mathcal{K}^{\alpha\beta}(q_2)}{\partial q_{2\rho}\partial q_{2\sigma}} + \frac{1}{2}\varepsilon_{1\rho\mu}\varepsilon_{2\sigma\nu}q_1^{\mu}q_1^{\nu}\frac{\partial^2\mathcal{K}^{\alpha\beta}(q_1)}{\partial q_{1\rho}\partial q_{1\sigma}}\right]
\end{aligned}
$$

Three graviton vertex $V^{(3)}$

- Next, we want to determine the 3 point coupling of soft gravitons. This can be obtained by expanding the Einstein-Hilbert action.

- The result is given by

$$
\begin{aligned}
&V^{(3)}_{\mu\nu}(\varepsilon_1, k_1, \varepsilon_2, k_2) \\
&\quad = \frac{i}{2}\,\varepsilon_{1,ab}\varepsilon_{2,cd}\Big[\Big\{\eta_{\mu\nu}\eta^{ac}\eta^{bd}k_1^{\rho}k_{2\rho} - 2\eta^{ad}\eta^{c}{}_{\nu}k_2^{b}k_{2\mu} - 2\eta^{cb}\eta^{a}{}_{\nu}k_1^{d}k_{1\mu} \\
&\qquad +2\eta^{ad}\eta^{c}{}_{\nu}k_{1\mu}k_2^{b} + 2\eta^{cb}\eta^{a}{}_{\mu}k_1^{d}k_{2\nu} - 2\eta^{ac}\eta^{bd}k_{1\mu}k_{2\nu} - 4\eta^{a}{}_{\nu}\eta^{c}{}_{\mu}k_1^{d}k_2^{b} \\
&\qquad +2\eta^{c}{}_{\mu}\eta^{d}{}_{\nu}k_2^{b}k_2^{a} + 2\eta^{a}{}_{\mu}\eta^{b}{}_{\nu}k_1^{d}k_1^{c}\Big\} \; + \; \{\mu \leftrightarrow \nu\}\Big]
\end{aligned}
$$

Propagators

- Finally, we need the propagator for the soft gravitons and the hard particles.

- The soft graviton propagator is given by

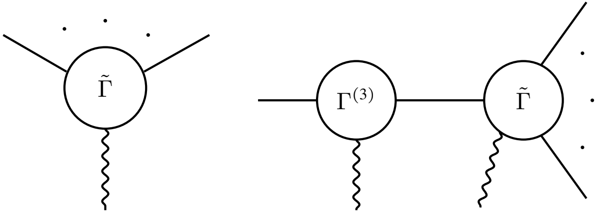
$$G_{\mu\nu,\rho\sigma}(k) = -\frac{1}{2}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma} - \frac{2}{d-2}\eta_{\mu\nu}\eta_{\rho\sigma}\right)\frac{i}{k^2}$$

- The propagator for the hard particles can be read from the quadratic part of the 1PI action to be

$$i\mathcal{K}^{-1}(q) \equiv \frac{\mathcal{N}_i(q)}{q^2 + M_i^2}$$

Evaluating $\tilde{\Gamma}$

- One more ingredient needed to evaluate the diagrams is the expression of $\tilde{\Gamma}$ which represents the sum of the diagrams in which external soft gravitons are not attached to an external line.



- For proving soft graviton theorem, we need an expression for $\tilde{\Gamma}$ in terms of the amplitude without the soft graviton.

Evaluating $\tilde{\Gamma}$

- To obtain this expression, we note that since all the vertices (i.e. full 1PI effective action) have been covariantized w.r.t. soft graviton fields, the amplitude computed using this must also reflect this fact.

- This means that $\tilde{\Gamma}$ can be obtained by the covariantization of the full amplitude without the soft graviton field.

- If $\Gamma^{\alpha_1\cdots\alpha_N}(q_1,\cdots,q_N)$ denotes the amplitude without the soft graviton, the result of covariantization (and hence $\tilde{\Gamma}$) is given by

$$\begin{aligned}
&\tilde{\Gamma}^{\alpha_1\cdots\alpha_N}(\varepsilon,k;q_1,\cdots q_N)\\
= \quad & -\sum_{i=1}^{N}\varepsilon_{\mu\nu}q_i^{\mu}\frac{\partial}{\partial q_{i\nu}}\Gamma^{\alpha_1\cdots\alpha_N}(q_1,\cdots,q_N)+\mathcal{O}(k_\mu)
\end{aligned}$$

Final Result

- The Feynman diagrams can now be evaluated following the standard procedure using the vertices and propagator given above.

- We just state the final result for the amplitude involving M soft gravitons and N finite energy particles carrying arbitrary mass and arbitrary spin in terms of the amplitude without the soft gravitons upto subleading order.

(Chakrabarti, Kashyap, Sahoo, Sen, MV)

Final Result

$$\left\{\prod_{i=1}^{N} \epsilon_{i,\alpha_i}(p_i)\right\} \Bigg[\left\{\prod_{r=1}^{M} S_r^{(0)}\right\} \; \Gamma^{\alpha_1\cdots\alpha_N} + \sum_{s=1}^{M} \left\{ \prod_{\substack{r=1\\ r\neq s}}^{M} S_r^{(0)} \right\} \; \left[S_s^{(1)}\Gamma\right]^{\alpha_1\cdots\alpha_N}$$

$$+ \sum_{\substack{r,u=1\\ r<u}}^{M} \left\{ \prod_{\substack{s=1\\ s\neq r,u}}^{M} S_s^{(0)} \right\} \left\{ \sum_{j=1}^{N} \{p_j\cdot(k_r+k_u)\}^{-1} \;\; \mathcal{M}(p_j;\varepsilon_r,k_r,\varepsilon_u,k_u) \right\} \Gamma^{\alpha_1\cdots\alpha_N} \Bigg]$$

where, $$S_r^{(0)} = \sum_{\ell=1}^{N} (p_\ell\cdot k_r)^{-1} \; \varepsilon_{r,\mu\nu}\, p_\ell^\mu p_\ell^\nu$$

$$\begin{aligned}
[S_s^{(1)}\Gamma]^{\alpha_1\cdots\alpha_N} \;\; &= \;\; \sum_{j=1}^{N} (p_j\cdot k_s)^{-1}\, \varepsilon_{s,b\mu}\, k_{sa}\, p_j^\mu \;\left(p_j^b \frac{\partial}{\partial p_{ja}} - p_j^a \frac{\partial}{\partial p_{jb}} \right) \Gamma^{\alpha_1\cdots\alpha_N} \\
&\quad + \sum_{j=1}^{N} (p_j\cdot k_s)^{-1}\, \varepsilon_{s,b\mu}\, k_{sa}\, p_j^\mu \; \left(J^{ab}\right)_{\beta_j}^{\;\alpha_j} \Gamma^{\alpha_1\cdots\alpha_{j-1}\beta_j\alpha_{j+1}\cdots\alpha_N}
\end{aligned}$$

Final Result

$$
\begin{aligned}
&\mathcal{M}(p_i;\varepsilon_1,k_1,\varepsilon_2,k_2)\\
&= (p_i\cdot k_1)^{-1}(p_i\cdot k_2)^{-1}\Big\{-(k_1\cdot k_2)\,(p_i\cdot\varepsilon_1\cdot p_i)\,(p_i\cdot\varepsilon_2\cdot p_i)\\
&\quad + 2\,(p_i\cdot k_2)\,(p_i\cdot\varepsilon_1\cdot p_i)\,(p_i\cdot\varepsilon_2\cdot k_1) + 2\,(p_i\cdot k_1)\,(p_i\cdot\varepsilon_2\cdot p_i)\,(p_i\cdot\varepsilon_1\cdot k_2)\\
&\quad -2\,(p_i\cdot k_1)\,(p_i\cdot k_2)\,(p_i\cdot\varepsilon_1\cdot\varepsilon_2\cdot p_i)\Big\}\\
&\quad + (k_1\cdot k_2)^{-1}\Big\{-(k_2\cdot\varepsilon_1\cdot\varepsilon_2\cdot p_i)(k_2\cdot p_i) - (k_1\cdot\varepsilon_2\cdot\varepsilon_1\cdot p_i)(k_1\cdot p_i)\\
&\qquad + (k_2\cdot\varepsilon_1\cdot\varepsilon_2\cdot p_i)(k_1\cdot p_i) + (k_1\cdot\varepsilon_2\cdot\varepsilon_1\cdot p_i)(k_2\cdot p_i)\\
&\qquad -\varepsilon_1^{\gamma\delta}\varepsilon_{2\gamma\delta}(k_1\cdot p_i)(k_2\cdot p_i) - \ 2(p_i\cdot\varepsilon_1\cdot k_2)(p_i\cdot\varepsilon_2\cdot k_1)\\
&\qquad +(p_i\cdot\varepsilon_2\cdot p_i)(k_2\cdot\varepsilon_1\cdot k_2) + (p_i\cdot\varepsilon_1\cdot p_i)(k_1\cdot\varepsilon_2\cdot k_1)\Big\}
\end{aligned}
$$

Some Consistency Checks

- The covariantized action possesses a gauge invariance under

$$S_{\mu\nu}(x) \to S_{\mu\nu}(x) + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

The above amplitude obeys this property.

- The above result matches with the result for two soft gravitons in specific theories derived earlier.

(Klose, McLoughlin, Nandan, Plefka, Travaglini; Saha)

- The full result for the case of arbitrary number of soft gravitons in Einstein's gravity derived using the CHY prescription also matches with this result.

(Chakrabarti, Kashyap, Sahoo, Sen, MV)

Beyond subleading order....

- The covariantization procedure as described earlier does not give rise to a unique 1PI action if we are interested in the subsubleading order results.

- To see this, we note that for promoting a flat space action to curved space, not only we need to replace ordinary derivatives by covariant derivatives, we also have the possibility of adding terms involving the coupling of the fields with the Riemann tensor.

- At subsubleading order, we shall need to keep terms upto $\mathcal{O}(k_\mu k_\nu)$ while doing the Feynman diagram calculations. Now, the Riemann tensor contains two derivatives. Hence, the coupling with the Riemann tensor can't be ignored.

Beyond subleading order....

- Upto subsubleading order, the most general form for the relevant coupling of Riemann tensor with the finite energy fields can be written as (Laddha, Sen)

$$\bar{S}^{(3)} = \frac{1}{2}\int \frac{d^d q_1}{(2\pi)^d}\frac{d^d q_2}{(2\pi)^d}(2\pi)^d \delta^d(q_1+q_2+k)\mathcal{R}_{\mu\rho\nu\sigma}(\varepsilon,k) \\ \Phi_\alpha(q_1)\mathcal{B}^{\alpha\beta,\mu\rho\nu\sigma}(q_2)\Phi_\beta(q_2)$$

  where, $\mathcal{R}_{\mu\rho\nu\sigma}$ is the Riemann tensor computed using the soft graviton metric $S_{\mu\nu}$ and $\mathcal{B}^{\alpha\beta,\mu\rho\nu\sigma}$ is an arbitrary tensor which may depend upon the theory we are considering.

Beyond subleading order....

- Once we allow for the arbitrary tensor $\mathcal{B}^{\alpha\beta,\mu\rho\nu\sigma}$, the universality of the theorem is lost (even though the factorization needed for the soft theorem may still work).

- For the case of single soft graviton, the explicit form for the subsubleading order result is known. This involves a universal part and a non universal part. (Laddha, Sen)

- For the case of multiple soft gravitons, one needs to work out more vertices. But, the procedure is straightforward.

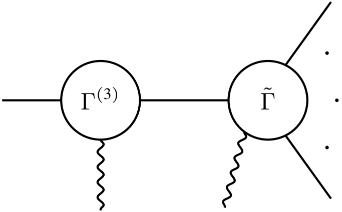
Beyond subleading order....

- However, even though the result depends upon the theory under consideration, at subsubleading order the amplitude with soft external gravitons can still be expressed in terms of the amplitude without the soft gravitons multiplied by some soft factors.

- However, beyond subsubleading order, the soft graviton theorem breaks down (i.e. we can't express the amplitude involving soft gravitons in terms of amplitudes without soft gravitons multiplied by some soft factors).

Beyond subleading order....

- To see this, we recall the following diagram for the two soft gravitons



- At subsubsubleading order, we shall need to evaluate the $\tilde{\Gamma}$ upto $\mathcal{O}(k_\mu k_\nu)$.
- Since the covariantization procedure was directly used for the evaluation of $\tilde{\Gamma}$, it will be affected by Riemann coupling term.

Beyond subleading order....

- Now, the Riemann coupling term will not be of the factorised form in which it is given by a soft factor times an amplitude without the soft gravitons. Thus, the soft theorem breaks down.

- **To summarize: The soft graviton theorem is universal for leading and subleading orders, non-universal at sub-subleading order and breaks down beyond it.**

Some comments about Yang-Mills theories

- One can follow the similar procedure to derive the soft theorems in the Yang-Mills theories.
- The leading result in this case is known to be universal.

  (Weinberg)
- Beyond leading order, there can be a non universal coupling with the field strength $F_{\mu\nu}$ which involves a single derivative only. The diagramatics now shows that the subleading result is non universal and beyond subleading order, the soft theorem breaks down.
- **To summarize: The soft photon/gluon theorem is universal for leading order, non-universal for subleading order and breaks down beyond it.**

Thank You